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A Contribution to Fixed Point Theory on Metric Space with Graph

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Dedication

This dissertation is dedicated to:

My loving parents Zohra and Mebarek, without them I would never have seen the light of this day.

My brothers Ahmed and Mohammed, My sisters Hadjer and Sara, who have always helped me and believed that I could do it...

The Soul of my uncle Abdessalem.

All my family and all my friends.

All those who have contributed directly or indirectly to the realization of this work. To those who -from near or far- have, with me, gone a long way, all those who supported, guided, advised, oriented and helped me.

Thank you!

Khadidja

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Contents

List of Symbols

sup	Supremum (or least upper bound)
inf	Infimum (or greatest lower bound)
max	Maximum
min	Minimum
$\mathbb{N} \cup 0$	Set of all natural numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^+$	Set of all non negative real numbers
Δ	The diagonal of the Cartesian product $X \times X$
$P(X)$	Family of all nonempty subsets of X
$P_{cl}(X)$	Family of all nonempty closed subsets of X
$P_{cp}(X)$	Family of all nonempty compact subsets of X
$P_{cv}(X)$	Family of all nonempty convex subsets of X
$C(I,\mathbb{R})$	Banach space of all continuous functions on bounded sets I equipped with the sup-norm
$d(x,y)$	Distance between x and y
$D_d(A,B)$	$\inf\{d(a,b) \mid a \in A, b \in B\}$, $A,B \in P(X)$
$e_d(A,B)$	$\sup\{D_d(a, B) \mid a \in A\}$, $A,B \in P(X)$
$H_d(A,B)$	$\max\{e_d(A,B), e_d(B,A)\}$
$M(x,y,t)$	Fuzzy set between x and y, $t > 0$
$M(a,B,t)$	$\sup_{b\in B} M(a,b,t)$, $t > 0$, $A \in P_{cp}(X)$
$H_M(A,B,t)$	$\min\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\}$, $t > 0$, $A,B \in P_{cp}(X)$

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Research Publications and Submission

1. A. Boudaoui, **K. Mebarki**, W. Shatanawi, and K. Abodayeh. Solution of some impulsive differential equations via coupled fixed point. Symmetry, 13(3):501, 2021.
2. **K. Mebarki**, A. Boudaoui, W. Shatanawi, K. Abodayeh, and T. A. Shatnawi. Solution of differential equations with infinite delay via coupled fixed point. Heliyon, 8(2):e08849, 2022.
3. **K. Mebarki**, A. Boudaoui. Coupled fixed points for edge preserving mappings in a b-fuzzy metric space with a graph (submitted).
4. **K. Mebarki**, A. Boudaoui. Coupled fixed point for multi-valued contraction in b-metric space endowed with graph (submitted).
5. **K. Mebarki**, A. Boudaoui. Solution of some fractal-fractional differential equations via coupled fixed point in b-fuzzy metric spaces (submitted).

Introduction

The Fixed Point Theory is one of the most powerful and productive tools from the nonlinear analysis. It is an important mathematical discipline because of its applications in different areas such as differential equations, optimization theory and variational analysis. We can model these problems by the equation

$$Tx = x,$$

where T is a nonlinear operator defined on a metric space. The solutions of this equation are called fixed points of T; see [1, 9, 11, 13, 37, 38, 57, 62].

For this reason, we have found numerous researches that assert the existence of fixed points of a mapping T of X into itself under certain conditions (on the mapping T and the space X).

In this context, several results have been found in two different fields: The topological fixed point (Brouwer's fixed point theorem, Schauder's fixed point theorem); see [28, 60, 90, 98], and the metric fixed point (Contraction mapping theorem); see [18, 20, 30, 31, 61, 69, 82].

The most important result of metric fixed point is the Banach fixed point theorem, also known as the Banach contraction principle, first appeared in 1922 in Banach's thesis [18]. This theorem quickly became one of the central results of analysis. Because of its major impact, this theorem is considered as the cornerstone of metric fixed point theory.

Many results have been obtained in recent years extending Banach's theorem to partially ordered spaces; see [2, 10, 29, 72, 73, 76, 79, 81]. Knaster and Tarski [63, 98] established the first fixed point result with considerations of order. Their theorem asserts that if $(X, \preceq)$ is a complete lattice and $T : X \to X$ is order-preserving, then T has a fixed point and the set of fixed points of T is a complete lattice. In 2004, Ran and Reurings [81] proved the existence of fixed point of a continuous and monotone mapping T, with some restriction, a classical contractive condition, and such that for some $x_0 \in X$, either $x_0 \preceq Tx_0$ or $Tx_0 \preceq x_0$ in complete metric space (X, d) endowed with a partial ordering $\preceq$. They also presented its applications to linear and nonlinear matrix equations.

Subsequently, Nieto and Rodríguez-López [72] extended Ran and Reurings result by replacing the continuity with an assumption ensuring that for every nondecreasing (or non-increasing) sequence $\{x_n\}$, if $x_n \to x$ then $x_n \preceq x$ (or $x \preceq x_n$) for every $n \in \mathbb{N}$. As an application, they obtained a theorem on the existence of a unique solution for periodic boundary problems relative to ordinary differential equations. Further improvements of the above results can be found in [73, 76]. The main characteristic of these works is that the contractivity condition is only assumed to hold on comparable elements with respect to the partial order, and their main strategy involves combining the ideas of iterative methods with those of monotone methods.

Nadler [70] is the first to generalize the contraction principle to multi-valued contractions with nonempty closed bounded values defined on a complete metric space.

In 2008, Jachymski [59] gave a more general unified version of the previous results by considering graphs instead of partial orders and by introducing the notion of single-valued G-contraction in complete metric spaces endowed with a graph. In recent years, a substantial amount of researches based on Jachymski's technique has been published, studying different contractions for single-valued [23–26, 51, 89, 95].

In 2015, Monder [5] extended the conclusion of the results given by Ran and Reurings [81] to the case of monotone multi-valued mappings in metric spaces endowed with a graph. There have also been various generalizations in different spaces in this direction; see [35, 42, 56, 71].

On the other hand, in 1987, Guo and Lakshmikantham [50] introduced the concept of Coupled fixed point. Several years later, Bhaskar and Lakshmikantham [21] presented the coupled fixed points results in the setting of ordered metric space. They also gave an application in which they proved the existence and uniqueness of a solution for a boundary value problem. Many researchers have obtained coupled Fixed Point results in metric space, ordered metric spaces, $b-$metric space, fuzzy metric spaces, and other spaces; see [74, 78, 87, 91, 102].

In 2014, Chifu and Petrusel [33] generalized the results obtained by Gnana Bhaskar and Lakshmikantham [21] to metric spaces endowed with a directed graph. As an application, they obtained the existence of a continuous solution for a system of Fredholm and Volterra integral equations. Several researchers have obtained coupled fixed point in metric spaces, $b-$metric space, fuzzy metric spaces, and other spaces endowed with directed graph; see [6, 14, 34, 36, 97, 100].

The purposes of this dissertation are to create a new concepts of contraction and prove a coupled fixed point theorems in b-metric space and b-fuzzy metric space endowed with a direct graph. As application, we apply our rsults to obtain the sufficient conditions for the existence and uniqueness of solutions different type of differential equations.

The dissertation has been organized into five chapters. The 1^{st} **Chapter**, Preliminaries, is a brief overview of the prerequisites of this dissertation. Throughout the three sections, we refer to the concepts of graphs, $b-$metric and $b-$fuzzy metric space, basic notions and some recent coupled fixed point theorem.

The other chapters are organized into two parts, each with two chapters. The first part deals with coupled fixed points of single-valued mappings, while the second part is devoted to coupled fixed points of multi-valued mappings.

Chapter 1 in **Part I** is devoted to coupled fixed points theorems in a b-metric space with a graph. In the **first Section** of the chapter, the concept of contraction given by Seshagiri Rao and Kalyani [93] has been extended by introducing $b-$contraction. Based on this notion, we obtain the existence and uniqueness of coupled fixed points in $b-$metric space with a graph. Additionally, to support the applicability of these results, an application to impulsive differential equations is given. **Section two** deals with the generalization of the results of Işik and Türkoğlu [58] by introducing a new contraction who that generalize their contraction in $b-$metric space endowed with a direct graph. Some coupled fixed point theorems in complete $b-$metric spaces have been investigated under this contraction. To demonstrate the usability of the presented results, we applied them in differential equations with infinite delay.

Chapter 2 is intended to establish some coupled fixed points results for mappings satisfying φ-fuzzy contraction in $b-$fuzzy metric space. The result is applied to prove the existence of a continuous solution for a system of fractional differential equations of the Caputo type.

In the **first Chapter** in **Part II**, we introduce the concept of $\mu - \psi$-contraction for multi-valued mapping in $b-$metric space endowed with the directed graph. Then, some coupled fixed point theorems have been established. Additionally, we present an application to fractional differential equations of the Caputo type.

The objective of the **second Chapter** is to define φ-multi-fuzzy contraction of multi-valued mapping and demonstrate an existing and unique coupled fixed point result in $b-$fuzzy metric space. We validate these results using an application to fractal-fractional differential equations.

At the end of this dissertation, we present the conclusion of the present study, with a suggestion of perspectives for future research.

Chapter 1

Preliminaries

To provide adequate background for consequent chapters. This preliminary chapter presents some properties of graph theory, basic definitions and some recent results connected to this work. However, some basic definitions will be rehashed in numerous chapters for convenience.

1.1 Graph theory

Graph theory is the study of mathematical structures used to model pairwise relations between objects from a certain collection. It is an essential part of many disciplines, including mathematics, engineering, physical, social, biological, computer science, linguistics, and many others (see [4, 32, 40, 45, 46, 49, 53, 55, 77, 96]). Many of the early concepts and theorems of graph theory came about indirectly, often from recreational mathematics, through puzzles or games or problems that could be phrased in graphs. The very first of these was a puzzle called the Königsberg Bridge Problem. This problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island but without crossing any bridge twice.

The history of graph theory can be traced back to 1735 when Leonhard Euler (1707-1782) published a paper in which he only involved references to the physical arrangement of the bridges. Still, he proved the first theorem in graph theory, which gave birth to graph theory. Because graph theory is thought to have begun in 1736 with the publication of

Euler's solution to the Königsberg bridge problem, Euler became known as the "Father of Graph Theory".

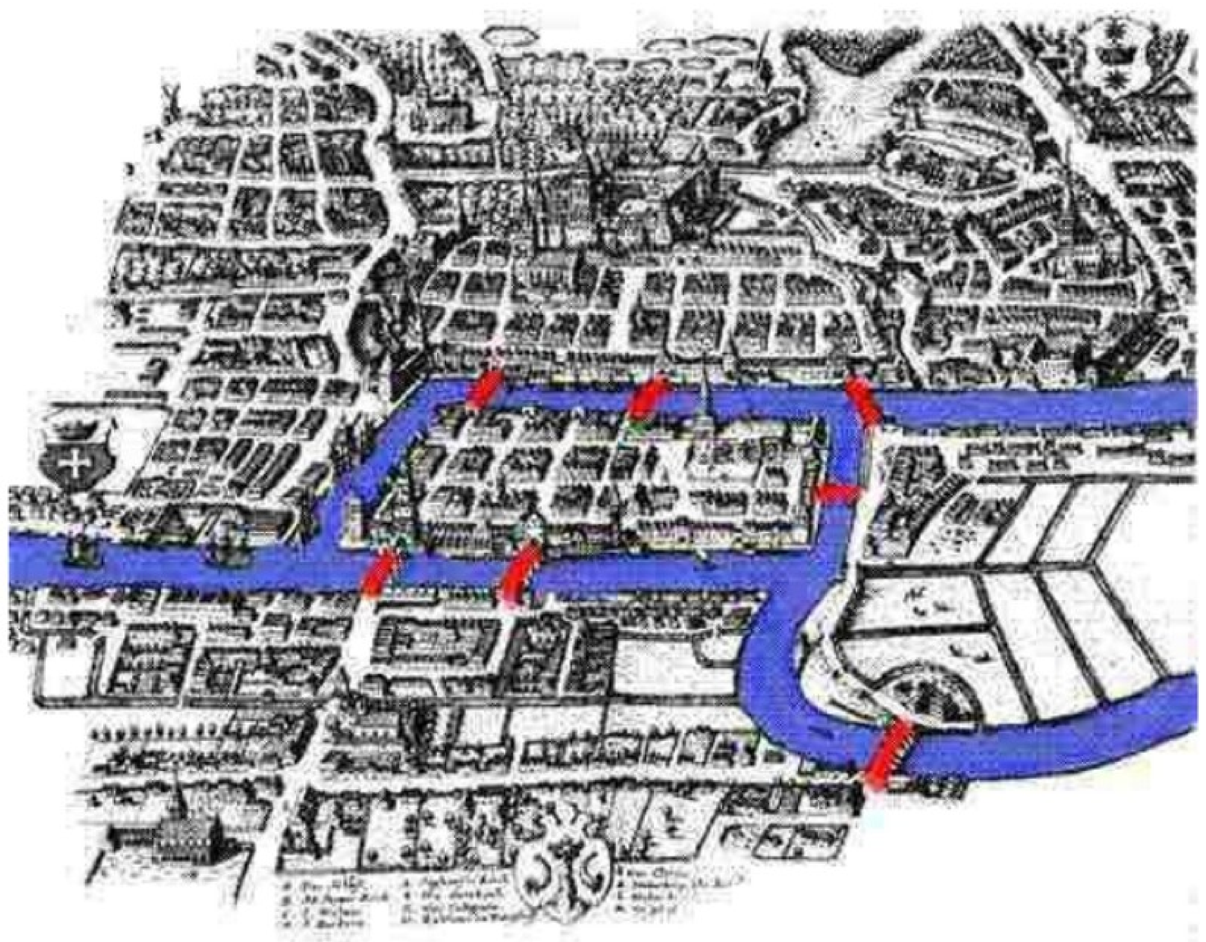

Figure 1.1: The seven bridges of Königsberg.

A graph is a pair $G = (V,E)$ of sets satisfying $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. The elements of V are the vertices (or nodes, or points) of the graph G, the elements of E are its edges (or lines). The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information which pairs of vertices form an edge and which do not.

A graph with vertex set V is said to be a graph on V. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. These conventions are independent of any actual names of these two sets: the vertex set W of a graph $H = (W,F)$ is still referred to as $V(H)$, not as $W(H)$. We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on.

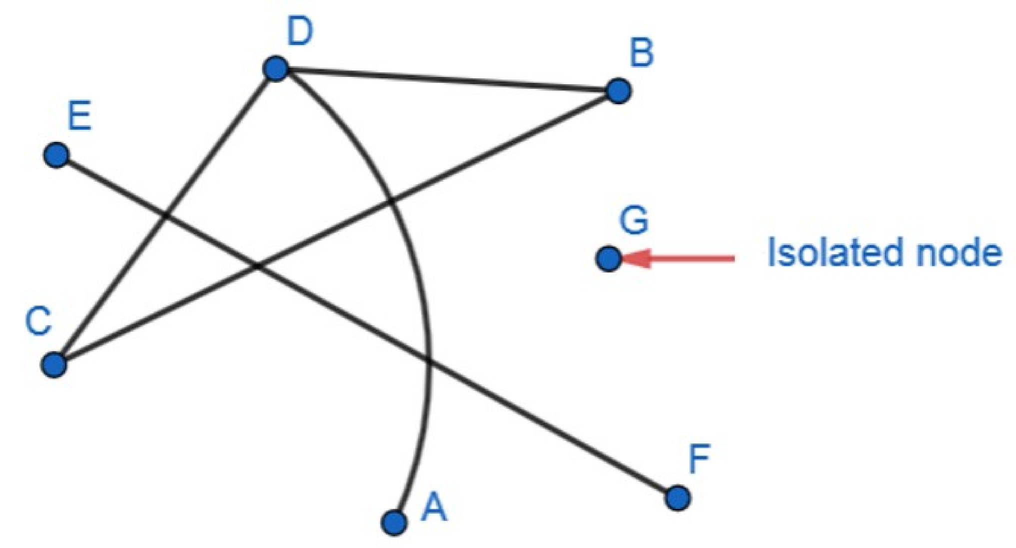

Figure 1.2: Graph H.

On $V = \{A,...,G\}$ with edge set $E = \{\{A,D\},\{B,C\},\{B,D\},\{C,D\},\{E,F\}\}$.

Definition 1.1. A directed graph (or digraph) is a graph that is made up of a set of vertices connected by edges, where the edges have a direction associated with them.

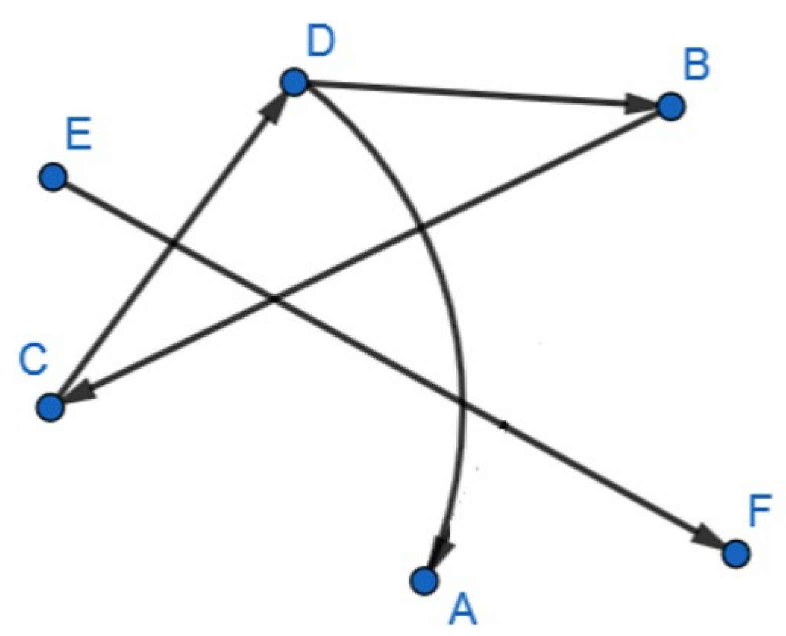

Figure 1.3: Directe graph G.

Definition 1.2. A weighted graph is a graph obtained by assigning to each edge the distance between the vertices.

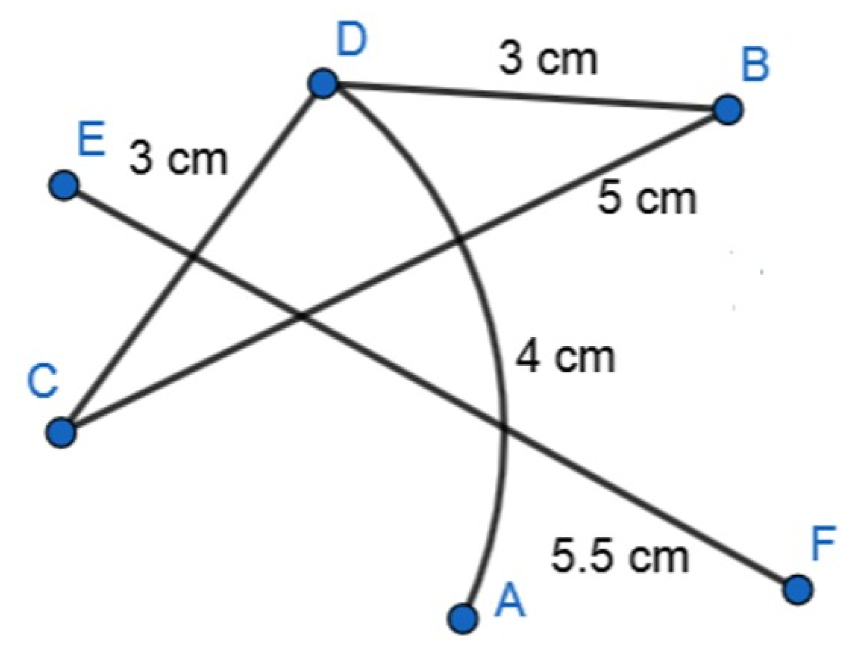

Figure 1.4: Weighted graph.

Definition 1.3. By G^{-1} we denote the conversion of a directed graph G that is a graph obtained by G reversing its edges; i.e.

$$E(G^{-1}) = \{(y,x) \in X \times X : (x,y) \in E(G)\}.$$

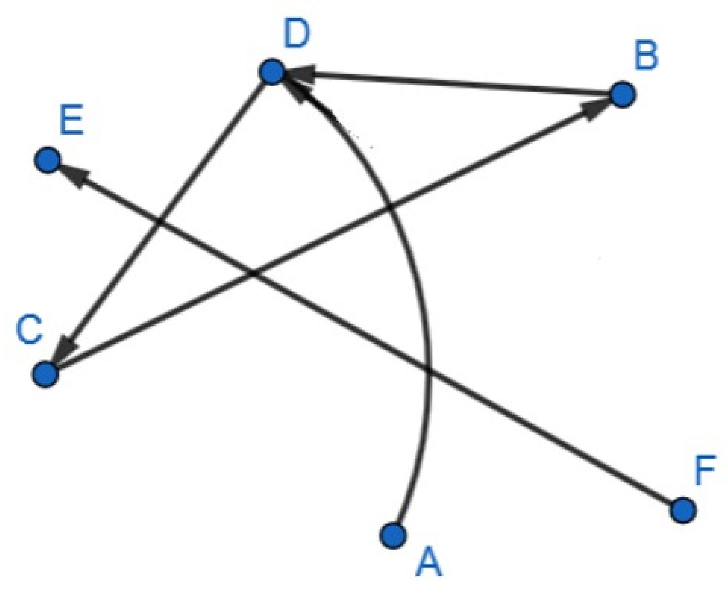

Figure 1.5: Conversion graph obtained from digraph G.

$\tilde{G}$ denotes the undirected graph obtained from G by ignoring the directions of the edges of G. We consider G as a directed graph whose set of edges is symmetric, thus we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

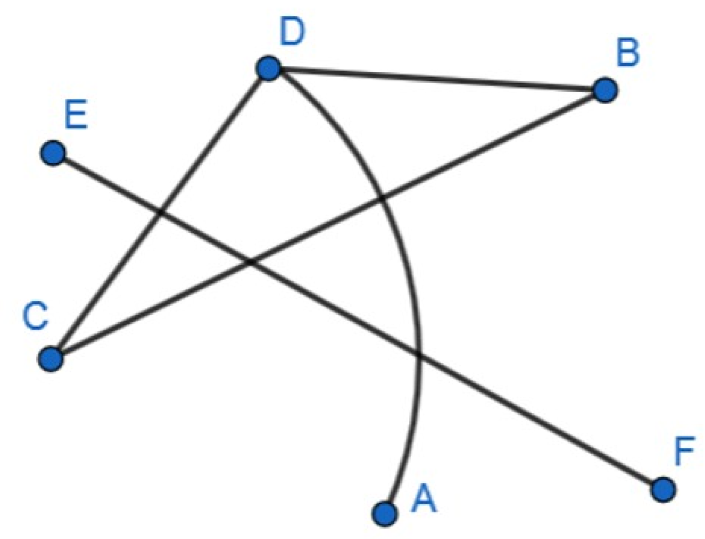

Figure 1.6: Undirected graph obtained from digraph G.

Definition 1.4. A subgraph of a graph H is a graph L such that $V(L) \subseteq V(H)$ and $E(L) \subseteq E(H)$.

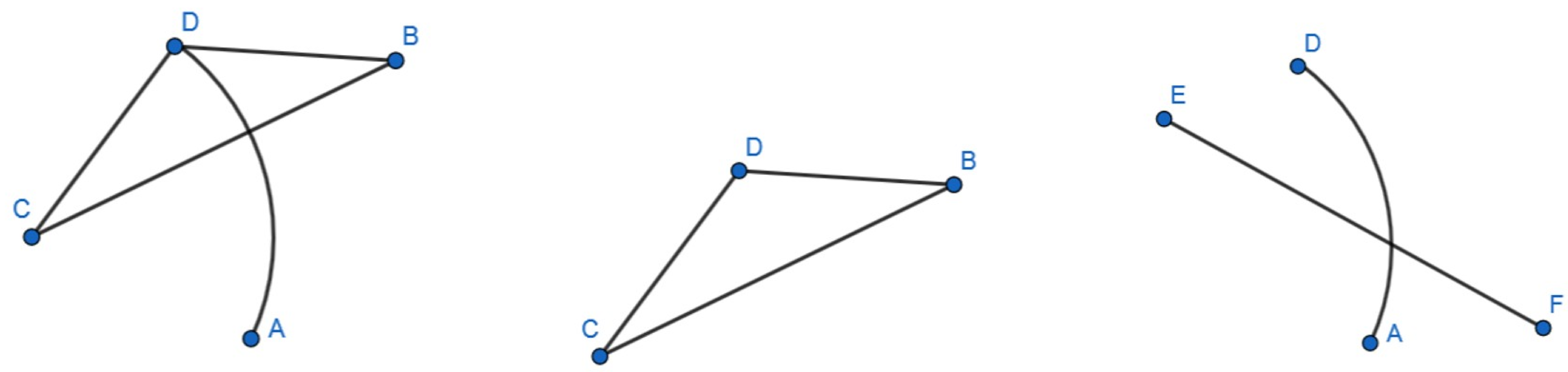

Figure 1.7: Subgraphs of graph H.

Definition 1.5. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that

$$x_0 = x \ , \ x_N = y,$$

and

$$(x_i, x_{i+1}) \in E(G), \ i = 1,....,N.$$

Definition 1.6. The number of edges in G constituting the path between any two vertices of G.

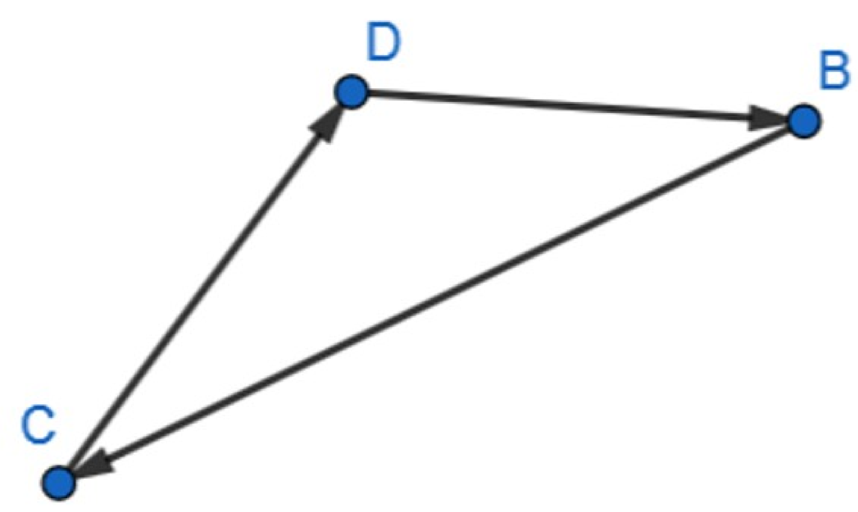

Figure 1.8: Path from 'B' to 'D' of length 2.

Definition 1.7. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

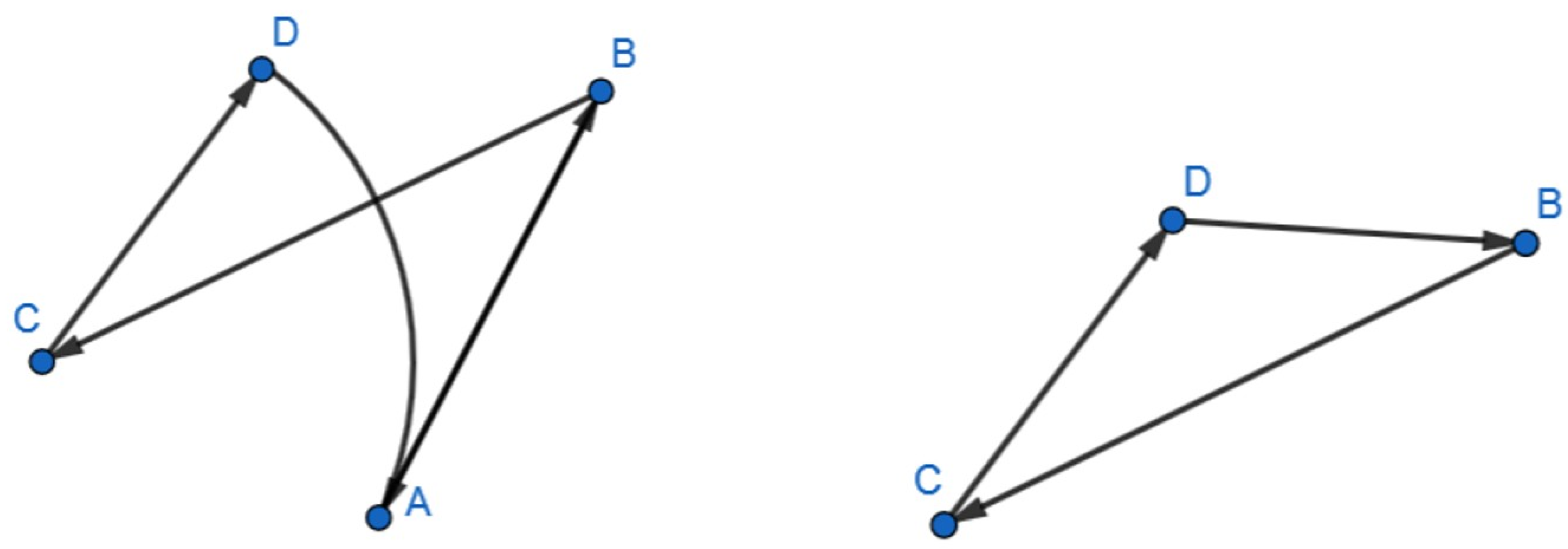

Figure 1.9: Connected digraphs.

If G is not connected then it is called disconnected and its different paths are called the components of G. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. Thus $V(G) = [x]_G$ where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected. For more information about this subject, see [41].

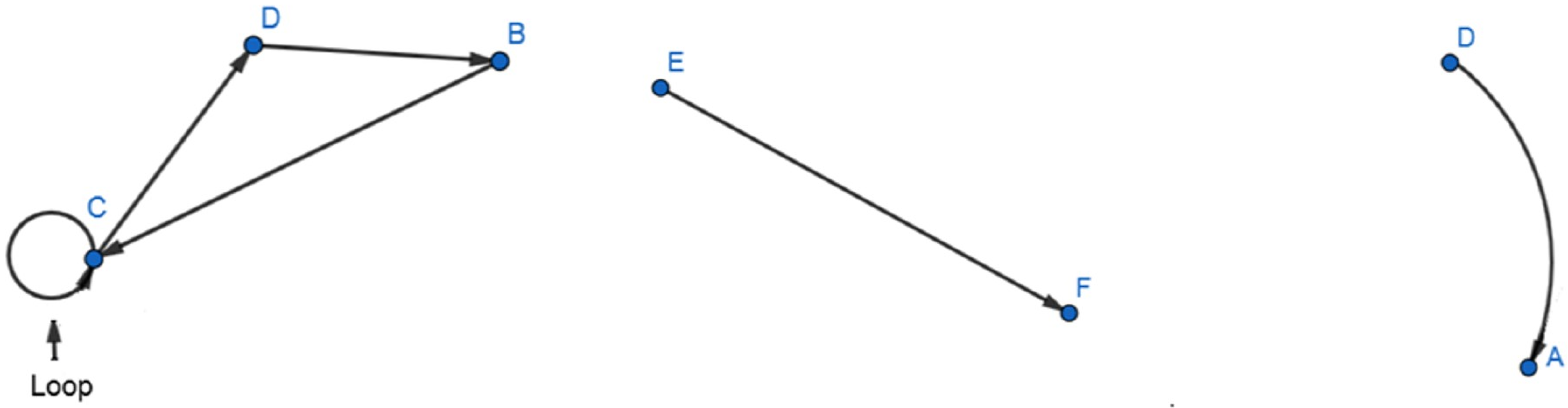

Figure 1.10: Components of the digraph G.

Each of which is a subgraph.

The following property and definition will be used to prove the main results in this dissertation.

Property 1.1.
1. for any $\{x_n\}_{n\in\mathbb{N}}$ in X such that $(x_n, x_{n+1}) \in E(G)$ and $\lim\limits_{n\to+\infty} x_n = x$, then $(x_n, x) \in E(G)$,

   and

2. for any $\{x_n\}_{n\in\mathbb{N}}$ in X such that $(x_{n+1}, x_n) \in E(G)$ and $\lim\limits_{n\to+\infty} x_n = x$, then $(x,x_n) \in E(G)$,

Alfuraidan and Khamsi [6] presented the following property of mixed G-monotone.

Definition 1.8. Endowed the complete metric space (X,d) with the direct graph G. The mapping $T\colon X \times X \to X$ possesses the mixed G-monotone property if

$$(x_1, x_2) \in E(G) \Rightarrow (T(x_1, y), T(x_2, y)) \in E(G),$$

for all $x_1, x_2, y \in X$, and

$$(y_1,y_2) \in E(G) \Rightarrow (T(x, y_2), T(x, y_1)) \in E(G),$$

for all $x, y_1,y_2 \in X$.

1.2 $b-$metric space and b-fuzzy metric space

In this section, we will recall the main notions we will need.

1.2.1 b-metric space

The study of b-metric spaces was initiated in 1993 by Czerwik [39], he put another axiom which is weaker than the triangle inequality.

Definition 1.9. Let X be a set, and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X, and the pair (X,d) is called a b-metric space if, for all $x,y,z \in X$, *i)* $d(x,y) = 0$ if and only if $x = y$,

ii) $d(x,y) = d(y,x)$,

iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

Example 1.1. let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where

$$l_p(\mathbb{R}) := \left\{ x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Then $d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$ is a b-metric on X with $s = 2^{1/p}$.

Obviously, each metric space is a b-metric space (for $s = 1$). However, b-metric on X need not be a metric on X. The following simple examples can be used to show this.

Example 1.2. Let $X = \{x_1,x_2,x_3\}$ and $d : X \times X \to \mathbb{R}^+$ be such that $d(x_1,x_2) = x \geq 3$, $d(x_1,x_3) = 2$, $d(x_2,x_3) = 1$, $d(x_n,x_n) = 0$, $d(x_n,x_k) = d(x_k,x_n)$ for $n,k = \{1,2,3\}$. Then

$$d(x_n,x_k) \leq \frac{x}{3}\left[d(x_n,x_i) + d(x_i,x_k)\right], \quad n,k,i = \{1,2,3\}.$$

Hence, (X,d) is a b-metric space (with $s = x/3$), and not a metric space if $x > 3$.

Note that, in general a b-metric is not continuous.

Example 1.3. Let $X = \mathbb{N} \cup \infty$ and let $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ \left|\frac{1}{x} - \frac{1}{y}\right| \text{ if } x, y \text{ are even or xy } = \infty \\ 5 \text{ if } x, y \text{ are odd and } x \neq y \\ 2 \text{ otherwise} \end{cases}$$

Then (X,d) is a b-metric with $s = 3$ but it is not continuous.

Definition 1.10. Let (X,d) be a b-metric space.

1. A sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

2. A sequence $\{x_n\}$ is said to be Cauchy if and only if $d\,(x_n,x_m) \to 0$ as $m,n \to \infty$.

Definition 1.11. When every Cauchy sequence is convergent, the b-metric space (X,d) is said to be complete.

Now, we define the continuity notion of multi-valued mappings.

Definition 1.12. Let $S : X \times X \to P(X)$ be a multi-valued mapping. We will say that S is continuous if for any sequences $\{x_n\}$ and $\{y_n\}$ which converge respectively to x and y, we have

$$\lim_{n\to\infty} H(S(x_n, y_n), S(x,y)) = 0.$$

Lemma 1.1. *[27] Let (X,d) be a b-metric space with constant $s \geq 1$, $A,B \in P(X)$ and $\lambda > 1$. Then, for every $a \in A$ there exists $b \in B$ such that $d(a,b) \leq \lambda e_d(A,B)$.*

1.2.2 b-Fuzzy metric space

The concept of fuzzy set theory was introduced by Zadeh [101]. After that, Kramosil and Michálek [64] introduce the concept of fuzzy metric space. Then, George and Veeramani [47] changed the definition of fuzzy metric, which was presented by Kramosil and Michálek.

Definition 1.13. Let $* : [0,1] \times [0,1] \to [0,1]$ is a binary operation. If it meets the following criteria, it is a continuous t-norm:

1. $*$ is commutative, associative and continuous,

2. $a * 1 = a$ for all $a \in [0,1]$,

3. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a,b,c,d \in [0,1]$.

Definition 1.14. [101] A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function which assigns to each object a grade of membership ranging between zero and one.

Definition 1.15. Let $*$ is a continuous t-norm, X is a non-empty arbitrary set and M is a fuzzy set on $X^2 \times (0,\infty)$ verifying the following conditions for each $x,y,z \in X$ and $t,s > 0$,

1. $M(x,y,t) > 0$,
2. $M(x,y,t) = 1$ if and only if $x = y$,
3. $M(x,y,t) = M(y,x,t)$,
4. $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
5. $M(x,y,.) : \mathbb{R}_+^* \to [0,1]$ is continuous.

The triple $(X,M,*)$ is called fuzzy metric space.

Then, Sedghi and Shobe [92] introduced the notion of b-fuzzy metric space in 2012, in which the fuzzy metric space triangle inequality is replaced by a weaker one.

Definition 1.16. Let $*$ is a continuous t-norm, X is a non-empty arbitrary set and M is a fuzzy set on $X^2 \times (0,\infty)$ verifying the following conditions for each $x,y,z \in X$ and $t,c > 0$ and a given real number $s \geq 1$,

1. $M(x,y,t) > 0$,
2. $M(x,y,t) = 1$ if and only if $x = y$,
3. $M(x,y,t) = M(y,x,t)$,
4. $M(x,y,\frac{t}{s}) * M(y,z,\frac{c}{s}) \leq M(x,z,t+c)$,
5. $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous,

Then, the triple $(X,M,*)$ is called a b-fuzzy metric space with $s \geq 1$.

It should be noted that the class of b-fuzzy metric spaces is effectively larger than that of fuzzy metric spaces, since a b-fuzzy metric space is a fuzzy metric space when $s = 1$.

We present an example that shows that a b-fuzzy metric on X need not be a fuzzy metric on X.

Example 1.4. Let $X = \mathbb{R}$ and $a * b = ab$ for all $a,b \in [0,1]$. For each $t \in (0,\infty)$ define

$$M(x,y,t) = e^{-\frac{|x-y|^p}{t}},$$

where $p > 1$ is a real number. Then $(X,M,*)$ is a b-fuzzy metric space with $s = 2^{p-1}$. We noted that if $p = 2$, $(X,M,*)$ is not a fuzzy metric space.

Definition 1.17. Let $(X,M,*)$ be a b-fuzzy metric space.

1. A sequence $\{x_n\}$ converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$ for each $t > 0$. We write $\lim_{n\to\infty} x_n = x$ in this situation.

2. If for all $0 < a < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_p, x_q, t) > 1 - a$ for all $p,q \geq n_0$, the $\{x_n\}$ is termed a Cauchy sequence.

Definition 1.18. When every Cauchy sequence is convergent, the b-fuzzy metric space $(X,M,*)$ is said to be complete.

Lemma 1.2. *[92] We have the following in a b-fuzzy metric space $(X,M,*)$:*

1. *If a sequence $\{x_n\}$ in X converges to x, then x is unique;*

2. *If a sequence $\{x_n\}$ in X converges to x, then it is a Cauchy sequence.*

Definition 1.19. Let $(X,M,*)$ be a b-fuzzy metric space with $s \geq 1$. Then $M(x,y,t)$ is b-nondecreasing with respect to t; that is,

$$\text{if } t > sb \text{ implies } M(x,y,t) \geq M(x,y,b).$$

Definition 1.20. Let $S : X \times X \to P(X)$ be a multivalued mapping. We will say that S is continuous if for any sequences $\{x_n\}$ and $\{y_n\}$ which converge respectively to x and y, we have

$$\lim_{n\to\infty} H_M(S(x_n, y_n), S(x, y),t) = 0, \quad \forall t > 0.$$

We will use the following lemmas to prove the main results in the second part of this dissertation.

Lemma 1.3. *[85] Allow* $(X,M,*)$ *to be a b-fuzzy metric space in which* M *is a continuous function on* $X \times X \times (0,+\infty)$*. Then, for each* $a \in X$*,* $B \in P_{cp}(X)$ *and* $t > 0$*, there is* $b_0 \in B$ *such that*

$$M(a,B,t) = M(a,b_0,t).$$

Proof. Let $a \in X$, $B \in P(X)$ and $t > 0$. By the continuity of M and by the compactness of B, there exists $b_0 \in B$ such that $\sup_{b \in B} M(a,b,t) = M(a,b_0,t)$, i.e.,
$M(a,B,t) = M(a,b_0,t)$. □

Lemma 1.4. *[94] Let* $(X,M,*)$ *be a complete b-fuzzy metric space, such that* $(P_{cp}(X),H_M,*)$ *is a hausdorff b-fuzzy metric space on* $P_{cp}(X)$*. Then, for all* A*,* $B \in P_{cp}(X)$*, for each* $a \in A$ *and for* $t > 0$*, there exists* $b_a \in B$*, satisfies* $M(a,B,t) = M(a,b_a,t)$*, then*

$$H_M(A,B,t) \leq M(a,b_a,t).$$

1.3 Some Elementary Coupled Fixed Point Theorems

in 1987, Guo and Lakshmikantham [50] introduced the notion of coupled fixed points.

Definition 1.21. The pair $(x,y) \in X \times X$ is called a coupled fixed point of
$T : X \times X \to X$ if

$$x = T(x,y) \text{ and } y = T(y,x).$$

A.Petruşel, G Petruşel, B.Samet and J.Yao [75] introduced the concept of coupled fixed point for multi-valued mappings.

Definition 1.22. The pair $(x,y) \in X \times X$ is called a coupled fixed point of
$S : X \times X \to P(X)$ if

$$x \in S(x,y) \text{ and } y \in S(y,x).$$

Denotes the graph of a multivalued mapping $S : X \times X \to P(X)$ by

$$Graph(T) := \{(x,y,z) \in X^3 : z \in S(x,y)\}.$$

We will present in the first chapter of PartI a generalization of the following result obtained by Seshagiri Rao and Kalyani.

Theorem 1.1. *[93] Endowed the set X with partial order $\preceq$. On $(X,d,\preceq)$, let the continuous map $T\colon X\times X\to X$ with a strict mixed monotone property on X satisfies:*

$$d(T(x,y),T(u,v)) \leq \alpha\frac{d(x,T(x,y))[1+d(u,T(u,v))]}{1+d(x,u)} + \beta[d(x,T(x,y))+d(u,T(u,v))]+\gamma d(x,u),$$

where $\alpha,\beta,\gamma\in[0,1)$ such that $1>\alpha+2\beta+\gamma$. If there exist two points x_0, $y_0\in X$ with $x_0\preceq T(x_0,y_0)$ and $T(y_0,x_0)\preceq y_0$, then T possesses a coupled fixed point $(x,y)\in X\times X$.

Işik and Türkoğlu [58] introduced the set Υ of pair of functions (φ,ψ), where $\varphi,\psi\colon[0,\infty)\to[0,\infty)$ satisfying the following conditions:

1. φ is continuous and non-decreasing;
2. $\varphi(t)=0$ if and only if $t=0$;
3. $\varphi(t+s)\leq\varphi(t)+\varphi(s)$, $\forall t,s\in[0,\infty)$;
4. ψ is continuous;
5. $\varphi(t)>\psi(t)$ for all $t>0$.

And they proved the following results that we will offer a generalization of them in the first chapter of Part I.

Theorem 1.2. *Let $(X,\preceq,d)$ be a complete partially ordered metric space. Let $T:X\times X\to X$ be a mixed monotone mapping for which there exist $(\varphi,\psi)\in\Upsilon$ such that for all $x,y,u,v\in X$ with $x\succeq u, y\preceq v$,*

$$\varphi(d(T(x,y),T(u,v)))\leq\frac{1}{2}\psi(d(x,u)+d(y,v)).$$

Suppose either

1. *T is continuous*

   or

2. *X satisfies the following property*

$$\text{if } \{x_n\} \text{ is a nondecreasing sequence with } x_n \to x \text{ then } x_n \preceq x \text{ for all } n,$$
$$\text{if } \{y_n\} \text{ is a nonincreasing sequence with } y_n \to y \text{ then } y \preceq y_n \text{ for all } n.$$

If there exist $x_0, y_0 \in X$ *with* $x_0 \preceq T(x_0, y_0)$ *and* $T(y_0, x_0) \preceq y_0$*, then* T *has a coupled fixed point.*

Part I

Coupled Fixed Points of Single-valued Mappings

Chapter 2

Coupled Fixed Points Theorems in a b-metric Space with a Graph

In this chapter, we offer new coupled fixed point theorems that are a generalization of certain recent developments using directed graphs with the connotation of $b-$metric spaces.

Throughout this chapter, (X,d,G) stands to a complete b-metric space with $s \geq 1$, endowed with directed graph G such that the set $V(G) = X$, $\Delta \subseteq E(G)$, G is transitive ($(a,b) \in E(G)$ and $(b,c) \in E(G)$ implies that $(a,c) \in E(G)$) and has no parallel edges. The mapping $T\colon X \times X \to X$ possesses the mixed G-monotone property.

Further, we endow the product space $X \times X$ by another graph denoted also by G, such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for any $(x,y),(u,v) \in X \times X$.

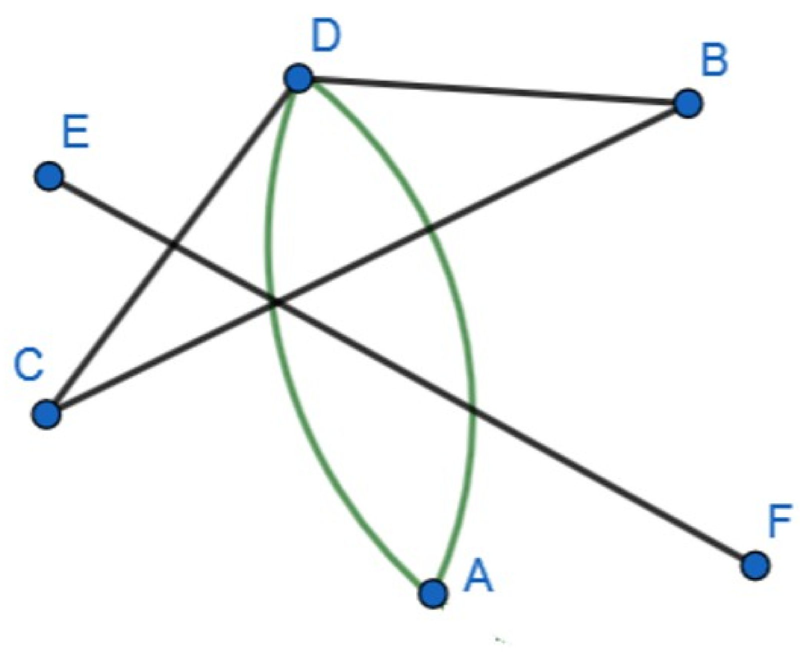

Figure 2.1: Graph with parallel edges.

2.1 Coupled Fixed Point Theorems for b-contraction Mappings

We considered the following contraction.

Definition 2.1. The mapping $T : X \times X \to X$ is called $b-$contraction if there exist $\alpha,\beta,\gamma \in [0,1)$ with

$$\sum_{i=0}^{\infty} s^i \left(\frac{\beta+\gamma}{1-\alpha-\beta}\right)^i < \infty$$

such that

$$d(T(x,y),T(u,v)) \leq \alpha \frac{d(x,T(x,y))[1+d(u,T(u,v))]}{1+d(x,u)} + \beta[d(x,T(x,y))+d(u,T(u,v))]+\gamma d(x,u),$$

holds for all $(x,y),(u,v) \in X \times X$ with $((x,y),(u,v)) \in E(G)$.

Using the previous contraction, we proved a generalization of Seshagiri Rao and Kalyani [93] results in b-metric space endowed with a directed graph.

2.1.1 Existence of Coupled Fixed Points

In the sequel of this section, we assume that d is continuous.

Theorem 2.1. *On (X,d,G), suppose that T is continuous and $b-$contraction mapping. If there exist $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.*

Proof. Set $x_1 = T(x_0,y_0)$ and $y_1 = T(y_0,x_0)$. The assumption implies that $((x_0,y_0),(x_1,y_1)) \in E(G)$. Hence

$$d(x_2,x_1) = d(T(x_1,y_1),T(x_0,y_0)) \leq \alpha\frac{d(x_1,T(x_1,y_1))[1+d(x_0,T(x_0,y_0))]}{1+d(x_1,x_0)} + \beta[d(x_0,T(x_0,y_0))+d(x_1,T(x_1,y_1))] + \gamma d(x_0,x_1).$$

So,

$$d(x_2,x_1) \leq \frac{\beta+\gamma}{1-\alpha-\beta}d(x_1,x_0).$$

Similarly, since $((y_1,x_1),(y_0,x_0)) \in E(G)$, then

$$d(y_2,y_1) \leq \frac{\beta+\gamma}{1-\alpha-\beta}d(y_1,y_0).$$

Further, for $n = 1,2,...$, we let

$$x_{n+1} = T(x_n,y_n), \text{ and } y_{n+1} = T(y_n,x_n).$$

Referring to the fact that T possesses the mixed G-monotone property on X, we have

$$((x_n,y_n),(x_{n+1},y_{n+1})) \in E(G) \text{ and } ((y_{n+1},x_{n+1}),(y_n,x_n)) \in E(G).$$

Then

$$d(x_{n+1},x_n) \leq \frac{\beta+\gamma}{1-\alpha-\beta}d(x_n,x_{n-1}),$$

and

$$d(y_{n+1},y_n) \leq \frac{\beta+\gamma}{1-\alpha-\beta}d(y_n,y_{n-1}).$$

Therefore, for $n \in \mathbb{N}$ we get

$$d(x_{n+1},x_n) \leq \left(\frac{\beta+\gamma}{1-\alpha-\beta}\right)^n d(x_1,x_0), \tag{2.1}$$

and

$$d(y_{n+1},y_n) \leq \left(\frac{\beta+\gamma}{1-\alpha-\beta}\right)^n d(y_1,y_0). \tag{2.2}$$

For $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, by (2.1) we gain

$$\begin{aligned} d(x_n,x_{n+p}) &\leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + ... + s^n d(x_{n+p-1},x_{n+p}) \\ &= \frac{1}{s^{n-1}}\sum_{i=n}^{n+p-1} s^i d(x_i,x_{i+1}) \\ &\leq \frac{1}{s^{n-1}}\sum_{i=n}^{n+p-1} s^i \left(\frac{\beta+\gamma}{1-\alpha-\beta}\right)^i d(x_0,x_1). \end{aligned}$$

By assumption, we get $\lim_{n\to\infty} d(x_n,x_{n+p}) = 0$.

By the same process we obtain

$$d(y_n,y_{n+p}) \leq \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i \left(\frac{\beta+\gamma}{1-\alpha-\beta}\right)^i d(y_0,y_1).$$

Then $\lim_{n\to\infty} d(y_n,y_{n+p}) = 0$.

This imply that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy. The completeness of X implies that $x^*,y^* \in X$ with

$$\lim_{n\to\infty} x_n = x^* \text{ and } \lim_{n\to\infty} y_n = y^*.$$

The continuity of T implies that

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1},y_{n-1}) = T\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}\right) = T(x^*,y^*),$$

$$y^* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T(y_{n-1},x_{n-1}) = T\left(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}\right) = T(y^*,x^*),$$

i.e., T possesses (x^*,y^*) as a couple fixed point. □

The continuity of T in Theorem 2.1 can be discarded by adding some new conditions. Now, assume that (X,d,G) possesses property1.1.

Theorem 2.2. *Endowed (X,d,G) with the property1.1. Suppose that T is b-contraction mapping. If there exist $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.*

Proof. By referring to the proof of Theorem 2.1, we need only to show that $x^* = T(x^*,y^*)$ and $y^* = T(y^*,x^*)$.

Since $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n,y_n) = x^*$, $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} T(y_n,x_n) = y^*$ and $(x_n,x_{n+1}) \in E(G)$ and $(y_{n+1},y_n) \in E(G)$, the property1.1 implies that

$$(x_n,x^*) \in E(G) \text{ and } (y^*,y_n) \in E(G).$$

So,

$$((x_n,y_n),(x^*,y^*)) \in E(G).$$

Thus, we get

$$\begin{aligned} d(T(x_n,y_n),T(x^*,y^*) &\leq \alpha \frac{d(x_n,T(x_n,y_n))[1+d(x^*,T(x^*,y^*))]}{1+d(x_n,x^*)} \\ &\quad + \beta[d(x_n,T(x_n,y_n)) + d(x^*,T(x^*,y^*))] + \gamma d(x_n,x^*) \\ &= \alpha \frac{d(x_n,x_{n+1})[1+d(x^*,T(x^*,y^*))]}{1+d(x_n,x^*)} + \beta[d(x_n,x_{n+1}) + d(x^*,T(x^*,y^*))] \\ &\quad + \gamma d(x_n,x^*). \end{aligned}$$

By the same way, we have

$$\begin{aligned} d(T(y_n,x_n),T(y^*,x^*)) &\leq \alpha \frac{d(y_n,y_{n+1})[1+d(y^*,T(y^*,x^*))]}{1+d(y_n,y^*)} \\ &\quad + \beta[d(y_n,y_{n+1}) + d(y^*,T(y^*,x^*))] + \gamma d(y_n,y^*). \end{aligned}$$

letting $n \to \infty$, we arrive $x^* = T(x^*,y^*)$ and $y^* = T(y^*,x^*)$, i.e., T possesses (x^*,y^*) as a couple fixed point. □

Referring to the fact that every metric space is a b-metric, we derive the next results:

Corollary 2.1. Endowed the complete metric space (X,d) with the direct graph G. Let $T\colon X \times X \to X$ be a mixed G-monotone property on X. Assume that there exist $\alpha,\beta,\gamma \in [0,1)$ with $1 > \alpha + \beta + \gamma$ such that

$$\begin{aligned} d(T(x,y),T(u,v)) &\leq \alpha \frac{d(x,T(x,y))[1+d(u,T(u,v))]}{1+d(x,u)} \\ &\quad + \beta[d(x,T(x,y)) + d(u,T(u,v))] + \gamma d(x,u), \end{aligned}$$

holds for all $(x,y),(u,v) \in X \times X$ with $((x,y),(u,v)) \in E(G)$. Suppose either

1. T is continuous

   or

2. (X,d,G) possesses property1.1.

If there exists $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.

2.1.2 Uniqueness of Coupled Fixed Points

Theorem 2.3. *Suppose T satisfies the hypotheses of Theorem 2.1 (Theorem 2.2). If the coupled fixed point (x^*,y^*) of T satisfies $((x^*,y^*),(x_0,y_0)) \in E(G)$, then (x^*,y^*) is unique.*

Proof. If we suppose that there is another coupled fixed point (u,v). By referring to the proof of Theorem 2.1 or Theorem 2.2, we construct two sequence $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such $x_{n+1} = T(x_n,y_n)$ and $y_{n+1} = T(y_n,x_n)$ for $n \in \mathbb{N}$ with $\lim\limits_{n\to\infty} x_n = x^*$ and $\lim\limits_{n\to\infty} y_n = y^*$. Since T possesses the mixed G-monotone, then $((u,v),(x_n,y_n)) \in E(G)$. Therefore,

$$\begin{aligned} d(u,x_{n+1}) =& d(T(u,v),T(x_n,y_n)) \\ & \leq \alpha \frac{d(u,T(u,v))[1 + d(x_n,x_{n+1})]}{1 + d(u,x_n)} + \beta[d(u,T(u,v)) + d(x_n,x_{n+1})] \\ & \quad + \gamma d(u,x_n) \\ & = \beta d(x_n,x_{n+1}) + \gamma d(u,x_n), \end{aligned}$$

and

$$\begin{aligned} d(v,y_{n+1}) =& d(T(v,u),T(y_n,x_n)) \\ & \leq \alpha \frac{d(v,T(v,u))[1 + d(y_n,y_{n+1})]}{1 + d(v,y_n)} + \beta[d(v,T(v,u)) + d(y_n,y_{n+1})] \\ & \quad + \gamma d(v,y_n) \\ & = \beta d(y_n,y_{n+1}) + \gamma d(v,y_n). \end{aligned}$$

On letting $n \to \infty$, we arrive to

$$x^* = u \text{ and } y^* = v.$$ □

Theorem 2.4. *Suppose T satisfies the hypothesis of Theorem 2.1 (Theorem 2.2). $(x^*,y^*) \in E(G)$, then $x^* = y^*$.*

Proof. Since $(x^*,y^*) \in E(G)$, we have $((x^*,y^*),(y^*,x^*)) \in E(G)$. Thus,

$$\begin{aligned} d(x^*,y^*) &= d(T(x^*,y^*),T(y^*,x^*)) \\ & \leq \alpha \frac{d(x^*,T(x^*,y^*))[1 + d(y^*,T(y^*,x^*))]}{1 + d(x^*,y^*)} \\ & \quad + \beta[d(x^*,T(x^*,y^*)) + d(y^*,T(y^*,x^*))] + \gamma d(x^*,y^*) \\ & \leq \gamma d(x^*,y^*). \end{aligned}$$

Hence $x^* = y^*$. □

2.1.3 Application

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics; see the monographs [19, 48, 66, 88].

us consider the following system

$$x'(t) = f(t,x(t),y(t)), \quad y'(t) = f(t,y(t),x(t)), \quad t \in J\backslash\{t_1\} \tag{2.3}$$

$$x(t_1^+) - x(t_1^-) = I_1(x(t_1),y(t_1)), \quad y(t_1^+) - y(t_1^-) = I_1(y(t_1),x(t_1)), \tag{2.4}$$

$$x(0) = x_0 = y(0), \tag{2.5}$$

where $0 < t_1 < 1$, $J := [0,1]$, $f\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $I_1 \in C(\mathbb{R} \times \mathbb{R},\mathbb{R})$. The notations $x(t^+) = \lim\limits_{h \to 0^+} x(t+h)$ and $x(t^-) = \lim\limits_{h \to 0^+} x(t-h)$.

In order to define a solutions for Problem (2.3)–(2.5), consider the space of piecewise continuous functions:

$$\begin{aligned} PC(J,\mathbb{R}) = &\{y\colon J \to \mathbb{R}, \quad y \in C(J\backslash\{t_1\},\mathbb{R}); \text{ such that } y(t_1^-) \text{ and } y(t_1^+) \\ &\text{exist and satisfy } y(t_1^-) = y(t_1)\}, \end{aligned}$$

endowed the following b-metric space

$$d(x,y) = \sup_{t \in J} |x(t) - y(t)|^2 \text{ with } s = 2.$$

Definition 2.2. A functions $(x,y) \in PC\times$ is said to be a solution of (2.3)–(2.5), if (x,y) satisfies (2.3)–(2.5).

Consider on $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ the partial order relation:

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1(t) \leq x_2(t) \text{ and } y_1(t) \geq y_2(t), \quad t \in J.$$

We define the graph G by

$$V(G) = PC(J,\mathbb{R}) \text{ and } E(G) = \{(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R}), \quad x \leq y\},$$

and we endow the product space $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ by another graph denoted also by G, such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for any $(x,y),(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$. We study the existence of a solution to the previous system.

Assumption 2.1. Assume the following assertions:

1. $f\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

2. For all $x,y,u,v \in PC(J,\mathbb{R})$, with $x \leq u$ and $v \leq y$, we have

$$f(t,x(t),y(t)) \leq f(t,u(t),v(t)) \text{ and } I_1(x(t_1),y(t_1)) \leq I_1(u(t_1),v(t_1)) \quad \forall\ t \in J;$$

3. There exist $\alpha,\beta,\gamma \in [0,1)$ with $\sum\limits_{i=0}^{\infty} 2^i \left(\frac{\beta+\gamma}{1-\alpha-\beta}\right)^i < \infty$ such that

$$\begin{aligned} |f(t,x(t),y(t)) - f(t,u(t),v(t)|^2 \leq\ & \frac{\alpha}{2} \frac{|x(t) - f(t,x(t),y(t))|^2[1 + |u(t) - f(t,u(t),v(t))|^2]}{1 + |x(t) - u(t)|^2} \\ & + \frac{\beta}{2} \left[|x(t) - f(t,x(t),y(t))|^2 + |u(t) - f(t,u(t),v(t))|^2\right], \end{aligned}$$

and

$$|I_1(x(t_1),y(t_1)) - I_1(u(t_1),v(t_1))|^2 \leq \frac{\gamma}{2} \left(|x(t) - u(t)|^2\right),$$

for each $t \in J$, $x,y,u,v \in PC(J,\mathbb{R})$, $x \leq u$ and $v \leq y$.

We shall obtain the solution of Eqs. (2.3)-(2.5). This problem is equivalent to the integral equations:

$$\begin{cases} x(t) = x_0 + \int_0^t f(s,x(s),y(s))ds + I_1(x(t_1),y(t_1)), \\ y(t) = y_0 + \int_0^t f(s,y(s),x(s))ds + I_1(y(t_1),x(t_1)), \end{cases} \quad t \in J. \tag{2.6}$$

We define for $t \in J$,

$$T(x,y)(t) = x_0 + \int_0^t f(s,x(s),y(s))ds + I_1(x(t_1),y(t_1)), \quad t \in J.$$

Note that if $(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ is a couple fixed point of T, then we have

$$x(t) = T(x,y)(t) \text{ and } y(t) = T(y,x)(t),$$

for all $t \in J$, and (x,y) is a solution of (2.6).

Theorem 2.5. *Assume the Assumption 2.1 holds. Assume that there exists* $(u_0,v_0) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ *such that*

$$u_0(t) \leq x_0 + \int_0^t f(s,u_0(s),v_0(s))ds + I_1(u(t_1),v(t_1))$$

and

$$v_0(t) \geq x_0 + \int_0^t f(s,v_0(s),u_0(s))ds + I_1(v(t_1),u(t_1)), \quad t \in J.$$

Then the system (2.3)–(2.5) *possesses a solution.*

Proof. We prove that the integral system (2.6) has a solution by showing that the operator $T\colon X \times X \to X$ has a coupled fixed point in $X \times X$. To do this, we have to show that T satisfies the conditions of Theorem 2.1 or Theorem 2.2.

By using Assumption 2.1, we obtain for all x, y, x_1, x_2, y_1, $y_2 \in PC(J,\mathbb{R})$, if $(x_1,x_2) \in E(G)$, then

$$\begin{aligned} T(x_1,y)(t) &= x_0 + \int_0^t f(s,x_1(s),y(s))ds + I_1(x_1(t_1),y(t_1)) \\ &\leq x_0 + \int_0^t f(s,x_2(s),y(s))ds + I_1(x_2(t_1),y(t_1)) = T(x_2,y)(t). \end{aligned}$$

Thus $(T(x_1,y),T(x_2,y)) \in E(G)$.
Also, if $(y_1,y_2) \in E(G)$ we have

$$\begin{aligned} T(x,y_2)(t) &= x_0 + \int_0^t f(s,x(s),y_2(s))ds + I_1(x(t_1),y_2(t_1)) \\ &\leq x_0 + \int_0^t f(s,x(s),y_1(s))ds + I_1(x(t_1),y_1(t_1)) = T(x,y_1)(t). \end{aligned}$$

Then $(T(x,y_2),T(x,y_1)) \in E(G)$.
Thus T possesses the mixed G-monotone property.

Now, let us consider $(x,y),(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that

$((x,y),(u,v)) \in E(G)$, then

$$\begin{aligned}
|T(x,y)(t) - T(u,v)(t)|^2 &= |\int_0^t f(t,x(s),y(s))ds + I_1(x(t_1),y(t_1)) \\
&\quad - \int_0^t f(t,u(s),v(s))ds - I_1(u(t_1),v(t_1))|^2 \\
&\leq 2\int_0^t |f(t,x(s),y(s)) - f(t,u(s),v(s))|^2 ds \\
&\quad + 2|I_1(x(t_1),y(t_1)) - I_1(u(t_1),v(t_1))|^2 \\
&\leq \int_0^t \alpha \frac{|x(s) - f(s,x(s),y(s))|^2[1 + |u(s) - f(s,u(s),v(s))|^2]}{1 + |x(s) - u(s)|^2} \\
&\quad + \beta[|x(s) - f(s,x(s),y(s))|^2 + |u(s) - f(s,u(s),v(s))|^2]ds \\
&\quad + \gamma|x(t) - u(t)|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(T(x,y),T(u,v)) \leq \alpha &\frac{d(x,T(x,y))[1 + d(u,T(u,v))]}{1 + d(x,u)} \\
&+ \beta[d(x,T(x,y)) + d(u,T(u,v))] + \gamma d(x,u).
\end{aligned}$$

Now, by hypotheses we can conclude that

$$((u_0,v_0),(T(u_0,v_0),T(v_0,u_0))) \in E(G).$$

Since T is a continuous mapping and (X,d,G) possesses the property1.1, which show that all hypotheses of Theorem 2.1 and Theorem 2.2 are satisfied. Thus T has a coupled fixed point in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$.

□

2.2 Coupled Fixed Point Theorems for φ−ψ-contraction Mappings

In this section, we establish a new generalization for coupled fixed point in partially ordered complete metric spaces, which was previously presented by Işik and Türkoğlu [58].

Firstly, we present the following contraction that we will use in this section.

Definition 2.3. The mapping $T : X \times X \to X$ is called $\varphi - \psi-$contraction if there exist a pair $(\varphi,\psi) \in \Upsilon$ such that

$$\varphi\left(s^2 d(T(x,y),T(u,v))\right) \leq \frac{1}{2}\psi\left(d(x,u) + d(y,v)\right), \tag{2.7}$$

holds for all $(x,y),(u,v) \in X \times X$ with $((x,y),(u,v)) \in E(G)$.

2.2.1 Existence of Coupled Fixed Points

Before introducing our main result in this subsection, we formulate and prove the following result:

Lemma 2.1. *Let (X,d) be a b-metric space with $s \geq 1$. Let $\{x_j\}$ and $\{y_j\}$ be two sequences in X. Assume that there exists $\alpha \in \left[0,\frac{1}{s}\right)$ satisfying*

$$d(x_j,x_{j+1}) + d(y_j,y_{j+1}) \leq \alpha[d(x_{j-1},x_j) + d(y_{j-1},y_j)], \tag{2.8}$$

for any $j \in \mathbb{N}$. Then $\{x_j\}$ and $\{y_j\}$ are Cauchy sequences.

Proof. Let $i, j \in \mathbb{N}$ and $i < j$. Then

$$\begin{aligned}
d(x_i,x_j) + d(y_i,y_j) &\leq s[d(x_i,x_{i+1}) + d(x_{i+1},x_j)] + s[d(y_i,y_{i+1}) + d(y_{i+1},y_j)] \\
&\leq s[d(x_i,x_{i+1}) + d(y_i,y_{i+1})] + s^2[d(x_{i+1},x_{i+2}) + d(y_{i+1},y_{i+2})] \\
&\quad + s^2[d(x_{i+2},x_j) + d(y_{i+2},y_j)] \\
&\leq \\
&\vdots \\
&\leq s[d(x_i,x_{i+1}) + d(y_i,y_{i+1})] + s^2[d(x_{i+1},x_{i+2}) + d(y_{i+1},y_{i+2})] \\
&\quad + \cdots + s^{j-i-1}[d(x_{j-2},x_{j-1}) + d(x_{j-1},x_j) + d(y_{j-2},y_{j-1}) + d(y_{j-1},y_j)] \\
&\leq s[d(x_i,x_{i+1}) + d(y_i,y_{i+1})] + s^2[d(x_{i+1},x_{i+2}) + d(y_{i+1},y_{i+2})] \\
&\quad + \cdots + s^{j-i-1}[d(x_{j-2},x_{j-1}) + d(y_{j-2},y_{j-1})] + s^{j-i}[d(x_{j-1},x_j) + d(y_{j-1},y_j)].
\end{aligned}$$

Using Equation (2.8) and the fact that $s\alpha < 1$, we have

$$\begin{aligned}
d(x_i,x_j) + d(y_i,y_j) &\leq [s\alpha^i + s^2\alpha^{i+1} + \cdots + s^{j-i-1}\alpha^{j-2} + s^{j-i}\alpha^{j-1}](d(x_0,x_1) + d(y_0,y_1)) \\
&= s\alpha^i[1 + s\alpha + \cdots + s^{j-i-2}\alpha^{j-i-2} + s^{j-i-1}\alpha^{j-i-1}](d(x_0,x_1) + d(y_0,y_1)) \\
&\leq \frac{s\alpha^i}{1 - s\alpha}(d(x_0,x_1) + d(y_0,y_1)).
\end{aligned}$$

Hence,

$$\lim_{i\to\infty}[d(x_i,x_j)+d(y_i,y_j)]=0,$$

and thus, $\{x_j\}$ and $\{y_j\}$ are Cauchy sequences. □

Theorem 2.6. *On (X,d,G), suppose that T is continuous and $\varphi-\psi-$contraction mapping. If there exist $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.*

Proof. Set $x_{n+1}=T(x_n,y_n)$, and $y_{n+1}=T(y_n,x_n)$, $n=1,2,...$. By assumption we get $((x_0,y_0),(x_1,y_1)) \in E(G)$ which implies that

$$\begin{aligned}\varphi\left(s^2 d(x_2,x_1)\right) &= \varphi\left(s^2 d(T(x_1,y_1),T(x_0,y_0))\right)\\ &\leq \frac{1}{2}\psi\left(d(x_1,x_0)+d(y_1,y_0)\right).\end{aligned}$$

Similarly, since $((y_1,x_1),(y_0,x_0)) \in E(G)$, we have

$$\varphi\left(s^2 d(y_2,y_1)\right) \leq \frac{1}{2}\psi\left(d(y_1,y_0)+d(x_1,x_0)\right).$$

Since T has the mixed G-monotone property on X, we get for $n=1,2,...$

$$((x_n,y_n),(x_{n+1},y_{n+1})) \in E(G), \quad ((y_{n+1},x_{n+1}),(y_n,x_{n+1})) \in E(G).$$

Then

$$\varphi\left(s^2 d(x_{n+1},x_n)\right) \leq \frac{1}{2}\psi\left(d(x_n,x_{n-1})+d(y_n,y_{n-1})\right) \tag{2.9}$$

and

$$\varphi\left(s^2 d(y_{n+1},y_n)\right) \leq \frac{1}{2}\psi\left(d(y_n,y_{n-1})+d(x_n,x_{n-1})\right). \tag{2.10}$$

Adding the above inequalities (2.9) to (2.10), we obtain

$$\varphi\left(s^2 d(x_{n+1},x_n)\right)+\varphi\left(s^2 d(y_{n+1},y_n)\right) \leq \psi\left(d(x_n,x_{n-1})+d(y_n,y_{n-1})\right).$$

Due to the properties of (φ,ψ), we have

$$\varphi\left(s^2\left(d(x_{n+1},x_n)+d(y_{n+1},y_n)\right)\right) < \varphi\left(d(x_n,x_{n-1})+d(y_n,y_{n-1})\right).$$

Since φ is non-decreasing, imply

$$s^2\left(d(x_{n+1},x_n)+d(y_{n+1},y_n)\right) < d(x_n,x_{n-1})+d(y_n,y_{n-1}).$$

Then

$$d(x_{n+1},x_n) + d(y_{n+1},y_n) < \frac{1}{s^2}\left(d(x_n,x_{n-1}) + d(y_n,y_{n-1})\right).$$

Since $0 \leq \frac{1}{s^2} < \frac{1}{s}$, then by Lemma 2.1 we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. The completeness of the space X implies that, there exist $x^*,y^* \in X$ such that

$$\lim_{n\to+\infty} x_n = x^* \text{ and } \lim_{n\to+\infty} y_n = y^*.$$

We have by the continuity of T

$$x^* = \lim_{n\to+\infty} x_n = \lim_{n\to+\infty} T(x_{n-1},y_{n-1}) = T(\lim_{n\to+\infty} x_{n-1}, \lim_{n\to+\infty} y_{n-1}) = T(x^*,y^*),$$

$$y^* = \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} T(y_{n-1},x_{n-1}) = T(\lim_{n\to+\infty} y_{n-1}, \lim_{n\to+\infty} x_{n-1}) = T(y^*,x^*),$$

i.e., (x^*,y^*) is a couple fixed point of T. □

Example 2.1. Let $X = \mathbb{R}$, $d(x,y) = |x-y|^2$ be a $b-$metric space with $s = 2$. Let G be the directed graph defined on X by

$$((x,y),(u,v)) \in E(G) \Leftrightarrow x \leq u \wedge v \leq y.$$

And let $T : X \times X \to X$ be defined by $T(x,y) = \frac{x+y}{4}$, $(x,y) \in X \times X$.

Then T is mixed G-monotone and satisfies condition 2.7 with $\varphi(t) = \frac{1}{2}t$ and $\psi(t) = \frac{1}{4}t$. Indeed, it is clear that $(\varphi,\psi) \in \Upsilon$, then for any (x,y), $(u,v) \in X \times X$ such that $((x,y),(u,v)) \in E(G)$, we have

$$\begin{aligned}
\varphi\left(s^2 d(T(x,y),T(u,v)\right) &= \frac{1}{2}\left(2^2\left(\frac{x+y}{4} - \frac{u+v}{4}\right)^2\right) \\
&\leq \frac{1}{8}\left((x-u)+(y-v)\right)^2 \\
&\leq \frac{1}{4}\left((x-u)^2+(y-v)^2\right) \\
&= \psi\left(d(x,u)+d(y,v)\right).
\end{aligned}$$

Notice that $((0,0),(0,0)) \in E(G)$. So by Theorem 2.6, we have that T has a coupled fixed point $(0,0)$.

Now, we give another sufficient condition for the existence of couple fixed point in the case where the mapping T is not continuous and the triple (X,d,G) has property1.1.

Theorem 2.7. *Endowed* (X,d,G) *with the property1.1. Suppose that* T *is* $\varphi-\psi-$*contraction mapping. If there exist* $x_0,y_0 \in X$ *such that* $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$*, then* T *possesses a coupled fixed point.*

Proof. We follow the proof of Theorem 2.6 and prove that $x^* = T(x^*,y^*)$ and $y^* = T(y^*,x^*)$.

Since $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n,y_n) = x^*$, $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} T(y_n,x_n) = y^*$ and (x_n,x_{n+1}), $(y_{n+1},y_n) \in E(G)$, then by Property 1.1 we have

$$(x_n,x^*) \in E(G) \text{ and } (y^*,y_n) \in E(G).$$

Then

$$((x_n,y_n),(x^*,y^*)) \in E(G),$$

and hence we get

$$\varphi\left(s^2 d(T(x_n,y_n),T(x^*,y^*))\right) \leq \frac{1}{2}\psi\left(d(x_n,x^*) + d(y_n,y^*)\right).$$

Similarly, we have

$$\varphi\left(s^2 d(T(y_n,x_n),T(y^*,x^*))\right) \leq \frac{1}{2}\psi\left(d(y_n,y^*) + d(x_n,x^*)\right).$$

Letting $n \to +\infty$ we have

$$\lim_{n\to+\infty} d(T(x_n,y_n),T(x^*,y^*)) = 0 \text{ and } \lim_{n\to+\infty} d(T(y_n,x_n),T(y^*,x^*)) = 0,$$

then

$$\lim_{n\to+\infty} x_{n+1} = T(x^*,y^*) \text{ and } \lim_{n\to+\infty} y_{n+1} = T(y^*,x^*).$$

Thus, $x^* = T(x^*,y^*)$ and $y^* = T(y^*,x^*)$, i.e., (x^*,y^*) is a coupled fixed point of T. □

If we take $\varphi(t) = t$ and $\psi(t) = kt$ in Theorem 2.6 and Theorem 2.7 with $s = 1$, we have the following result which is an extension of the results given by Alfuraidan and Khamsi [6].

Corollary 2.2. Let (X,d) be a complete metric space endowed with a direct graph G. Let $T\colon X \times X \to X$ be a mapping that has the mixed G-monotone property on X for which there exists $k \in [0,1)$ such that

$$d(T(x,y),T(u,v)) \leq \frac{k}{2}\left(d(x,u) + d(y,v)\right)$$

for any $(x,y),(u,v) \in X \times X$ with $((x,y),(u,v)) \in E(G)$. Assume that either

1. T is continuous

   or

2. the triple (X,d,G) has the property 1.1.

If there exist $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T has a coupled fixed point.

From Theorem 2.6 and Theorem 2.7 we deduce the following result.

Corollary 2.3. Let (X,d) be a complete metric space endowed with a direct graph G. Assume the mapping $T\colon X \times X \to X$ has the mixed G-monotone property on X. Let $(\varphi,\psi) \in \Upsilon$ such that

$$\varphi(d(T(x,y),T(u,v))) \leq \frac{1}{2}\varphi\left(d(x,u)+d(y,v)\right) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$

for all $(x,y),(u,v) \in X \times X$ with $((x,y),(u,v)) \in E(G)$. Assume that either

1. T is continuous

   or

2. the triple (X,d,G) has the property 1.1.

If there exist $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T has a coupled fixed point.

Note that, the above result is an extension of the results obtained by Luong and Thuan [99] in metric space endowed with a graph.

2.2.2 Uniqueness of Coupled Fixed Points

Theorem 2.8. *In addition to the hypothesis of Theorem 2.6 and Theorem 2.7, suppose that for any $(x,y),(x^*,y^*) \in X \times X$ there exists $(u,v) \in X \times X$ such that $((x,y),(u,v)) \in E(G)$ and $((x^*,y^*),(u,v)) \in E(G)$. Then T has a unique coupled fixed point.*

Proof. Suppose that there exist two couple fixed point (x,y) and (x^*,y^*) of T. By assumption, there exists $(u,v) \in X \times X$ such that $((x,y),(u,v)),((x^*,y^*),(u,v)) \in E(G)$. We define the sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = T(u_n, v_n) \text{ and } v_{n+1} = T(v_n, u_n) \text{ for all } n \in \mathbb{N}.$$

Since $((x,y),(u,v)) \in E(G)$ and T has the mixed G-monotone property we can prove that $((x,y),(u_n,v_n)) \in E(G)$. Then

$$\begin{aligned}\varphi\left(s^2 d(x,u_{n+1})\right) &= \varphi\left(s^2 d(T(x,y),T(u_n,v_n))\right) \\ &\leq \frac{1}{2}\psi\left(d(x,u_n) + d(y,v_n)\right),\end{aligned}$$

and

$$\begin{aligned}\varphi\left(s^2 d(v_{n+1},y)\right) &= \varphi\left(s^2 d(T(v_n,u_n),T(y,x))\right) \\ &\leq \frac{1}{2}\psi\left(d(v_n,y) + d(u_n,x)\right).\end{aligned}$$

Adding the above two inequalities and using the property of (φ,ψ), we get

$$\varphi\left(s^2(d(x,u_{n+1}) + d(y,v_{n+1}))\right) \leq \psi(d(x,u_n) + d(y,v_n)). \tag{2.11}$$

Since φ is a nondecreasing function and $\varphi(t) > \psi(t)$ for $t > 0$, we have

$$s^2(d(x,u_{n+1}) + d(y,v_{n+1})) < d(x,u_n) + d(y,v_n).$$

Since $s \geq 1$

$$d(x,u_{n+1}) + d(y,v_{n+1}) < d(x,u_n) + d(y,v_n).$$

This gives us $\{d(x, u_n) + d(y, v_n)\}$ is a nonnegative decreasing sequence, and consequently, there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} d(x, u_n) + d(y, v_n) = \delta.$$

Since φ and ψ are continuous functions, letting $n \to \infty$ in (2.11), we have

$$\varphi(s^2\delta) \leq \psi(\delta).$$

It implies, by the properties of φ and ψ that $\delta = 0$. Hence

$$\lim_{n\to\infty} d(x, u_n) + d(y, v_n) = 0.$$

Then,

$$\lim_{n\to\infty} d(x, u_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(y, v_n) = 0.$$

Following the same process, we can show that

$$\lim_{n\to\infty} d(x^*, u_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(y^*, v_n) = 0.$$

By letting $n \to \infty$ in the following inequality

$$d(x, x^*) \leq s[d(x, u_n) + d(u_n, x^*)],$$

$$d(y, y^*) \leq s[d(y, v_n) + d(v_n, y^*)].$$

Therefore, $d(x, x^*) = d(y, y^*) = 0$ and hence $x = x^*$ and $y = y^*$. □

Theorem 2.9. *If (x^*,y^*) is a coupled fixed point of T. Assume that $(x^*,y^*) \in E(G)$ and the hypothesis of Theorem 2.6 and Theorem 2.7 hold. Then $x^* = y^*$.*

Proof. Since $(x^*,y^*) \in E(G)$ we have $((x^*,y^*),(y^*,x^*)) \in E(G)$. Hence,

$$\begin{aligned}\varphi(s^2 d(x^*,y^*)) &= \varphi(s^2 d(T(x^*,y^*),T(y^*,x^*))) \\ &\leq \frac{1}{2}\psi(d(x^*,y^*) + d(x^*,y^*)) \\ &< \varphi(d(x^*,y^*)).\end{aligned}$$

Since φ is non-decreasing function, we have

$$s^2 d(x^*,y^*) < d(x^*,y^*).$$

Hence $d(x^*,y^*) = 0$ and so $x^* = y^*$. □

2.2.3 Application

Delay differential equations are a type of functional differential equations where the evolution of the system depends not only on the current state of the system but also on its past history. Such equations are frequently encountered as mathematical models of population dynamics, epidemiology, immunology, physiology, and neural networks , etc. For details, see the monographs [17, 22, 65, 80, 83, 84]. For example, any time when in physics or technology we consider a problem of a force, acting on a material point, that depends on the velocity and position of the point not only at the given moment but at some moment preceding the given moment.

Consider the following system of differential equations:

$$x'(t) = f(t,x_t,y_t), \quad y'(t) = f(t,y_t,x_t), \quad t \in J \tag{2.12}$$

$$x(t) = \omega_1(t), \quad y(t) = \omega_2(t), \quad t \in (-\infty,0], \tag{2.13}$$

where $J := [0,L]$, $f\colon J \times \Omega \times \Omega \to \mathbb{R}^n$, $\omega_1,\omega_2 \in \Omega$, and x_t,y_t present the history of the state from $-\infty$ up to the present time t. We assume that the histories x_t,y_t belong to Ω. Where $(\Omega, ||.||_\Omega)$ the seminormed linear space of functions mapping $x\colon (-\infty,0] \to \mathbb{R}^n$ and satisfying the following axioms that were introduced by Hale and Kato[52] for ordinary differential equations.

1. If $x\colon (-\infty,L] \to \mathbb{R}^n$, $L > 0$ is continuous on J and $x_0 \in \Omega$, then there exist constants α, β, $\gamma > 0$ such that for every $t \in [0, L)$ the following conditions hold:

   i) x_t is in Ω,

   ii) $||x|| \leq \alpha||x_t||_\Omega$,

   iii) $||x_t||_\Omega \leq \beta \sup\{||x(s)||, \quad 0 \leq s \leq t\} + \gamma||x_0||_\Omega$.

2. For the function $x(.)$ in 1, x_t is a Ω-valued continuous function on $[0, L)$.

3. The space Ω is complete.

In order to define a solutions for Problem (2.12)-(2.13), consider the following space

$$X = \{x, \quad x\colon (-\infty,L] \to \mathbb{R}^n, \quad x \in C(X,\mathbb{R}^n), \quad x(t) = \omega(t), \quad t \in (-\infty,0], \quad \omega \in \Omega\},$$

endowed with the following seminorm

$$||x||_X = ||x_0||_\Omega + \sup_{t\in J} ||y(t)||.$$

Definition 2.4. A functions $(x,y) \in X \times X$, is said to be a solution of (2.12)-(2.13) if (x,y) satisfies (2.12)-(2.13).

We define the operator $T\colon X \times X \to X$ by

$$T(x,y)(t) = \begin{cases} \omega_1(t), & \text{if } t \in (-\infty,0) \\ \omega_1(0) + \int_0^t f(s,x_s,y_s)ds & \text{if } t \in J \end{cases},$$

$$T(y,x)(t) = \begin{cases} \omega_2(t), & \text{if } t \in (-\infty,0) \\ \omega_2(0) + \int_0^t f(s,y_s,x_s)ds & \text{if } t \in J \end{cases}.$$

Let $\hat{\omega}_1$, $\hat{\omega}_2\colon (-\infty; L) \to \mathbb{R}^n$ be the functions defined by

$$\hat{\omega}_1(t) = \begin{cases} \omega_1(t), & \text{if } t \in (-\infty,0) \\ \omega_1(0), & \text{if } t \in J \end{cases}, \hat{\omega}_2(t) = \begin{cases} \omega_2(t), & \text{if } t \in (-\infty,0) \\ \omega_2(0) & \text{if } t \in J \end{cases}$$

Then $\hat{\omega}_{1,0} = \omega_1$ and $\hat{\omega}_{2,0} = \omega_2$. For each ϑ_1, $\vartheta_2 \in C(J,\mathbb{R}^n)$ with $\vartheta_1(0) = 0$ and $\vartheta_2(0) = 0$, we denote by $\overline{\vartheta}$ and $\overline{\vartheta}'$ the functions defined by

$$\overline{\vartheta_1}(t) = \begin{cases} 0, & \text{if } t \in (-\infty,0) \\ \vartheta_1(t), & \text{if } t \in J \end{cases}, \overline{\vartheta}_2(t) = \begin{cases} 0, & \text{if } t \in (-\infty,0) \\ \vartheta_2(t) & \text{if } t \in J \end{cases}$$

If $x(.)$, $y(.)$ satisfy the integral equations,

$$x(t) = \omega_1(t) + \int_0^t f(s,x_s,y_s)ds$$

and

$$y(t) = \omega_2(t) + \int_0^t f(s,y_s,x_s)ds,$$

we can decompose $x(.)$ as $x(t) = \overline{\vartheta_1}(t) + \hat{\omega}_1(t)$ and $y(.)$ as $y(t) = \overline{\vartheta}_2(t) + \hat{\omega}_2(t)$ for every $t \in J$, and the function $\vartheta_1(.)$, and ϑ_2 satisfies

$$\vartheta_1(t) = \int_0^t f(s,\overline{\vartheta_1}_s + \hat{\omega}_{1s},\overline{\vartheta}_{2s} + \hat{\omega}_{2s})ds$$

and

$$\vartheta_2(t) = \int_0^t f(s,\overline{\vartheta}_{2s} + \hat{\omega}_{2s},\overline{\vartheta}_{1s} + \hat{\omega}_{1s})ds.$$

Set $C_0 = \{\vartheta \in C(J,\mathbb{R}^n), \vartheta(0) = 0\}$ endowed with the following b-metric with $s = 2$

$$d(x,y) = \sup_{t\in J} ||x(t) - y(t)||^2.$$

Consider on $C_0 \times C_0$ the partial order relation:

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1(t) \leq x_2(t) \text{ and } y_1(t) \geq y_2(t), \quad t \in J,$$

the graph G such that $V(G) = C_0$, and

$$E(G) = \{(x,y) \in C_0 \times C_0, \quad x \leq y\}.$$

We endow the product space $C_0 \times C_0$ by another graph denoted also by G, such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for all $(x, y), (u, v) \in C_0 \times C_0$.

We define the operator $F\colon C_0 \times C_0 \to C_0$ by

$$F(x_1, x_2)(v) = \int_0^v f(s,\overline{x}_{1s} + \hat{\omega}_{1s},\overline{x}_{2s} + \hat{\omega}_{2s})ds.$$

Clearly, if T has a Couple fixed point, then F has a Couple fixed point and vice versa. So we will show that F has a Couple fixed point by using Theorem 2.6 and Theorem 2.7.

Assumption 2.2.

1. $f\colon J \times \Omega \times \Omega \to \mathbb{R}^n$ is continuous;
2. For all $x,y,u,v \in \mathbb{R}^n$, with $x \leq u$ and $v \leq y$,

$$f(t,x,y) \leq f(t,u,v);$$

3. For each $t \in J$, $x,y,u,v \in \mathbb{R}^n$, $x \leq u$ and $v \leq y$ we get

$$||f(t,x,y) - f(t,u,v)||^2 \leq \frac{1}{8L^2} \ln\Big(\frac{1}{\beta}||x - u||^2 + \frac{1}{\beta}||y - v||^2 + 1\Big).$$

Theorem 2.10. *Consider the system* (2.12)-(2.13) *suppose that the Assumption 2.2 is satisfied. Assume that there exists* $(u,v) \in C_0 \times C_0$ *such that*

$$u(t) \leq \int_0^t f(s,\overline{u}_s + \hat{\omega}_{1s},\overline{v}_s + \hat{\omega}_{2s})ds,$$

and

$$v(t) \geq \int_0^t f(s,\overline{v}_s + \hat{\omega}_{1s},\overline{u}_s + \hat{\omega}_{2s})ds, \quad t \in J.$$

Then the integral system (2.12)-(2.13) *has unique solution.*

Proof. We obtain the existence solution of the integral system (2.12)-(2.13) by showing that the operator $F\colon C_0 \times C_0 \to C_0$ has a coupled fixed point in $C_0 \times C_0$. To do this, we verify that F satisfies the hypotheses of Theorem 2.6 or Theorem 2.7.

Using Assumption 2.2, we obtain for all x, y, m, n, $w \in C_0$, if $(x,m) \in E(G)$ then we have

$$\begin{aligned} F(x,y)(t) &= \int_0^t f(s,\overline{x}_s + \hat{\omega}_{1s},\overline{y}_s + \hat{\omega}_{2s})ds \\ &\leq \int_0^t f(s,\overline{m}_s + \hat{\omega}_{1s},\overline{y}_s + \hat{\omega}_{2s})ds \\ &= F(m,y)(t). \end{aligned}$$

Therefore, $(F(x,y),F(m,y)) \in E(G)$.
Also, if $(n,w) \in E(G)$ we have

$$\begin{aligned} F(x,w)(t) &= \int_0^t f(s,\overline{x}_s + \hat{\omega}_{1s},\overline{w}_s + \hat{\omega}_{2s})ds \\ &\leq \int_0^t f(s,\overline{x}_s + \hat{\omega}_{1s},\overline{n}_s + \hat{\omega}_{2s})ds \\ &= F(x,n)(t). \end{aligned}$$

Hence, $(F(x,w),F(x,n)) \in E(G)$ and thus F has the mixed G-monotone property.

Now, let us consider $(x,y),(u,v) \in C_0 \times C_0$ such that $((x,y),(u,v)) \in E(G)$. Then

$$\begin{aligned}||F(x,y)(t) - F(u,v)(t)||^2 &= ||\int_0^t f(s,\overline{x}_s + \hat{\omega}_{1s},\overline{y}_s + \hat{\omega}_{2s})ds - \int_0^t f(s,\overline{u}_s + \hat{\omega}_{1s},\overline{t}_s + \hat{\omega}_{2s})ds||^2 \\ &\leq L \int_0^t ||f(s,\overline{x}_s + \hat{\omega}_{1s},\overline{y}_s + \hat{\omega}_{2s}) - f(s,\overline{u}_s + \hat{\omega}_{1s},\overline{v}_s + \hat{\omega}_{2s})||^2 ds \\ &\leq \frac{1}{8L}\int_0^t \ln\left(\frac{1}{\beta}||\overline{x}_s - \overline{u}_s||^2 + \frac{1}{\beta}||\overline{y}_s - \overline{v}_s||^2 + 1\right) ds \\ &\leq \frac{1}{8}\ln\left(\sup_{t\in J}||x(t) - u(t)||^2 + \sup_{t\in J}||y(t) - v(t)||^2 + 1\right).\end{aligned}$$

Therefore,

$$\varphi\left(s^2 d(F(x,y),F(u,v))\right) \leq \frac{1}{2}\psi\left(d(x,u) + d(y,v)\right),$$

where $\varphi(x) = x$ and $\psi(x) = \ln(x+1)$. Clearly $(\varphi,\psi) \in \Upsilon$.

Now, by hypotheses we can conclude that

$$((u,v),(F(u,v),F(v,u))) \in E(G).$$

Since F is a continuous mapping and the triple (C_0,d,G) has the Property 1.1, which show that all hypotheses of Theorem 2.6 and Theorem 2.7 are satisfied. Therefore, F has a coupled fixed point in $C_0 \times C_0$. And by Theorem 2.8 we obtain the uniqueness of coupled fixed point. □

Chapter 3

Coupled Fixed Points Theorems in a b-fuzzy Metric Space with a Graph

This chapter deals with presenting new contractive mappings and gives sufficient conditions for the existence of coupled fixed points in $b-$fuzzy metric spaces endowed with a directed graph. Finally, we use our results to confirm the presence of a continuous solution for a system of fractional differential equations.

Let $(X,M,*,G)$ stands to a complete b-fuzzy metric space with constant $s \geq 1$ such that $a * a \geq a^2$ and $\lim_{t\to\infty} M(x,y,t) = 1$, endowed with directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and G has no parallel edges. Let $T\colon X \times X \to X$ be mapping.

Further, we endow the product space $X \times X$ by another graph denoted also by G, such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for any $(x,y), (u,v) \in X \times X$.

3.1 Coupled Fixed Point Theorems for φ-fuzzy contraction Mappings

We denote by Ω the set of function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ that meets all of the following criteria:

1. φ is nondecreasing;

2. for all $a \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$ we have $\varphi(at) = a\varphi(t)$;

3. $\sum_{i=0}^{\infty} \varphi^i(t)$ converges for all $t > 0$.

Obviously, if $\varphi \in \Omega$, then $\varphi(t) < t$ for each $t > 0$.

Definition 3.1. The mapping $T : X \times X \to X$ is called φ-fuzzy contraction if there exist $\varphi \in \Omega$ such that:

1. for all $x,y,u,v \in X$, T is edge preserving, i.e.,

$$((x,y),(u,v)) \in E(G) \text{ then } ((T(x,y),T(y,x)),(T(u,v),T(v,u))) \in E(G);$$

2. for all $(x,y),(u,v) \in X \times X$ such that $((x,y),(u,v)) \in E(G)$,

$$M(T(x,y),T(u,v),\varphi(t)) \geq M(x,u,st)^{\frac{1}{2}} * M(y,v,st)^{\frac{1}{2}}. \tag{3.1}$$

3.1.1 Existence of Coupled Fixed Points

Theorem 3.1. *On $(X,M,*,G)$, suppose that T is continuous mapping and φ-fuzzy contraction mapping. If there exist $x_0,y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.*

Proof. Set $x_{n+1} = T(x_n,y_n)$, and $y_{n+1} = T(y_n,x_n)$, for $n = 1,2,...$. By assumption we get $((x_0,y_0),(x_1,y_1)) \in E(G)$ and $((y_1,x_1),(y_0,x_0)) \in E(G)$ which implies

$$\begin{aligned} M(x_1,x_2,\varphi(t)) &= M(T(x_0,y_0),T(x_1,y_1),\varphi(t)) \\ &\geq M(x_0,x_1,st)^{\frac{1}{2}} * M(y_0,y_1,st)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} M(y_2,y_1,\varphi(t)) &= M(T(y_1,x_1),T(y_0,x_0),\varphi(t)) \\ &\geq M(y_1,y_0,st)^{\frac{1}{2}} * M(x_1,x_0,st)^{\frac{1}{2}}. \end{aligned}$$

Since T is edge preserving, we get

$$((x_1,y_1),(x_2,y_2)) \in E(G) \text{ and } ((y_2,x_2),(y_1,x_1)) \in E(G),$$

then we obtain

$$\begin{aligned} M(x_2,x_3,\varphi^2(t)) &= M(T(x_1,y_1),T(x_2,y_2),\varphi^2(t)) \\ &\geq M(x_1,x_2,s\varphi(t))^{\frac{1}{2}} * M(y_1,y_2,s\varphi(t))^{\frac{1}{2}} \\ &\geq M(x_0,x_1,s^2t)^{\frac{1}{2}} * M(y_0,y_1,s^2t)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} M(y_3,y_2,\varphi^2(t)) &= M(T(y_2,x_2),T(y_1,x_1),\varphi^2(t)) \\ &\geq M(y_2,y_1,s\varphi(t))^{\frac{1}{2}} * M(x_2,x_1,s\varphi(t))^{\frac{1}{2}} \\ &\geq M(y_1,y_0,s^2t)^{\frac{1}{2}} * M(x_1,x_0,s^2t)^{\frac{1}{2}}. \end{aligned}$$

By the same way, for $n = 1,2,...,$ we have

$$((x_n,y_n),(x_{n+1},y_{n+1})) \in E(G) \text{ and } ((y_{n+1},x_{n+1}),(y_n,x_n)) \in E(G).$$

Then

$$M(x_n,x_{n+1},\varphi^n(t)) \geq M(x_0,x_1,s^nt)^{\frac{1}{2}} * M(y_0,y_1,s^nt)^{\frac{1}{2}}$$

and

$$M(y_{n+1},y_n,\varphi^n(t)) \geq M(y_1,y_0,s^nt)^{\frac{1}{2}} * M(x_1,x_0,s^nt)^{\frac{1}{2}}.$$

Since $\varphi \in \Omega$, for all $t_0 > 0$ there exist $m > 0$ such that $t_0 > \sum_{k=n_0}^{\infty} \varphi^k(t)$. So for all $m \geq n \geq n_0$ we have

$$\begin{aligned} M(x_n,x_m,t_0) &\geq M\left(x_n,x_m,\frac{\sum_{k=n}^{m-1}\varphi^k(t)}{s}\right) \\ &\geq M\left(x_n,x_{n+1},\frac{\varphi^n(t)}{s^2}\right) * M\left(x_{n+1},x_{n+2},\frac{\varphi^{n+1}(t)}{s^3}\right) \\ &\quad * M\left(x_{n+2},x_{n+3},\frac{\varphi^{n+2}(t)}{s^4}\right) * \cdots * M\left(x_{m-1},x_m,\frac{\varphi^{m-1}(t)}{s^{m-n+1}}\right) \\ &\geq M\left(x_0,x_1,\frac{s^n}{s^2}t\right)^{\frac{1}{2}} * M\left(y_0,y_1,\frac{s^n}{s^2}t\right)^{\frac{1}{2}} * M\left(x_0,x_1,\frac{s^{n+1}}{s^3}t\right)^{\frac{1}{2}} \\ &\quad * M\left(y_0,y_1,\frac{s^{n+1}}{s^3}t\right)^{\frac{1}{2}} * M\left(x_0,x_1,\frac{s^{n+2}}{s^4}t\right)^{\frac{1}{2}} * M\left(y_0,y_1,\frac{s^{n+2}}{s^4}t\right)^{\frac{1}{2}} \\ &\quad * \cdots * M\left(x_0,x_1,\frac{s^{m-1}}{s^{m-n+1}}t\right)^{\frac{1}{2}} * M\left(y_0,y_1,\frac{s^{m-1}}{s^{m-n+1}}t\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$, we have

$$\lim_{n\to\infty} M(x_n,x_m,t) \geq 1 * 1 * 1 * \cdots * 1 = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. By the same way we can prove that $\{y_n\}$ is also a Cauchy sequence.

Since X is complete space then there exists $x^*,y^* \in X$ such that

$$\lim_{n\to+\infty} x_n = x^* \text{ and } \lim_{n\to+\infty} y_n = y^*.$$

By the continuity of T we have

$$x^* = \lim_{n\to+\infty} x_n = \lim_{n\to+\infty} T(x_{n-1},y_{n-1}) = T(\lim_{n\to+\infty} x_{n-1}, \lim_{n\to+\infty} y_{n-1}) = T(x^*,y^*),$$

$$y^* = \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} T(y_{n-1},x_{n-1}) = T(\lim_{n\to+\infty} y_{n-1}, \lim_{n\to+\infty} x_{n-1}) = T(y^*,x^*).$$

Thus, (x^*,y^*) is a coupled fixed point of T. □

Theorem 3.2. *Endowed (X,M,G) with the property1.1. Suppose that T is φ-fuzzy contraction. If there exist x_0, $y_0 \in X$ such that $((x_0,y_0),(T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.*

Proof. By following the proof of Theorem 3.1, we only prove that $x^* = T(x^*,y^*)$ and $y^* = T(y^*,x^*)$.

We constructed two sequences $\{x_n\}$ and $\{y_n\}$ such that

1. $x_{n+1} = T(x_n,y_n)$ and $y_{n+1} = T(y_n,x_n)$;

2. $((x_n,y_n),(x_{n+1},y_{n+1})) \in E(G)$;

3. $$M(x_n,x_{n+1},\varphi^n(t)) \geq M(x_0,x_1,s^n t)^{\frac{1}{2}} * M(y_0,y_1,s^n t)^{\frac{1}{2}}$$

   and

   $$M(y_{n+1},y_n,\varphi^n(t)) \geq M(y_1,y_0,s^n t)^{\frac{1}{2}} * M(x_1,x_0,s^n t)^{\frac{1}{2}};$$

4. There exist $x^*,y^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$ and $\lim_{n\to+\infty} y_n = y^*$;

for $n = 1,2,\ldots$ By property 1.1 we have

$$(x_n,x^*) \in E(G) \text{ and } (y^*,y_n) \in E(G),$$

then

$$((x_n,y_n),(x^*,y^*)) \in E(G).$$

Thus

$$\begin{aligned} M(x^*,T(x^*,y^*),t) &\geq M\left(x^*, x_{n+1}, \frac{t-\varphi(t)}{s}\right) * M\left(x_{n+1},T(x^*,y^*),\frac{\varphi(t)}{s}\right) \\ &\geq M\left(x^*, x_{n+1}, \frac{t-\varphi(t)}{s}\right) * M\left(x_n, x^*,t\right)^{\frac{1}{2}} * M\left(y_n,y^*,t\right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} M(y^*,T(y^*,x^*),t) &\geq M\left(y^*,y_{n+1},\frac{t-\varphi(t)}{s}\right) * M\left(y_{n+1},T(y^*,x^*),\frac{\varphi(t)}{s}\right) \\ &\geq M\left(y^*, y_{n+1},\frac{t-\varphi(t)}{s}\right) * M\left(y_n, y^*,t\right)^{\frac{1}{2}} * M\left(x_n, x^*,t\right)^{\frac{1}{2}}. \end{aligned}$$

Letting $n \to +\infty$ we have

$$M(x^*,T(x^*,y^*),t) = 1$$

and

$$M(y^*,T(y^*,x^*),t) = 1.$$

Then

$$x^* = T(x^*,y^*) \text{ and } y^* = T(y^*,x^*),$$

i.e, (x^*,y^*) is a coupled fixed point of T. □

If we take $\varphi(t) = kt$, where $k \in (0,1)$ in Theorem 3.1 and Theorem 3.2, we have the following corollary whose we can show that us an extension of the results obtained by Zhu and Xiao [102] in b-fuzzy metric space endowed with graph.

Corollary 3.1. Let $(X,M,*,G)$ be a complete b-fuzzy metric space with $s \geq 1$ endowed with the direct graph G such that $a * a \geq a^2$. Let $T : X \times X \to X$ is edge preserving. Assume there exists $k \in (0,1)$ such that

$$M(T(x,y),T(u,v),kt) \geq M(x,u,st)^{\frac{1}{2}} * M(y,v,st)^{\frac{1}{2}},$$

for any $(x,y),(u,v) \in X \times X$ such that $((x,y),(u,v)) \in E(G)$. Suppose that

1. T is a continuous mapping

   or

2. the triple (X,M,G) has property 1.1.

If there exist $x_0,y_0 \in X$ such that $((x_0,y_0), (T(x_0,y_0),T(y_0,x_0))) \in E(G)$, then T possesses a coupled fixed point.

Example 3.1. Let $X = [0,1]$ and $x * y = xy$ for all $x,y \in X$. Then $(X,M,*)$ is a complete b-fuzzy metric space with $s = 2$, where

$$M(x,y,t) = \exp\left(-\frac{|x-y|^2}{t}\right), \quad \forall x,y \in X, \quad t > 0.$$

Let $T : X \times X \to X$ be define by

$$T(x,y) = \frac{x-y}{4}, \quad \forall x,y \in X.$$

Let G be a graph defined by

$$E(G) = \{(x,y) \in X \times X, \quad x \leq y\}.$$

It is clear that T is edge preserving.

Now, since

$$\begin{aligned}
\frac{|T(x,y) - T(u,v)|^2}{\frac{1}{2}t} &= \frac{2}{t}\left|\frac{x-y}{4} - \frac{u-v}{4}\right|^2 \\
&= \frac{2}{t}\left|\frac{x-u}{4} + \frac{v-y}{4}\right|^2 \\
&\leq \frac{4}{t}\left(\frac{|x-u|^2}{16} + \frac{|v-y|^2}{16}\right) \\
&= \frac{1}{t}\left(\frac{|x-u|^2}{4} + \frac{|v-y|^2}{4}\right) \\
&= \frac{1}{2}\left(\frac{|x-u|^2}{2t} + \frac{|v-y|^2}{2t}\right),
\end{aligned}$$

for all $((x,y),(u,v)) \in E(G)$. Thus

$$M(T(x,y),T(u,v),\varphi(t)) \geq M(x,u,st)^{\frac{1}{2}} * M(y,v,st)^{\frac{1}{2}},$$

where $\varphi(t) = \frac{1}{2}t$. Notice that $\varphi \in \Omega$, T is a continuous mapping, the triple (X,d,G) has the Property 1.1 and $((0,1),(T(1,0),T(0,1))) \in E(G)$. So by Thereom 3.1 (Theorem 3.2), we have that T possesses a coupled fixed point (0,0).

3.1.2 Uniqueness of Coupled Fixed Points

Theorem 3.3. *In addition to the hypothesis of both Theorem 3.1 and Theorem 3.2, suppose that* $((x_0,y_0),(x^*,y^*)) \in E(G)$. *Then* T *has a unique coupled fixed point.*

Proof. If we suppose that (x,y) is another coupled fixed point of T. By the previous proof we construct two sequence $\{x_n\}$ and $\{y_n\}$, then

$$\begin{aligned}
M(x_{n+1},x,t) &= M(T(x_n,y_n),T(x,y),t) \\
&\geq M\left(T(x_n,y_n),T(x,y),\frac{\varphi^n(t)}{s}\right) \\
&\geq M(x_n, x,\varphi^{n-1}(t))^{\frac{1}{2}} * M(y_n, y,\varphi^{n-1}(t))^{\frac{1}{2}} \\
&\geq M(x_{n-1}, x,s\varphi^{n-2}(t))^{\frac{1}{2}} * M(y_{n-1}, y,s\varphi^{n-2}(t))^{\frac{1}{2}} \\
&\vdots \\
&\geq M(x_0, x,s^n)^{\frac{1}{2}} * M(y_0, y,s^n)^{\frac{1}{2}}.
\end{aligned}$$

By the similair way we can get

$$M(y_{n+1},y,t) \geq M(y_0,y,s^n)^{\frac{1}{2}} * M(x_0, x,s^n)^{\frac{1}{2}}.$$

Letting $n \to +\infty$ we get

$$\lim_{n\to+\infty} x_{n+1} = x \text{ and } \lim_{n\to+\infty} y_{n+1} = y,$$

then

$$x^* = x \text{ and } y^* = y.$$ □

Theorem 3.4. *In addition to the hypothesis of both Theorem 3.1 and Theorem 3.2, suppose that for every* (x,y), $(x^*,y^*) \in X \times X$ *there exists* $(u,v) \in X \times X$ *such that* $((x,y),(u,v)) \in E(G)$ *and* $((x^*,y^*),(u,v)) \in E(G)$. *Then* T *has unique coupled fixed point.*

Proof. Let us suppose that there exists two couple fixed point (x,y), (x^*,y^*) of T and we show that $x = x^*$ and $y = y^*$. By assumption, there exists $(u,v) \in X \times X$ such that $((x,y),(u,v)) \in E(G)$ and $((x^*,y^*),(u,v)) \in E(G)$. We define sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = T(u_n, v_n) \text{ and } v_{n+1} = T(v_n, u_n), \quad n \in \mathbb{N}.$$

Since $((x,y),(u,v)) \in E(G)$ and $((x^*,y^*),(u,v) \in E(G)$ with T is edge preserving, we can prove that $((x,y),(u_n,v_n)) \in E(G)$ and $((x^*,y^*),(u_n,v_n) \in E(G)$, $n \in \mathbb{N}$. Then, for all $t > 0$

$$\begin{aligned} M(x,u_{n+1},t) &\geq M\left(x,u_{n+1},\frac{\varphi^n(t)}{s}\right) \\ &\geq M(x,u_n,\varphi^{n-1}(t)) \\ &\vdots \\ &\geq M(x,u_0,s^{n-1}t) \end{aligned}$$

and

$$\begin{aligned} M(y,v_{n+1},t) &\geq M\left(y,v_{n+1},\frac{\varphi^n(t)}{s}\right) \\ &\geq M(y,v_n,\varphi^{n-1}(t)) \\ &\vdots \\ &\geq M(y,v_0,s^{n-1}t). \end{aligned}$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty} M(x,u_{n+1},t) = 1 \text{ and } \lim_{n\to\infty} M(y,v_{n+1},t) = 1.$$

Thus

$$\lim_{n\to\infty} u_{n+1} = x \text{ and } \lim_{n\to\infty} v_{n+1} = y.$$

By the same process we can show

$$\lim_{n\to\infty} u_{n+1} = x^* \text{ and } \lim_{n\to\infty} v_{n+1} = y^*.$$

Therefore

$$x = x^* \text{ and } y = y^*.$$ □

Theorem 3.5. *Assume that $(x^*,y^*) \in E(G)$ with the hypothesis of both Theorem 3.1 and Theorem 3.2. Then $x^* = y^*$.*

Proof. Since $(x^*,y^*) \in E(G)$ we have $\big((x^*,y^*),(y^*,x^*)\big) \in E(G)$, then

$$\begin{aligned} M(x^*,y^*,t) &= M(T(x^*,y^*),T(y^*,x^*),t) \\ &\geq M\left(T(x^*,y^*),T(y^*,x^*),\frac{\varphi^n(t)}{s}\right) \\ &\geq M(x^*,y^*,\varphi^{n-1}(t))^{\frac{1}{2}} * M(y^*,x^*,\varphi^{n-1}(t))^{\frac{1}{2}} \\ &= M(x^*,y^*,\varphi^{n-1}(t)) \\ &\geq M(x^*,y^*,s\varphi^{n-2}(t))^{\frac{1}{2}} * M(y^*,x^*,s\varphi^{n-2}(t))^{\frac{1}{2}} \\ &\vdots \\ &\geq M(x^*,y^*,s^{n-1}t), \end{aligned}$$

we get by letting $n \to \infty$ that $x^* = y^*$. □

3.1.3 Application

For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians.

However, in the last few decades many authors pointed out that derivatives of non-integer order provide an excellent instrument for the description of memory and hereditary properties of varions materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of this derivatives become apparent in its demonstrated applications in a wide spectrum of fields ranging from fluid and solid mechanics, control theory and dynamical systems to signal/image processing, economics and biomathematics, one might wish to look up [43, 54, 67, 68, 86].

Definition 3.2. Given a function $x : [a,+\infty[\to \mathbb{R}$, then the Caputo fractional derivative of x of order α is defined by

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{x'(s)}{(t-s)^\alpha} ds, \tag{3.2}$$

where Γ is the gamma function; that is,

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} \exp(-s) ds.$$

So, in this subsection we will study the existence of a continuous solution for a system of fractional differential equations. Let us consider the following system

$$D^{\alpha}x(t) = f(t,x(t),y(t)), \quad D^{\alpha}y(t) = f(t,y(t),x(t)), \quad t \in J \tag{3.3}$$

$$x(0) = x_0 = y(0). \tag{3.4}$$

Where the symbol D^{α} denotes the Caputo fractional derivative of order $\alpha \in (0,1)$ defined in (3.2), $J := [0,L]$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later.

In order to define a solutions for Problem (3.3)-(3.4), let us consider $X = C(J,\mathbb{R})$ endowed with the following b-fuzzy metric space

$$M(x,y,t) = e^{\frac{\sup_{t\in J} |x(t)-y(t)|^2}{t}} \text{ with } s = 2.$$

Consider on $X \times X$ the partial order relation:

$$(x_1,y_1) \le (x_2,y_2) \Leftrightarrow x_1(t) \le x_2(t) \text{ and } y_1(t) \ge y_2(t), \quad t \in J.$$

We shall obtain the unique solution of Eqs. (3.3)-(3.4). This problem is equivalent to the integral equations:

$$\begin{cases} x(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))ds \\ y(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s,y(s),x(s))ds \end{cases} t \in J, \tag{3.5}$$

where Γ is the gamma function.

Assumption 3.1. 1. $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is contionous;

2. For all $x,y,u,v \in \mathbb{R}$, with $x \le u$ and $v \le y$ we have

$$f(t,x,y) \le f(t,u,v), \quad \text{for all } t \in J;$$

3. For each $t \in J$, $x,y,u,v \in \mathbb{R}$, $x \le u$ and $v \le y$, we have

$$|f(t,x,y) - f(t,u,v)|^2 \le |x-u|^2 + |y-v|^2;$$

4. We suppose that
$$K = \frac{8L^{2\alpha}}{\Gamma(\alpha)^2} < 1.$$

We define for $t \in J$,

$$T(x,y)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))ds.$$

Note that if $(x,y) \in X \times X$ is a couple fixed point of T, then we have

$$x(t) = T(x,y)(t) \text{ and } y(t) = T(y,x)(t),$$

for all $t \in J$, and (x,y) is a solution of (3.5).

Now, we shall prove the main result of this section.

Theorem 3.6. *Consider the system* (3.3)-(3.4) *suppose that the Assumption 3.1 is satisfied. Assume that there exists* $(u_0,v_0) \in X \times X$ *such that*

$$u_0(t) \leq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u_0(s),v_0(s))ds$$

and

$$v_0(t) \geq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,v_0(s),u_0(s))ds, \quad t \in J.$$

Then, there exists a unique solution of the integral system (3.3)-(3.4)*.*

Proof. We can prove the existence of a solution of the integral system (3.5) by proving that the operator $T : X \times X \to X$ possesses a coupled fixed point in $X \times X$. We will do this by using Theorem 3.1 and Theorem 3.4.

Consider the graph G with

$$V(G) = X \text{ and } E(G) = \{(x,y) \in X \times X, \quad x \leq y\},$$

and we endow the product space $X \times X$ by another graph denoted also by G, such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for any $(x,y),(u,v) \in X \times X$.

It's easy to prove by using Assumption 3.1 that T is edge preserving.

Now, let us consider $(x,y),(u,v) \in X \times X$ such that $((x,y),(u,v)) \in E(G)$ and by using the Cauchy–Schwarz inequality, then

$$\begin{aligned}
\frac{|T(x,y)(t) - T(u,v)(t)|^2}{\frac{1}{2}t} &= \frac{2}{t}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))ds \right.\\
&\quad \left. - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),v(s))ds\right|^2 \\
&= \frac{2}{t}\frac{1}{\Gamma(\alpha)^2}\left|\int_0^t (t-s)^{\alpha-1}[f(s,x(s),y(s)) - f(s,u(s),v(s))]ds\right|^2 \\
&\leq \frac{2}{t}\frac{L}{\Gamma(\alpha)^2}\int_0^t \left|(t-s)^{\alpha-1}[f(s,x(s),y(s)) - f(s,u(s),v(s))]\right|^2 ds \\
&\leq \frac{2}{t}\frac{L^{2\alpha-1}}{\Gamma(\alpha)^2}\int_0^t |f(s,x(s),y(s)) - f(s,u(s),v(s))|^2 ds,
\end{aligned}$$

then

$$\begin{aligned}
\sup_{t\in J}\frac{|T(x,y)(t) - T(u,v)(t)|^2}{\frac{1}{2}t} &\leq \frac{2L^{2\alpha}}{\Gamma(\alpha)^2}\sup_{t\in J}\frac{|f(t,x(t),y(t)) - f(t,u(t),v(t))|^2}{t} \\
&\leq \frac{8L^{2\alpha}}{2\Gamma(\alpha)^2}\sup_{t\in J}\left(\frac{|x(t)-u(t)|^2}{2t} + \frac{|y(t)-v(t)|^2}{2t}\right) \\
&\leq \frac{K}{2}\sup_{t\in J}\left(\frac{|x(t)-u(t)|^2}{2t} + \frac{|y(t)-v(t)|^2}{2t}\right).
\end{aligned}$$

Therefore, we obatin that for all $((x,y),(u,v)) \in E(G)$

$$M(T(x,y),T(u,v),\varphi(t)) \geq M(x,u,st)^{\frac{1}{2}} * M(y,v,st)^{\frac{1}{2}},$$

where $\varphi(t) = \frac{1}{2}t \in \Omega$ and $t * s = ts$.

Now, by hypotheses we can conclued that

$$((u_0,v_0),(T(u_0,v_0),T(v_0,u_0))) \in E(G).$$

Since T is a continuous mapping and the triple (X,M,G) has the property1.1, which show that all hypotheses of Theorem 3.1 and Theorem 3.2 are satisfied then T possesses a coupled fixed point in $X \times X$.

We obtain from Theorem 3.4 the uniqueness of solution of the integral system (3.5).□

Part II

Coupled Fixed Points of Multi-valued Mappings

Chapter 4

Coupled Fixed Points Theorems in a b-metric Space with a Graph

In this chapter, we establish the existence and uniqueness of the coupled fixed point of the $\mu - \psi$-contraction in b-metric space supplied with a directed graph. Furthermore, we investigated the presence of a continuous solution for a system of fractional differential equations as an application.

Throughout this chapter, (X,d,G) stands to a complete b-metric space with $s \geq 1$, endowed with directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and G has no parallel edges. And let $S : X \times X \to P_{cl}(X)$ be a multi-valued mapping.

We also provide the product space $X \times X$ with another graph, likewise denoted by G, so that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for every $(x,y),(u,v) \in X \times X$.

4.1 Coupled Fixed Point Theorems for $\mu{-}\psi$-contraction Mappings

Inspired by Işik and Türkoğlu [58], we note by Ψ the set of pair of functions (μ,ψ), where $\mu,\psi : [0,\infty) \to [0,\infty)$, verifying

1. μ is continuous and non-decreasing;
2. $\mu(c) = 0$ iff $c = 0$;
3. For $\alpha,\beta \in \mathbb{R}^+$ be a fixed number, $\mu(\alpha c + \beta e) \leq \alpha\mu(c) + \beta\mu(e)$, $\forall c,e \in [0,\infty)$;
4. ψ is continuous;
5. $\mu(c) > \psi(c)$ for all $c > 0$.

Now, we will present new notion of multi-valued contractive type mappings in the framework of $b-$metric spaces.

Definition 4.1. The multi-valued S is called $\mu - \psi$-contraction if there exist $k \in (0,\frac{1}{s})$ and a pair $(\mu,\psi) \in \Psi$ such that

1. S is edge preserving; for each $x,y,u,v \in X$ such that $((x,y),(u,v)) \in E(G)$, for each $x^{'} \in S(x,y)$ and $y^{'} \in S(y,x)$, there exist $u^{'} \in S(u,v)$ and $v^{'} \in S(v,u)$ such that $\left((x^{'},y^{'}),(u^{'},v^{'})\right) \in E(G)$;
2. $$\mu\left(e_d(S(x,y),S(u,v)) + e_d(S(y,x),S(v,u))\right) \leq \psi\left(k(d(x,u) + d(y,v))\right), \tag{4.1}$$
   for all $((x,y),(u,v)) \in E(G)$.

4.1.1 Existence of Coupled Fixed Points

Theorem 4.1. *On (X,d,G), assume that S is a continuous multi-valued mapping and $\mu - \psi$-contraction. If there exists x_0, $y_0 \in X$ for which there exist $(x_1,y_1) \in S(x_0,y_0) \times S(y_0,x_0)$ such that $((x_0,y_0),(x_1,y_1)) \in E(G)$, then S possesses a coupled fixed point.*

Proof. Set $x_1 \in S(x_0,y_0)$ and $y_1 \in S(y_0,x_0)$ such that $((x_0,y_0),(x_1,y_1)) \in E(G)$. By lemma 1.1, for any $\lambda > 1$ there exist $(x_2,y_2) \in S(x_1,y_1) \times S(y_1,x_1)$ such that

$$d(x_1,x_2) \leq \lambda e_d\left(S(x_0,y_0),S(x_1,y_1)\right),$$

$$d(y_1,y_2) \leq \lambda e_d\left(S(y_0,x_0),S(y_1,x_1)\right).$$

Then

$$d(x_1,x_2) + d(y_1,y_2) \leq \lambda\left(e_d\left(S(x_0,y_0),S(x_1,y_1)\right) + e_d\left(S(y_0,x_0),S(y_1,x_1)\right)\right).$$

By the properties of μ and ψ and since S is a $\mu - \psi$-contraction, we have

$$\begin{aligned}\mu\left(d(x_1,x_2) + d(y_1,y_2)\right) &\leq \mu\left(\lambda\left(e_d\left(S(x_0,y_0),S(x_1,y_1)\right) + e_d\left(S(y_0,x_0),S(y_1,x_1)\right)\right)\right)\\ &\leq \lambda\mu\left(e_d\left(S(x_0,y_0),S(x_1,y_1)\right) + e_d\left(S(y_0,x_0),S(y_1,x_1)\right)\right)\\ &\leq \lambda\psi\left(k(d(x_0,x_1) + d(y_0,y_1))\right)\\ &< \lambda\mu\left(k(d(x_0,x_1) + d(y_0,y_1))\right).\end{aligned}$$

Then

$$d(x_1,x_2) + d(y_1,y_2) < k\lambda\left(d(x_0,x_1) + d(y_0,y_1)\right).$$

Again, By lemma 1.1, for any $\lambda > 1$ there exist $(x_3,y_3) \in S(x_2,y_2) \times S(y_2,x_2)$ such that

$$d(x_2,x_3) \leq \lambda e_d\left(S(x_1,y_1),S(x_2,y_2)\right),$$

$$d(y_2,y_3) \leq \lambda e_d\left(S(y_1,x_1),S(y_2,x_2)\right).$$

Then

$$d(x_2,x_3) + d(y_2,y_3) \leq \lambda\left(e_d\left(S(x_1,y_1),S(x_2,y_2)\right) + e_d\left(S(y_1,x_1),S(y_2,x_2)\right)\right).$$

Since S is edge preserving, we have $((x_1,y_1),(x_2,y_2)) \in E(G)$.
Then by the properties of μ and ψ and since S is a $\mu - \psi$-contraction, we have

$$\begin{aligned}\mu\left(d(x_2,x_3) + d(y_2,y_3)\right) &\leq \mu\left(\lambda\left(e_d\left(S(x_1,y_1),S(x_2,y_2)\right) + e_d\left(S(y_1,x_1),S(y_2,x_2)\right)\right)\right)\\ &\leq \lambda\mu\left(e_d\left(S(x_1,y_1),S(x_2,y_2)\right) + e_d\left(S(y_1,x_1),S(y_2,x_2)\right)\right)\\ &\leq \lambda\psi\left(k(d(x_1,x_2) + d(y_1,y_2))\right)\\ &< \lambda\mu\left(k(d(x_1,x_2) + d(y_1,y_2))\right).\end{aligned}$$

Then

$$d(x_2,x_3) + d(y_2,y_3) < k^2\lambda^2 \left(d(x_0,x_1) + d(y_0,y_1)\right).$$

Further, for $n = 1,2,...,$ we let

$$x_{n+1} \in S(x_n,y_n), \text{ and } y_{n+1} \in S(y_n,x_n).$$

Since S is edge preserving, we get

$$\left((x_n,y_n), (x_{n+1},y_{n+1})\right) \in E(G).$$

Then

$$d(x_n,x_{n+1}) + d(y_n,y_{n+1}) < k^n\lambda^n \left(d(x_0,x_1) + d(y_0,y_1)\right). \tag{4.2}$$

We choose now $1 < \lambda < \frac{1}{sk}$. Let $n \geq 1$ and $p \geq 1$, we have

$$\begin{aligned} d(x_n,x_{n+p}) + d(y_n,y_{n+p}) \leq & s(d(x_n,x_{n+1}) + d(y_n,y_{n+1})) + s^2(d(x_{n+1},x_{n+2}) + d(y_{n+1},y_{n+2})) \\ & + \cdots + s^{p-1}(d(x_{n+p-2},x_{n+p-1}) + d(y_{n+p-2},y_{n+p-1})) \\ & + s^p(d(x_{n+p-1},x_{n+p}) + d(y_{n+p-1},y_{n+p})) \\ & < sk^n\lambda^n \left(d(x_0,x_1) + d(y_0,y_1)\right) + s^2k^{n+1}\lambda^{n+1} \left(d(x_0,x_1) + d(y_0,y_1)\right) \\ & + \cdots + s^{p-1}k^{n+p-2}\lambda^{n+p-2} \left(d(x_0,x_1) + d(y_0,y_1)\right) \\ & + s^pk^{n+p-1}\lambda^{n+p-1} \left(d(x_0,x_1) + d(y_0,y_1)\right) \\ & = sk^n\lambda^n \left[1 + sk\lambda + \cdots + s^{p-2}k^{p-2}\lambda^{p-2} \right. \\ & \left. + s^{p-1}k^{p-1}\lambda^{p-1}\right] \left(d(x_0,x_1) + d(y_0,y_1)\right). \end{aligned}$$

Since $sk\lambda < 1$, we get $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. So there exists $x^*,y^* \in X$ as X is complete space, such that

$$\lim_{n\to+\infty} x_n = x^* \text{ and } \lim_{n\to+\infty} y_n = y^*.$$

Since S is continuous, we get

$$\lim_{n\to\infty} H(S(y_n, x_n), S(y^*, x^*)) = 0.$$

Since $y_{n+1} \in S(y_n, x_n)$, Lemma 1.1 implies the existence of $b_n \in S(y^*, x^*)$ such that

$$d(y_{n+1}, b_n) \leq \lambda H(S(y_n, x_n), S(y, x)).$$

Clearly, we have $\lim_{n\to\infty} b_n = y^*$. Since $S(y^*, x^*)$ is closed, we conclude $y^* \in S(y^*, x^*)$. Similarly, we will show that $x^* \in S(x^*, y^*)$, i.e., (x^*,y^*) is a coupled fixed point of S.

□

Now, we will utilize another sufficient condition for the existence of couple fixed point in the case where the triple (X, d,G) owns the property 1.1.

Theorem 4.2. *Endowed (X,d,G) with the property1.1. Assume that S is $\mu-\psi$-contraction. If there exists $x_0,y_0 \in X$ for which there exist $(x_1,y_1) \in S(x_0,y_0) \times S(y_0,x_0)$ such that $((x_0,y_0), (x_1,y_1)) \in E(G)$, then S possesses a coupled fixed point.*

Proof. we follow the proof of Theorem 4.1 and we prove that $x^* \in S(x^*,y^*)$ and $y^* \in S(y^*,x^*)$.

Since $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} S(x_n,y_n) = x^*$, $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} S(y_n,x_n) = y^*$ and $(x_n,x_{n+1}) \in E(G)$ and $(y_{n+1},y_n) \in E(G)$, then by property 1.1 we have

$$(x_n,x^*) \in E(G) \text{ and } (y^*,y_n) \in E(G),$$

then

$$((x_n,y_n),(x^*,y^*)) \in E(G).$$

Since S is edge preserving, there exist $y_n^* \in S(y^*,x^*)$ and $x_n^* \in S(x^*,y^*)$ such that

$$((y_{n+1},x_{n+1}),(y_n^*,x_n^*)) \in E(G).$$

Then, for any $n \geq 1$

$$\begin{aligned}
\mu\left(d(x_{n+1},x_n^*) + d(y_{n+1},y_n^*)\right) &\leq \mu\left(\lambda\left(e_d\left(S(x_n,y_n),S(x^*,y^*)\right) + e_d\left(S(y_n,x_n),S(y^*,x^*)\right)\right)\right) \\
&\leq \lambda\mu\left(e_d\left(S(x_n,y_n),S(x^*,y^*)\right) + e_d\left(S(y_n,x_n),S(y^*,x^*)\right)\right) \\
&\leq \lambda\psi\left(k(d(x_n,x^*) + d(y_n,y^*))\right).
\end{aligned}$$

This will imply

$$\lim_{n\to\infty} d(x_{n+1},x_n^*) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(y_{n+1},y_n^*) = 0.$$

Therefore, we get

$$\lim_{n\to\infty} x_n^* = x^* \quad \text{and} \quad \lim_{n\to\infty} y_n^* = y^*.$$

Since $S(x^*, y^*)$ and $S(y^*, x^*)$ are closed, we conclude that $x^* \in S(x^*, y^*)$ and $y^* \in S(y^*, x^*)$, i.e., (x^*,y^*) is a coupled fixed point of S. $\square$

Theorem 4.3. *On (X,d,G), we assume that S having a closed graph and be a $\mu - \psi$-contraction. If there exists $x_0,y_0 \in X$ for which there exist $(x_1,y_1) \in S(x_0,y_0) \times S(y_0,x_0)$ such that $((x_0,y_0), (x_1,y_1)) \in E(G)$, then S possesses a coupled fixed point.*

Proof. we follow the proof of Theorem 4.1 and we prove that $x^* \in S(x^*,y^*)$ and $y^* \in S(y^*,x^*)$.

Since $x_{n+1} \in S(x_n,y_n)$ and $y_{n+1} \in S(y_n,x_n)$, then

$$(x_n,y_n,x_{n+1}) \in Graph(S) \quad \text{and} \quad (y_n,x_n,y_{n+1}) \in Graph(S).$$

Because we know that $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$ and in view of the graph of S is closed, we have

$$x^* \in S(x^*, y^*) \quad \textit{and} \quad y^* \in S(y^*, x^*),$$

i.e., (x^*,y^*) is a coupled fixed point of S. $\square$

4.1.2 Uniqueness of Coupled Fixed Points

Theorem 4.4. *In addition to the hypotheses of Theorem 4.1, Theorem 4.2 and Theorem 4.3, we suppose that there exists $(x^*,y^*) \in X \times X$ such that*

$$S(x^*,y^*) = \{x^*\} \quad \textit{and} \quad S(y^*,x^*) = \{y^*\},$$

and if for any coupled fixed point (x,y) of S verify $((x^*,y^*),(x,y)) \in E(G)$, then S have a unique coupled fixed point (x^*,y^*).

Proof. Let (x^*,y^*) be as in our hypothesis and (x,y) be another coupled fixed point of S. Then

$$\begin{aligned}\mu\left(d(x,x^*) + d(y,y^*)\right) &\leq \mu\left(e_d(S(x,y),S(x^*,y^*)) + e_d(S(y,x),S(y^*,x^*))\right)\\ &\leq \psi\left(k(d(x,x^*) + d(y,y^*))\right)\\ &< \mu\left(k(d(x,x^*) + d(y,y^*))\right).\end{aligned}$$

Then,

$$d(x^*,x) + d(y^*,y) < k(d(x^*,x) + d(y^*,y)).$$

Thus, $d(x^*,x) + d(y^*,y) = 0$ which implies that $x^* = x$ and $y^* = y$. □

Theorem 4.5. *Suppose that all the hypotheses of Theorem 4.4 take place. If (x^*,y^*) denotes the unique coupled fixed point of S and we suppose that $(x^*,y^*) \in E(G)$, then x^* is a unique "double" fixed point for S, i.e., $S(x^*,x^*) = \{x^*\}$.*

Proof. By Theorem 4.4 we know that $S(x^*,y^*) = \{x^*\}$ and $S(y^*,x^*) = \{y^*\}$.

Then,

$$\begin{aligned}
\mu\left(d(x^*,y^*)\right) &\leq \frac{1}{2}\mu\left(d(x^*,y^*) + d(y^*,x^*)\right) \\
&= \frac{1}{2}\mu\left(e_d(S(x^*,y^*),S(y^*,x^*)) + e_d(S(y^*,x^*),S(x^*,y^*))\right) \\
&\leq \frac{1}{2}\psi\left(k(d(x^*,y^*) + d(y^*,x^*))\right) \\
&< \frac{1}{2}\mu\left(k(d(x^*,y^*) + d(y^*,x^*))\right).
\end{aligned}$$

Then, $d(x^*,y^*) = 0$, and thus $x^* = y^*$. □

Remark 4.1. Since $e_d(A,B) \leq H_d(A,B)$ in any metric space, the above results take also if we remplace the equation (4.1) by

$$\mu\left(H_d(S(x,y),S(u,v)) + H_d(S(y,x),S(v,u))\right) \leq \psi\left(k(d(x,u) + d(y,v))\right),$$

for all $((x,y),(u,v)) \in E(G)$.

4.1.3 Application

Now, we will apply our results obtained in this section in the fractional differential equation of Caputo type.

Let us consider the following system:

$$D^\alpha x(t) \in F(t,x(t),y(t)), \quad D^\alpha y(t) \in F(t,y(t),x(t)), \quad t \in I \tag{4.3}$$

$$x(0) = x_0 = y(0), \tag{4.4}$$

where the symbol D^α denotes the Caputo fractional derivative of order $\alpha \in (0,1)$ defined in (3.2), $J := [0,L]$, $F : J \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is a multi-valued operator satisfying some appropriate conditions.

In order to define a solutions for Problem (4.3)-(4.4), let us consider $X = C(J,\mathbb{R})$ endowed with the following b-metric space

$$d(x,y) = \sup_{t\in J} |x(t) - y(t)|^2 \text{ with } s = 2.$$

Consider on $X \times X$ the partial order relation:

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1(t) \leq x_2(t) \text{ and } y_1(t) \geq y_2(t), \quad t \in J.$$

We define the graph G by $V(G) = X$, and

$$E(G) = \{(x,y) \in X \times X, \quad x \leq y\},$$

and we endow the product space $X \times X$ by another graph denoted also by G, such that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for any $(x, y), (u, v) \in X \times X$.

We shall obtain the unique solution of Eqs. (4.3)-(4.4). This problem is equivalent to the integral equations:

$$\begin{cases} x(t) \in x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\nu)^{\alpha-1} F(\nu,x(\nu),y(\nu))d\nu \\ y(t) \in x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\nu)^{\alpha-1} F(\nu,y(\nu),x(\nu))d\nu \end{cases} t \in J, \tag{4.5}$$

where Γ is the gamma function.

Assumption 4.1.

1. $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cl,cv}(\mathbb{R})$ is measurable in the first variable;
2. F is integrably bounded, i.e., there exists a mapping $r \in L^1(J,\mathbb{R})$ such that for each $(t,u,v) \in J \times \mathbb{R} \times \mathbb{R}$ and for any $f \in F(t,u,v)$, we have $|f| \leq r(t)$, $t \in J$;
3. $F(t,.,.)$ is edge preserving with respect to the last two variables, for all $t \in J$;
4. For each $t \in J$, $x,y,u,v \in \mathbb{R}$, $x \leq u$ and $v \leq y$, we have
$$e_{||}\left(F(t,x,y),F(t,u,v)\right) \leq |x-u| + |y-v|;$$
5. We suppose that
$$K = \frac{8L^{2\alpha}}{\Gamma(\alpha)^2} < 1.$$

We define $S : X \times X \to P(X)$ by

$$S(x,y) = \{s \in X, \quad s(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} f_{xy}(\nu) d\nu, \quad f_{xy} \in S_{F(.,x(.),y(.))}\},$$

where

$$S_{F(.,x(.),y(.))} = \{f \in L^1(J,\mathbb{R}) | \quad f(t) \in F(t,x(t),y(t)), t \in J\}.$$

Note that if $(x,y) \in X \times X$ is a couple fixed point of S, then we have

$$x(t) \in S(x,y)(t) \text{ and } y(t) \in S(y,x)(t),$$

for all $t \in J$, and (x,y) is a solution of (4.5).

Now, we shall prove the main result of this section.

Theorem 4.6. *Consider the system* (4.3)-(4.4) *suppose that the Assumption 4.1 is satisfied. Assume that there exists* $(u_0,v_0) \in X \times X$ *and two measurable selections* $f_{u_0,v_0} : J \to \mathbb{R}$ *of* $F(.,u_0(.),v_0(.))$ *and* $f_{v_0,u_0} : J \to \mathbb{R}$ *of* $F(.,v_0(.)u_0(.))$ *such that*

$$u_0(t) \leq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} f_{u_0,v_0}(\nu) d\nu$$

and

$$v_0(t) \geq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} f_{v_0,u_0}(\nu) d\nu, \quad t \in J.$$

Then there exists at least one solution of the integral system (4.3)-(4.4).

Proof. We will obtain the existence of solution of the integral system (4.5) by showing that the operator $S : X \times X \to P(X)$ has a coupled fixed point in $X \times X$. To do this, we verify that S satisfies the hypotheses of Theorem 4.2.

1. $S(x,y)$ is nonempty, for each $(x, y) \in X \times X$. Indeed, by the first assumption via Kuratowski and Ryll-Nardzewski selection theorem [15], there exist a measurable selection f for F. Then, $S_{F(.,x(.),y(.))}$ is nonempty. By the integrably bounded property of F in the second assumption we get that f is integrable. Hence $S_{F(.,x(.),y(.))} \subset L^1(J,\mathbb{R})$. Thus S is well defined.

2. S has closed values. Indeed, let $x, y \in X$ and let (s_n) be a sequence in $S(x, y)$ with $s_n \to s$ in (X, d). We have to show that $s \in S(x, y)$, i.e., there exists $f_{xy} \in S_{F(.,x(.),y(.))}$ such that
$$s(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} f_{xy}(\nu) d\nu.$$
Since $s_n \in S(x, y)$ there exists $f_{xy}^{(n)} \in S_{F(.,x(.),y(.))}$ such that
$$s_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} f_{xy}^{(n)}(\nu) d\nu, \text{ for } n \in \mathbb{N}, \quad t \in J.$$
Since $(s_n) \to s$ in X, for all t, we have $(s_n(t)) \to s(t)$. Since
$$s_n(t) \in x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} F(\nu,x(\nu),y(\nu))(\nu) d\nu, t \in J,$$
and F has a convex images and is integrably bounded, then, by Theorem 8.6.4 in [15] we get the set $x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} F(\nu,x(\nu),y(\nu))(\nu) d\nu$ is closed. Then
$$s(t) \in x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} F(\nu,x(\nu),y(\nu))(\nu) d\nu.$$
Thus, there exists $f_{xy} \in S_{F(.,x(.),y(.))}$ such that
$$s(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\nu)^{\alpha-1} f_{xy}(\nu) d\nu.$$

3. We will show that
$$\mu \left(e_d(S(x,y),S(u,v)) + e_d(S(y,x),S(v,u)) \right) \leq \psi \left(k'(d(x,u) + d(y,v)) \right),$$

for all $((x,y),(u,v)) \in E(G)$, such that $k' = \frac{1}{4}$, $\mu(t) = \frac{k}{2}t$ and $\psi(t) = \frac{kK}{2}t$ with $k \in (0,1)$. We note that $(\mu,\psi) \in \Psi$.
We will prove first that for each $w \in S(x,y)$ there exists $z \in S(u,v)$ such that

$$\mu(d(w,z)) \leq \frac{\psi}{2}(k'(d(x,u) + d(y,v))).$$

By the fourth assumption, we have

$$e_{||}\left(F(t,x,y),F(t,u,v)\right) \leq \frac{1}{2}\left(|x-u| + |y-v|\right),$$

for each $t \in J$, $x,y,u,v \in \mathbb{R}$ with $((x,y),(u,v)) \in E(G)$.

If $w \in S(x,y)$, then there exists $f_{xy} \in S_{F(.,x(.),y(.))}$ such that

$$w(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1} f_{xy}(\nu)d\nu, \text{ for } t \in J.$$

Since $f_{xy}(t) \in F(t,x(t),y(t))$ for $t \in J$, we can find $b \in F(t,u(t),v(t))$ such that

$$|f_{xy}(t) - b| \leq \frac{1}{2}\left(|x(t)-u(t)| + |y(t)-v(t)|\right).$$

Thus, if we define the multi-valued operator $A(t) := F(t,u(t),v(t)) \cap B(t)$, where $B(t) := \{b, \quad |f_{xy}(t) - b| \leq \frac{1}{2}\left(|x(t)-u(t)| + |y(t)-v(t)|\right)\}$, then $A(t)$ is nonempty for $t \in J$ and A is measurable (as an intersection of two measurable multi-valued operators). Thus, A has measurable selections and let $f_{uv}(t) \in A(t)$, for $t \in J$. Hence, for $t \in J$, we have $f_{uv}(t) \in F(t,u(t),v(t))$ and

$$|f_{xy}(t) - f_{uv}(t)| \leq \frac{1}{2}\left(|x(t)-u(t)| + |y(t)-v(t)|\right).$$

Define now

$$z(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1} f_{uv}(\nu)d\nu, \text{ for } t \in J.$$

Obvious, $z \in S(u,v)$.

Thus

$$\begin{aligned}
\frac{k}{2}|w(t)-z(t)|^2 &= \frac{k}{2}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1} f_{xy}(\nu)d\nu - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1} f_{uv}(\nu)d\nu\right|^2 \\
&= \frac{k}{2}\frac{1}{\Gamma(\alpha)^2}\left|\int_0^t (t-\nu)^{\alpha-1}[f_{xy}(\nu)-f_{uv}(\nu)]d\nu\right|^2 \\
&\leq \frac{k}{2}\frac{L}{\Gamma(\alpha)^2}\int_0^t \left|(t-\nu)^{\alpha-1}[f_{xy}(\nu)-f_{uv}(\nu)]\right|^2 d\nu \\
&\leq \frac{k}{2}\frac{L^{2\alpha-1}}{\Gamma(\alpha)^2}\int_0^t |f_{xy}(\nu)-f_{uv}(\nu)|^2 d\nu,
\end{aligned}$$

then

$$\begin{aligned}
\sup_{t\in J}\frac{k}{2}|w(t)-z(t)|^2 &\leq \frac{kL^{2\alpha}}{2\Gamma(\alpha)^2}\sup_{t\in J}|f_{xy}(t)-f_{uv}(t)|^2 \\
&\leq \frac{K}{16}\sup_{t\in J}\left(|x(t)-u(t)|^2+|y(t)-v(t)|^2\right).
\end{aligned}$$

Hence, we get

$$\mu(d(w,z)) \leq \frac{1}{2}\psi(k'(d(x,u)+d(y,v))).$$

By interchanging the roles of x and y, respectively u and v, we get

$$\mu\left(e_d(S(y,x),S(v,u))\right) \leq \frac{1}{2}\psi(k'(d(x,u)+d(y,v))).$$

Therefore, we obatin that for all $((x,y),(u,v)) \in E(G)$

$$\mu\left(e_d(S(x,y),S(u,v)) + e_d(S(y,x),S(v,u))\right) \leq \psi\left(k'(d(x,u)+d(y,v))\right).$$

Now, by hypotheses we can conclude that

$$((u_0,v_0),(u_1,v_1)) \in E(G),$$

such that $(u_1,v_1) \in S(u_0,v_0) \times S(v_0,u_0)$.

Since S is continuous and the triple (X,d,G) has the property 1.1, which show that all hypotheses of Theorem 4.1 and Theorem 4.2 are satisfied then S have a couple fixed point in $X \times X$. □

Chapter 5

Coupled Fixed Points Theorems in a b-fuzzy Metric Space with a Graph

This chapter's goal is to provide some coupled fixed point theorems for multi-valued operators satisfying a φ-fuzzy contraction on a $b-$ Fuzzy metric space with a graph. The proven results on the existence of a continuous solution for a system of fractal-fractional differential equations are then used.

We'll assume for all this chapter that $(X,M,*,G)$ is a complete b-fuzzy metric space with $s \geq 1$, such that $a * a \geq a^2$ and M is continuous, verify $\lim_{t\to\infty} M(x,y,t) = 1$ and endowed with directed graph G such that $V(G) = X$, $E(G) \supseteq \Delta$ and G has no parallel edges. And let $S : X \times X \to P_{cl,cp}(X)$ be a multi-valued mapping.

We also provide the product space $X \times X$ with another graph, likewise denoted by G, so that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for every $(x,y),(u,v) \in X \times X$.

5.1 Coupled Fixed Point Theorems for φ-multi-fuzzy Mappings

Definition 5.1. The multi-valued S is called φ-multi-fuzzy contraction if there exists a function $\varphi \in \Omega$ such that:

1. S is edge preserving; for each $x,y,u,v \in W$ such that $((x,y),(u,v)) \in E(G)$, for each $x' \in S(x,y)$ and $y' \in S(y,x)$, there exist $u' \in S(u,v)$ and $v' \in S(v,u)$ such that $\big((x',y'),(u',v')\big) \in E(G)$.

2. $$H_M\left(S(x,y),S(u,v),\varphi(t)\right) \geq M\left(x,u,st\right)^{\frac{1}{2}} * M\left(y,v,st\right)^{\frac{1}{2}},$$
   for all $((x,y),(u,v)) \in E(G)$.

5.1.1 Existence of Coupled Fixed Points

Theorem 5.1. *On $(X,M,*,G)$, suppose that S is continuous multi-valued mapping and φ-multi-fuzzy contraction. If there exist $x_0,y_0 \in X$ for which there exist $(x_1,y_1) \in S(x_0,y_0) \times S(y_0,x_0)$ such that $((x_0,y_0),(x_1,y_1)) \in E(G)$, then S possesses coupled fixed point.*

Proof. Set $x_1 \in S(x_0,y_0)$ and $y_1 \in S(y_0,x_0)$ such that $((x_0,y_0),(x_1,y_1)) \in E(G)$. By lemma 1.4, we can choose $(x_2,y_2) \in S(x_1,y_1) \times S(y_1,x_1)$ such that

$$\begin{aligned} M(x_1,x_2,\varphi(t)) &\geq H_M\left(S(x_0,y_0),S(x_1,y_1),\varphi(t)\right) \\ &\geq M(x_0,x_1,st)^{\frac{1}{2}} * M(y_0,y_1,st)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} M(y_1,y_2,\varphi(t)) &\geq H_M\left(S(y_0,x_0),S(y_1,x_1),\varphi(t)\right) \\ &\geq M(y_0,y_1,st)^{\frac{1}{2}} * M(x_0,x_1,st)^{\frac{1}{2}}. \end{aligned}$$

Again, By lemma 1.4, we can choose $(x_3,y_3) \in S(x_2,y_2) \times S(y_2,x_2)$ such that

$$\begin{aligned} M(x_2,x_3,\varphi^2(t)) &\geq H_M\left(S(x_1,y_1),S(x_2,y_2),\varphi^2(t)\right) \\ &\geq M(x_1,x_2,s\varphi(t))^{\frac{1}{2}} * M(y_1,y_2,s\varphi(t))^{\frac{1}{2}} \\ &\geq M(x_0,x_1,s^2t)^{\frac{1}{2}} * M(y_0,y_1,s^2t)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned}M(y_2,y_3,\varphi^2(t)) &\geq H_M\left(S(y_1,x_1),S(y_2,x_2),\varphi^2(t)\right)\\ &\geq M(y_1,y_2,s\varphi(t))^{\frac{1}{2}} * M(x_1,x_2,s\varphi(t))^{\frac{1}{2}}\\ &\geq M(y_0,y_1,s^2t)^{\frac{1}{2}} * M(x_0,x_1,s^2t)^{\frac{1}{2}}.\end{aligned}$$

Further, for $n = 1,2,...$, we let

$$x_{n+1} \in S(x_n,y_n), \text{ and } y_{n+1} \in S(y_n,x_n).$$

Since S is edge preserving, we get

$$\left((x_n,y_n), (x_{n+1},y_{n+1})\right) \in E(G)$$

then

$$M(x_n,x_{n+1},\varphi^n(t)) \geq M(x_0,x_1,s^nt)^{\frac{1}{2}} * M(y_0,y_1,s^nt)^{\frac{1}{2}}, \tag{5.1}$$

$$M(y_n,y_{n+1},\varphi^n(t)) \geq M(y_0,y_1,s^nt)^{\frac{1}{2}} * M(x_0,x_1,s^nt)^{\frac{1}{2}}. \tag{5.2}$$

Since $\varphi \in \Omega$, for all $t_0 > 0$ there exist $m > 0$ such that $t_0 > \sum_{k=n_0}^{\infty} \varphi^k(t)$. So for all $m \geq n \geq n_0$ we have:

$$\begin{aligned}M(x_n,x_m,t_0) &\geq M\left(x_n,x_m,\frac{\sum_{k=n}^{m-1}\varphi^k(t)}{s}\right)\\ &\geq M\left(x_n,x_{n+1},\frac{\varphi^n(t)}{s^2}\right) * M\left(x_{n+1},x_{n+2},\frac{\varphi^{n+1}(t)}{s^3}\right)\\ &\quad * M\left(x_{n+2},x_{n+3},\frac{\varphi^{n+2}(t)}{s^4}\right) * \cdots * M\left(x_{m-1},x_m,\frac{\varphi^{m-1}(t)}{s^{m-n+1}}\right)\\ &\geq M\left(x_0,x_1,\frac{s^n}{s^2}t\right)^{\frac{1}{2}} * M\left(y_0,y_1,\frac{s^n}{s^2}t\right)^{\frac{1}{2}} * M\left(x_0,x_1,\frac{s^{n+1}}{s^3}t\right)^{\frac{1}{2}}\\ &\quad * M\left(y_0,y_1,\frac{s^{n+1}}{s^3}t\right)^{\frac{1}{2}} * M\left(x_0,x_1,\frac{s^{n+2}}{s^4}t\right)^{\frac{1}{2}} * M\left(y_0,y_1,\frac{s^{n+2}}{s^4}t\right)^{\frac{1}{2}}\\ &\quad * \cdots * M\left(x_0,x_1,\frac{s^{m-1}}{s^{m-n+1}}t\right)^{\frac{1}{2}} * M\left(y_0,y_1,\frac{s^{m-1}}{s^{m-n+1}}t\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\lim\limits_{t\to\infty} M(x,y,t) = 1$ for all x, $y \in X$, we have

$$\lim_{t\to\infty} M(x_n,x_{n+p},t) \geq 1 * 1 * 1 * \cdots * 1 = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. By the same way we can prove that $\{y_n\}$ is also a Cauchy sequence.

Since X is complete space then there exists x^*,$y^* \in X$ such that

$$\lim_{n\to+\infty} x_n = x^* \text{ and } \lim_{n\to+\infty} y_n = y^*.$$

Since S is continuous, we get

$$\lim_{n\to\infty} H(S(y_n, x_n), S(x^*, y^*),t) = 0, \quad \forall t > 0.$$

Since $y_{n+1} \in S(y_n, x_n)$, Lemma 1.4 implies the existence of $b_n \in S(x^*, y^*)$ such that

$$M(x_{n+1}, b_n,t) \geq H_M(S(x_n, y_n), S(x^*, y^*),t).$$

Clearly, we have $\lim_{n\to\infty} b_n = x^*$. Since $S(x^*, y^*)$ is closed, we conclude $x^* \in S(x^*, y^*)$. Similarly, we will show that $y^* \in S(y^*, x^*)$, i.e., (x^*,y^*) is a coupled fixed point of S. □

Now, we will use another necessary condition for the presence of a coupled fixed point in the case when the triple (X, M,G) has the property1.1,

Theorem 5.2. *Endowed $(X,M, * ,G)$ with the property1.1. Suppose that S is a φ-multi-fuzzy contraction. If there exist x_0,$y_0 \in X$ for which there exist $(x_1,y_1) \in S(x_0,y_0) \times S(y_0,x_0)$ such that $((x_0,y_0), (x_1,y_1)) \in E(G)$, then S possesses coupled fixed point.*

Proof. We prove that $y^* \in S(y^*,x^*)$ and $x^* \in S(x^*,y^*)$ by following the proof of Theorem 5.1.

Since $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} S(x_n,y_n) = x^*$, $\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} S(y_n,x_n) = y^*$ and $(x_n,x_{n+1}) \in E(G)$ and $(y_{n+1},y_n) \in E(G)$, then by property 1.1 we have

$$(x_n,y^*) \in E(G) \text{ and } (x^*,y_n) \in E(G),$$

then

$$((x_n,y_n),(y^*,x^*)) \in E(G).$$

Since S is edge preserving, there exist $x_n^* \in S(x^*,y^*)$ and $y_n^* \in S(y^*,x^*)$ such that

$$((y_{n+1},x_{n+1}),(x_n^*,y_n^*)) \in E(G).$$

Then, for any $n \geq 1$

$$\begin{aligned} M\left(x_n^*,x^*,t\right) &\geq M\left(x_n^*,x^*,\frac{\varphi(t)}{s}\right) \\ &\geq H_M\left(S(x_n,y_n),S(x^*,y^*),\frac{\varphi(t)}{s}\right) \\ &\geq M\left(x_n,x^*,t\right)^{\frac{1}{2}} * M\left(y_n,y^*,t\right)^{\frac{1}{2}}. \end{aligned}$$

This will imply,

$$\lim_{n\to\infty} M\left(x_n^*,x^*,t\right) = 1.$$

Therefore, we get

$$\lim_{n\to\infty} x_n^* = x^*.$$

By the same process we can get

$$\lim_{n\to\infty} y_n^* = y^*.$$

Since $S(x^*, y^*)$ and $S(y^*, x^*)$ are closed, we conclude that $x^* \in S(x^*, y^*)$ and $y^* \in S(y^*, x^*)$, i.e., (x^*,y^*) is a coupled fixed point of S. □

Theorem 5.3. *On $(X,M,*,G)$, suppose that S having a closed graph and be a φ-multi-fuzzy contraction. If there exists $x_0,y_0 \in X$ for which there exist $(x_1,y_1) \in S(x_0,y_0) \times S(y_0,x_0)$ such that $((x_0,y_0),(x_1,y_1)) \in E(G)$, then S possesses coupled fixed point.*

Proof. We will prove that $x^* \in S(x^*,y^*)$ and $y^* \in S(y^*,x^*)$ also by following the proof of Theorem 5.1.

Since $x_{n+1} \in S(x_n,y_n)$ and $y_{n+1} \in S(y_n,x_n)$, then

$$(x_n,y_n,x_{n+1}) \in Graph(S) \quad \text{and} \quad (y_n,x_n,y_{n+1}) \in Graph(S).$$

Since $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$ and in view of the graph of S is closed, we have

$$x^* \in S(x^*, y^*) \quad and \quad y^* \in S(y^*, x^*),$$

i.e., (x^*,y^*) is a coupled fixed point of S. □

5.1.2 Uniqueness of Coupled Fixed Points

Theorem 5.4. *If in addition to the hypothesis of Theorem 5.1, Theorem 5.2 and Theorem 5.3 that, we suppose that $((x_0,y_0),(x^*,y^*)) \in E(G)$. Then S have a unique coupled fixed point.*

Proof. If we suppose that (x,y) is another coupled fixed point of S.

By the pervious proof we construct two sequence $\{x_n\}$ and $\{y_n\}$, then

$$\begin{aligned}
M(x_{n+1},x,t) &\geq H_M(S(x_n,y_n),S(x,y),t) \\
&\geq H_M\left(S(x_n,y_n),S(x,y),\frac{\varphi^n(t)}{s}\right) \\
&\geq M(x_n,x,\varphi^{n-1}(t))^{\frac{1}{2}} * M(y_n,y,\varphi^{n-1}(t))^{\frac{1}{2}} \\
&\geq H_M(S(x_{n-1},y_{n-1}),S(x,y),\varphi^{n-1}(t))^{\frac{1}{2}} * H_M(S(y_{n-1},x_{n-1}),S(y,x),\varphi^{n-1}(t))^{\frac{1}{2}} \\
&\geq M(x_{n-1},x,s\varphi^{n-2}(t))^{\frac{1}{2}} * M(y_{n-1},y,s\varphi^{n-2}(t))^{\frac{1}{2}} \\
&\vdots \\
&\geq M(x_0,x,s^{n-1}t)^{\frac{1}{2}} * M(x_0,y,s^{n-1}t)^{\frac{1}{2}}.
\end{aligned}$$

By the similair way we can get

$$M(y_{n+1},y,t) \geq M(y_0,y,s^{n-1}t)^{\frac{1}{2}} * M(x_0,x,s^{n-1}t)^{\frac{1}{2}}.$$

Letting $n \to +\infty$ we get

$$\lim_{n\to+\infty} x_{n+1} = x \text{ and } \lim_{n\to+\infty} y_{n+1} = y,$$

then

$$x^* = x \text{ and } y^* = y.$$ □

5.1.3 Application

The main novelty of the present article is to apply the fractal-fractional derivatives on the classical model. Fractal-fractional derivative is a new class of fractional derivative with power Law kernel which has many applications in real world problems. This operator is used for the first time in such kind of fluid flow. The big advantage of this operator is that we can formulate models describing much better the systems with memory effects. Furthermore, in real world there are many problems where we need to know that how much information the system carries that is why need memory in a system which is explained by fractal-fractional derivatives with power law kernel, see [3, 7, 8, 12, 44].

Definition 5.2. If x is fractal differentiable on (a, b) with order β and $x(t)$ is continuous on the opened interval (a, b), then the Fractal-Fractional derivative of x of order α in Riemann-Liouville sense with power law is given as

$$ {}_{a}^{FFP}D_t^{\alpha,\beta}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d}{dy^{\beta}}\int_a^t x(y)(t-y)^{n-\alpha-1}dy, $$
$$ n-1<\alpha\leq n\in\mathbb{N}, \quad 0<n-1<\beta\leq n $$

with

$$ \frac{dx(y)}{dy^{\beta}} = \lim_{y_1\to y}\frac{x(y_1)-x(y)}{y_1^{\beta}-y^{\beta}}. $$

Let us consider the following system of fractional differential equations

$$ {}_{0}^{FFP}D_t^{\alpha,\beta}x(t)\in F(t,x(t),y(t)), \quad {}_{0}^{FFP}D_t^{\alpha,\beta}y(t)\in F(t,y(t),x(t)), \quad t\in J \tag{5.3} $$

$$ x(0)=x_0=y(0), \tag{5.4} $$

where, the symbol ${}_{0}^{FFP}D_t^{\alpha,\beta}x(t)$ denotes the Fractal-Fractional derivative of x of order α in Riemann-Liouville sense with power law such that $x(t)$ is continuous in opened interval $(0, 1)$ fractal differentiable on (0,1) with order β, $J := [0,L]$, $F : J\times\mathbb{R}\times\mathbb{R}\to P(\mathbb{R})$ is a multi-valued operator satisfying some appropriate conditions.

Consider $X = C(J,\mathbb{R})$ supply with the following b-fuzzy metric space,

$$ M(x,y,t) = e^{-\frac{\sup_{t\in J}|x(t)-y(t)|^2}{t}} \text{ with } s=2. $$

Let on $X\times X$ the partial order relation:

$$ (y_1,x_1)\leq(y_2,x_2)\Leftrightarrow y_1(t)\leq y_2(t) \text{ and } x_1(t)\geq x_2(t), \quad t\in J. $$

We define the graph G by $V(G)=X$, and

$$ E(G)=\{(x,y)\in X\times X, \quad x\leq y\}. $$

Now we endow the product space $X \times X$ with another graph, also denoted by G, so that

$$((x,y),(u,v)) \in E(G) \Leftrightarrow (x,u) \in E(G) \text{ and } (v,y) \in E(G),$$

for any $(x,y),(u,v) \in X \times X$.

We'll figure out how to solve Eqs.(5.3)-(5.4). The following integral equations are equivalent to this problem:

$$\begin{cases} x(t) \in x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} F(\nu,x(\nu),y(\nu))d\nu \\ y(t) \in x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} F(\nu,y(\nu),x(\nu))d\nu \end{cases} t \in J, \tag{5.5}$$

where Γ is the gamma function;

Assumption 5.1.

1. $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cl,cv}(\mathbb{R})$ is measurable in the first variable;
2. F is integrably bounded, i.e., there exists a mapping $r \in L^1(J,\mathbb{R})$ such that for each $(t,u,v) \in J \times \mathbb{R} \times \mathbb{R}$ and for any $f \in F(t,u,v)$, we have $|f| \leq r(t)$, $t \in J$;
3. $F(t,.,.)$ is edge preserving with respect to the last two variables, for all $t \in J$;
4. For each $t \in J$, $x,y,u,v \in \mathbb{R}$, $x \leq u$ and $v \leq y$, we have
$$H_M\left(F(t,x,y),F(t,u,v),t\right) \leq |x-u| + |y-v|;$$
5. We suppose that
$$K = \frac{8\beta^2 L^{2\beta+2\alpha-2}}{\Gamma(\alpha)^2} < 1.$$

We define $S : X \times X \to P(X)$ by

$$S(x,y) = \{s \in X, \quad s(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{xy}(\nu)d\nu, \quad f_{xy} \in S_{F(.,x(.),l(.))}\},$$

where

$$S_{F(.,x(.),l(.))} = \{f \in L^1(J,\mathbb{R})| \quad f(t) \in F(t,x(t),y(t)), t \in J\}.$$

Note that if $(x,y) \in X \times X$ is a couple fixed point of S, then we have

$$x(t) \in S(x,y)(t) \text{ and } y(t) \in S(y,x)(t),$$

for all $t \in J$, and (x,y) is a solution of (5.5).

We'll now demonstrate the section's principal result.

Theorem 5.5. *Assume that the Assumption 5.1 is satisfied for the system* (5.3)-(5.4). *Assume that in* $X \times X$ *there is* (u_0,v_0) *and two measurable selections* $f_{u_0,v_0} : J \to \mathbb{R}$ *of* $F(., u_0(.), v_0(.))$ *and* $f_{v_0,u_0} : J \to \mathbb{R}$ *of* $F(., v_0(.)u_0(.))$ *such that*

$$u_0(t) \leq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{u_0,v_0}(\nu)d\nu$$

and

$$v_0(t) \geq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{v_0,u_0}(\nu)d\nu, \quad t \in J.$$

The integral system (5.5) *then has a solution.*

Proof. We'll prove that the integral system (5.5) has a solution by demonstrating that the operator $S : X \times X \to P(X)$ has a coupled fixed point in $X \times X$. To do so, we check that S fulfills the Theorem 5.2 hypotheses.

1. $S(x,y)$ is nonempty, for each $(x,y) \in X \times X$. Indeed, by the first assumption via Kuratowski and Ryll-Nardzewski selection theorem [15], there exist a measurable selection f for F. Then, $S_{F(.,x(.),y(.))}$ is nonempty. By the integrably bounded property of F in the second assumption we get that f is integrable. Hence $S_{F(.,x(.),y(.))} \subset L^1(J,\mathbb{R})$. Thus S is well defined.

2. S has closed values. Indeed, let $x,y \in X$ and let (s_n) be a sequence in $S(x,y)$ with $s_n \to s$ in (X, d). We have to show that $s \in S(x,y)$, i.e., there exists $f_{xy} \in S_{F(.,x(.),y(.))}$ such that
$$s(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{xy}(\nu)d\nu.$$
Since $s_n \in S(x,y)$ there exists $f_{xy}^{(n)} \in S_{F(.,x(.),l(.))}$ such that
$$s_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{xy}^{(n)}(\nu)d\nu, \text{ for} n \in \mathbb{N}, t \in J.$$
Since $(s_n) \to s$ in X, for all t, we have $(s_n(t)) \to s(t)$. Since
$$s_n(t) \in x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} F(\nu,x(\nu),y(\nu))(\nu)d\nu, t \in J,$$

and F has a convex images and is integrably bounded, then, by Theorem 8.6.4 in [15] we get the set $x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1}F(\nu,x(\nu),y(\nu))(\nu)d\nu$ is closed. Then

$$s(t) \in x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1}F(\nu,x(\nu),y(\nu))(\nu)d\nu.$$

Thus, there exists $f_{xy} \in S_{F(.,x(.),y(.))}$ such that

$$s(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1}f_{xy}(\nu)d\nu.$$

3. S has compact values. Indeed, since F is closed and integrably bounded, then by Theorem 4 in [16] we get that S has compact values.

4. We will show that

$$H_M\left(S(x,y),S(u,v),\varphi(t)\right) \geq M\left(x,u,st\right)^{\frac{1}{2}} * M\left(y,v,st\right)^{\frac{1}{2}},$$

for all $((x,y),(u,v)) \in E(G)$, such that $\varphi(t) = \frac{1}{2}t$ and $a * b = ab$.

We will prove first that for each $w \in S(x,y)$ there exists $z \in S(u,v)$ such that

$$M(w,z,\varphi(t)) \geq M\left(x,u,st\right)^{\frac{1}{2}} * M\left(y,v,st\right)^{\frac{1}{2}}.$$

By the fourth assumption, we have

$$H_M\left(F(t,x,y),F(t,u,v),t\right) \leq \frac{1}{8}\left(|x-u| + |y-v|\right);$$

for each $t \in J$, $x,y,u,v \in \mathbb{R}$ with $((x,y),(u,v)) \in E(G)$.

If $w \in S(x,y)$, then there exists $f_{xy} \in S_{F(.,x(.),y(.))}$ such that

$$w(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1}f_{xy}(\nu)d\nu, \text{ for } t \in J.$$

Since $f_{xy}(t) \in F(t,x(t),y(t))$ for $t \in J$, we can find $b \in F(t,u(t),v(t))$ such that

$$|f_{xy}(t) - b| \leq \frac{1}{8}\left(|x(t)-u(t)| + |y(t)-v(t)|\right).$$

Thus, if we define the multi-valued operator $A(t) := F(t,u(t),v(t)) \cap B(t)$, where $B(t) := \{b, \quad |f_{xy}(t) - b| \leq \frac{1}{8}\left(|x(t)-u(t)| + |y(t)-v(t)|\right)\}$, then $A(t)$ is nonempty for $t \in J$ and A is measurable (as an intersection of two measurable multi-valued operators). Thus, A has measurable selections and let $f_{uv}(t) \in A(t)$, for $t \in J$. Hence, for $t \in J$, we have $f_{uv}(t) \in F(t, u(t), v(t))$ and

$$|f_{xy}(t) - f_{uv}(t)| \leq \frac{1}{8}\left(|x(t) - u(t)| + |y(t) - v(t)|\right).$$

Define now

$$z(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{uv}(\nu)d\nu, \text{ for } t \in J.$$

Obvious, $z \in S(u, v)$.

Thus

$$\begin{aligned}
\frac{|w(t) - z(t)|^2}{\frac{t}{2}} &= \frac{2}{t}\left|\frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{el}(\nu)d\nu\right. \\
&\left.-\frac{1}{\Gamma(\alpha)} \int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1} f_{uv}(\nu)d\nu\right|^2 \\
&= \frac{2}{t}\frac{1}{\Gamma(\alpha)^2}\left|\int_0^t \beta\nu^{\beta-1}(t-\nu)^{\alpha-1}[f_{xy}(\nu) - f_{uv}(\nu)]d\nu\right|^2 \\
&\leq \frac{2}{t}\frac{\beta^2 L^{2\beta+2\alpha-3}}{\Gamma(\alpha)^2}\int_0^t |f_{xy}(\nu) - f_{uv}(\nu)|^2 d\nu \\
&\leq \frac{2}{t}\frac{\beta^2 L^{2\beta+2\alpha-3}}{\Gamma(\alpha)^2}\int_0^t |x(\nu) - u(\nu)|^2 + |y(\nu) - v(\nu)|^2 d\nu,
\end{aligned}$$

then

$$\begin{aligned}
\sup_{t\in J}\frac{|w(t) - z(t)|^2}{\frac{t}{2}} &\leq \frac{1}{2}\frac{8\beta^2 L^{2\beta+2\alpha-2}}{\Gamma(\alpha)^2}\sup_{t\in J}\left(\frac{|x(t) - u(t)|^2}{2t} + \frac{|y(t) - v(t)|^2}{2t}\right) \\
&\leq \frac{1}{2}\sup_{t\in J}\left(\frac{|x(t) - u(t)|^2}{2t} + \frac{|y(t) - v(t)|^2}{2t}\right).
\end{aligned}$$

Hence, we get

$$M(w, z, \varphi(t)) \geq M\left(x, u, st\right)^{\frac{1}{2}} * M\left(y, v, st\right)^{\frac{1}{2}}.$$

By interchanging the roles of x and y, respectively u and v, we get for all $((x,y),(u,v)) \in E(G)$

$$H_M\left(S(x,y), S(u,v), \varphi(t)\right) \geq M\left(x, u, st\right)^{\frac{1}{2}} * M\left(y, v, st\right)^{\frac{1}{2}},$$

Now, by hypotheses we can conclude that

$$((u_0,v_0),(u_1,v_1)) \in E(G),$$

such that $(u_1,v_1) \in S(u_0,v_0) \times S(v_0,u_0)$.

Because the multi-valued S is continuous and the triple (X,d,G) possesses the property 1.1, which indicates that all hypotheses of Theorem 5.1 and Theorem 5.2 are fulfilled, S has a couple fixed point in $X \times X$.

□

Conclusion

In this dissertation, we attempted to present the coupled fixed point theory in two different spaces: $b-$metric spaces and $b-$fuzzy metric spaces endowed with directed graph, We based our work on selected contractive properties and applied them to obtain the sufficient conditions for the existence and uniqueness of solutions different type of differential equations.

Based on the present research study, we will try to study coupled common fixed point and coupled coincidence point theorems for two mappings, or more in different abstract spaces. We also will study different kinds of contraction and expansion mappings like weakly commuting mappings, compatible mappings, etc. Moreover, we will apply them in various fields of mathematics and other technical sciences.

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ملخص

ركزنا في هذه الاطروحة على دراسة نظرية النقطة الثابتة المزدوجة المفيدة للغاية في التحليل غير الخطي من خلال مساهمتها في حل الكثير من المشكلات في مجالات تطبيقية مختلفة. اهتممنا بإثبات وجود و وحدانية النقاط الثابتة المزدوجة للتقلصات أحادية القيمة ومتعددة القيم المحددة على الفضاءات $b-$متريّة و متريّة $b-$ضبابية مزودة برسوم بيانيّة موجّهة. و لدعم قابلية تطبيق نتائجنا، قمنا بتطبيقها على أنواع مختلفة من المعادلات التفاضلية.

الكلمات المفتاحية: رسم بياني موجه ، نقطة ثابتة مزدوجة ، فضاءات $b-$ متريّة، فضاءات متريّة $b-$ضبابية، معادلات تفاضلية، مختلطة $G-$ رتيبة.

Abstract

The aim of this dissertation focuses on the study of coupled fixed point theory which is very useful in nonlinear analysis due to its contribution to solving many problems in different applications. We were interested in proving the existence and uniqueness of coupled fixed points for single-valued and multi-valued contractions defined on $b-$metric spaces and $b-$fuzzy metric spaces endowed with directed graphs. Additionally, to support the presented results, we applied them to different types of differential equations.

Key words : Directed Graph, Coupled Fixed point, b-metric space, $b-$fuzzy Metric Space, Differential Equations, Mixed $G-$monotone.

Résumé

Dans cette thése on a fait une contribution du point fixe couplé dans des différents espaces, qui est très utile dans l'analyse non-linéaire par sa contribution à résoudre beaucoup de problèmes dans des différentes applications. Nous nous sommes intéressés à prouver l'existence et l'unicité des points fixes couplés pour des contractions univoque et multivoque, sur les espaces $b-$métriques et $b-$fuzzy métriques équipés par des graphes orientés. Et pour soutenir nos résultats, nous les avons appliqués sur différents types d'équations différentielles.

Mots clés : Graphe orienté, point fixe couplé, espace b-métrique, espace $b-$fuzzy métrique, équations differentielles, mixte $G-$monotone.