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Thème

Quelques résultats de point fixe sur les espaces généralisés

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devant le jury composé de:

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Dedicace

It is with deep gratitude and sincere words that I dedicate this modest graduation work to my dear parents; who sacrificed their lives for my success and I have lit the way with their wise advice.

I hope that one day I can give them back some of what they have done for me may god give them happiness and long life. To my dear husband for his moral and material support.

To my brothers and sisters, my uncles and my aunts.

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Contents

Symbol Description

The most frequently used notations, symbols, and abbreviations are listed below

$\mathbb{R}_+$	Set of all nonnegative real numbers.
$\mathbb{R}^n$	Set of all $n-$ tuples $x = (x_1, x_2, \ldots, x_n)$.
$\mathbb{R}^n_+$	Set of all $x \in \mathbb{R}^n$ with $x_i \geq 0$ for all i.
$x \leq y$	Natural order relation in $\mathbb{R}^n$: $x_i \leq y_i$ for all i.
$x < y$	Strict order relation in $\mathbb{R}^n$: $x_i < y_i$ for all i.
$x \leq a$	For $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^+$: $x_i \leq a$ for all i.
$d(x, y)$	The distance between x and y with x,$y \in \mathbb{R}^n_+$.
(X, d)	Generalized metric space X.
$(X, \|.\|)$	Generalized Banach space X.
$B(x_0, r)$	The open ball centered in x_0 with radius r ,$\{x \in X \| d(x_0, x) < r\}$.
$\tilde{B}(x_0, r)$	The closed ball centered in x_0 with radius r ,$\{x \in X \| d(x_0, x) \leq r\}$.
$Fix(f)$	The set of all fixed points of f.
$\mathcal{M}_{n\times n}(\mathbb{R}_+)$	The set of all $n \times n$ matrices with positive elements.
$\overline{\tilde{B}_q(x_0, r)}^p$	The closure of $\tilde{B}_q(x_0, r)$ in $(X,\mathcal{P})$.where $\tilde{B}_q(x_0, r) = \{x \in X : q_\beta(x_0, x) \leq r_\beta, \forall \beta \in \Gamma\}$.
$(X,\mathcal{P})$	Generalized gauge space.
Γ,Λ	Directed sets

Introduction

Fixed point theory is a powerful and fruitful and an important tool in modern mathematics due to its applications to ensure the existence of solutions for mathematical problems under certain conditions. The knowledge of the existence of fixed points has prevalent applications in many disciplines, including mathematical analysis and topology, physics, biology, chemistry, engineering and other sciences ([3],[5],[18],[4]).

The theory of fixed points deals with the conditions which guarantee the existence of points x of a set X which solve an operator equation $x = Nx$, where N is a transformation defined on a set X. The solution set of such a problem can be empty, a finite set, a countable or uncountable infinite set.

The last century was the golden age of this theory; indeed, in 1922, S. Banach proved a theorem which focused on the existence and uniqueness of a fixed point in a complete metric space[20]. This theorem later was known as Banach's contraction principle and was extended in several directions.

One of this research directions involved the so-called generalized contractions mappings. The most influential works is [15] of Perov 1964 that show Banach Contraction Principle in generalized complete metric spaces by using the concept of M-contraction map where M is matrix converge to zero[22].

The content of the memoir is organized in three chapters. Every chapter contains a number of section of theorems and applications. In chapter 1, We define the generalized metric spaces in the sense of Perov and we give some definition and properties in this space. After that, we show the definition of gauge spaces and generalized gauge spaces.

In chapter 2, we present Perov's fixed point theory and some extension of classical Banach contraction principle for contractive maps on spaces endowed with vector-valued metric spaces. Next, with this fixed point results, we try the proof the existence and uniqueness the following system of differential equation with impulse effects.

$$\begin{gathered} x'(t) = F_1(t, x(t), y(t)), \quad y'(t) = F_2(t, x(t), y(t)), \text{ a.e. } t \in [0,1] \\ x\left(\tau^+\right) - x\left(\tau^-\right) = I_1(x(\tau), y(\tau)), \quad y\left(\tau^+\right) - y\left(\tau^-\right) = I_2(x(\tau), y(\tau)) \\ x(0) = x_0, \quad y(0) = y_0 \end{gathered} \tag{1}$$

where $0 < \tau < 1, i = 1; 2$; J $:= [0; 1]$; $F_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a function, $I_1; I_2 \in C(\mathbb{R} \times \mathbb{R}; \mathbb{R})$: The notations $x(\tau^+) = \lim_{h \to 0^+} x(\tau + h)$ and $x(\tau^-) = \lim_{h \to 0^+} x(\tau - h)$ stand for the right and the left limits of the function y at $t = \tau$, respectively.

In the last chapter, we discuss Perov type fixed point theorems for contractive mappings in Gheorghiu's sense on spaces endowed with a family of vector-valued pseudo-metrics. Applications to systems of integral equations are given to illustrate the theory. The examples also prove the advantage of using vector-valued pseudo-metrics and matrices that are convergent to zero, for the study of systems of equations.

This system is reduced to a fixed point problem

$$\begin{cases} A(x,y) = x, \\ B(x,y) = y. \end{cases} \tag{2}$$

It is obvious that system (2) can be viewed as a fixed point problem,

$$T(u) = u. \tag{3}$$

in the space X^2, where $u = (x, y)$ and $T = (A,B)$. Therefore, we may think to apply to (3) in X^2 endowed with the gauge structure induced by that of X. Finally, we show that T verify some condition and we present the existence of a solution of the following system :

$$\begin{cases} x(t) = \displaystyle\int_{t-\tau_1}^{t} f\left(s, x\left(\sigma_1(s)\right), y\left(\sigma_2(s)\right)\right) \mathrm{ds}, \\ y(t) = \displaystyle\int_{t-\tau_2}^{t} g\left(s, x\left(\sigma_1(s)\right), y\left(\sigma_2(s)\right)\right) \mathrm{ds}. \end{cases} \tag{4}$$

which present a mathematical model for the spread of two interacted infectious diseases with contact rates that vary seasonally. In these equations $x(t), y(t)$ represent the proportion of infectives in a population at time t, and $\tau_1,\ \tau_2 \in \mathbb{R}$, with $f,\ g,\ \sigma_1,\ \sigma_2$ are given and verified some assumption.

Chapter 1

Basic Concepts

This chapter provides the basic of notions and properties of generalized metric spaces ,matrix convergence and generalized gauge space which will be used throughout the memoir.

1.1 Generalized Metric and Banach Spaces

1.1.1 Generalized Metric Spaces

In this section we define generalized metric space (or vector metric spaces) and prove some properties. if $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also $|x| = (|x_1|, ..., |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), ..., \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each i = 1, . . . , n. For $x \in \mathbb{R}^n, (x)_i = x_i, i = 1, \dots, n$.

Definition 1.1. [14] Let X be a nonempty set. By a generalized metric on X (or vector-valued metric) we mean a map $d : X \times X \longrightarrow \mathbb{R}^n$ with the following properties:

1. $d(u, v) \geq 0$, for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$.

2. $d(u, v) = d(v, u)$ for all $u, v \in X$.

3. $d(u, v) \leq d(u,w) + d(w, v)$ for all $u, v,w \in X$.

Note that for any $i \in 1, \ldots, n; (d(u,v))_i = d_i(u,v)$ is a metric space in X.

We call the pair (X, d) a generalized metric space. For $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0,r) = \{x \in X : d(x_0,x) < r\}$$

the open ball centered at x_0 with radius r and

$$\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \leq r\}$$

the closed ball centered at x_0 with radius $r = (r_1, \ldots, r_n) > 0$; $r_i > 0$, $i = 1, \ldots, n$.

Definition 1.2. [14] Let (X, d) be a generalized metric space. A subset $A \subseteq X$ is called open if, for any $x_0 \in A$, there exists $r \in \mathbb{R}^n_+$ with $r > 0$ such that $B(x_0, r) \subseteq A$.

Any open ball is an open set and the collection of all open balls of X generates the generalized metric topology on X.

Definition 1.3. [14] Let (X, d) be a generalized metric space

A. A sequence (x_p) in X converges (or $\mathbb{R}^n_+$ converges) to some $x \in X$, if for every $\epsilon \in \mathbb{R}^n_+$, $\epsilon > 0$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that for each $d(x_p, x) \leq \epsilon$ for all $p \geq p_0(\epsilon)$.

B. A sequence (x_p) is called a Cauchy sequence if for every $\epsilon \in \mathbb{R}^n_+$, $\epsilon > 0$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that for each $d(x_p, x_q) \leq \epsilon$ for all $p, q \geq p_0(\epsilon)$.

C. A generalized metric space X is called complete if each Cauchy sequence in X converges to a limit in X.

D. A subset Y of a generalized metric space X is said to be closed whenever $(x_p) \subseteq Y$ and $x_p \to x$, as $p \to \infty$ imply $x \in Y$.

Using the above definitions, we have the following properties: If $x_p \to x$ as $p \to \infty$, then:

i) The limit x is unique.

ii) Every subsequence of (x_p) converges to x.

iii) If also $y_p \to y$ as $p \to \infty$, then

$$d(x_p, y_p) \to d(x, y) \text{ as } p \to \infty$$

.

Theorem 1.1. *[14] For the generalized metric space (X, d) the following hold:*

a) Every convergent sequence is an Cauchy sequence,

b) Every Cauchy sequence is bounded,

c) If a Cauchy sequence (x_p) has a subsequence (x_{p_k}) such that

$$x_{p_k} \to x \text{ as } p_k \to \infty$$

then

$$x_p \to x \text{ as } p \to \infty$$

.

Proof. a) Let$(x_p)_{p\in\mathbb{N}}$ be a convergent sequence in X . The for every $\epsilon \in \mathbb{R}^n_+$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that $d(x_p, x) \leq \epsilon/2$ for all $p \geq p_0(\epsilon)$. Then for every $p, q \geq p_0(\epsilon)$ we have $d(x_p, x_q) \leq d(x_p, x) + d(x_q, x) \Rightarrow d(x_p, x_q) \leq \epsilon$. Hence $(x_p)_p \in \mathbb{N}$ is a Cauchy sequence in X.

b) Let $(x_p)_{p\in\mathbb{N}}$ be a Cauchy sequence. Fix $\epsilon \in \mathbb{R}^n_+$. There exists $p_0(\epsilon) \in \mathbb{N}$ such that $d(x_p, x_q) \leq \epsilon$, for all $p, q \geq p_0(\epsilon)$ .Hence for each $p \in_n n$, we get

$$x_p \in B(x_{p_0(\epsilon)}, \epsilon + r), r = \max_{1\leq i,j\leq p_0(\epsilon)-1} d(x_i,x_j),$$

this implies that $(x_p)_p \in \mathbb{N}$ bounded in X.

c) Let $(x_p)_p \in \mathbb{N}$ be a Cauchy sequence and let $(x_{p_k})_{p_k} \in \mathbb{N}$ be a subsequence of $(x_p)_p \in \mathbb{N}$ such that $\lim_{p_k\to\infty} x_{p_k} = x$. The for every $\epsilon \in \mathbb{R}^n_+$ there exist $p_*(\epsilon), q_*(\epsilon) \in \mathbb{N}$ such that

$$d(x_p, x_q) \leq \frac{\epsilon}{2} \text{ for all } p, q \geq p_*(\epsilon)$$

and

$$d(x_{p_k}, x) \leq \frac{\epsilon}{2} \text{ for all } p_k \geq q_*(\epsilon)$$

then

$$d(x_p, x) \leq d(x_p, x_{p_k}) + d(x_{p_k}, x) \leq \epsilon \text{ for all } p \geq \max(q_*(\epsilon), p_*(\epsilon)).$$

Hence

$$x_p \to x \text{ as } p \to \infty$$

. □

Definition 1.4. [14] Let (X, d) and (Y,ρ) be generalized metric spaces, and let $x \in X$. A function $f : X \to Y$ is said to be continuous (or topologically continuous) at x if for every $\epsilon \in \mathbb{R}^n_+$, $\epsilon > 0$ there exists some $\delta(\epsilon) \in \mathbb{R}^n_+$, $\delta(\epsilon) > 0$ such that

$$\rho(f(x), f(y)) < \epsilon$$

whenever $x, y \in X$ and $d(x, y) < \delta(\epsilon)$.

The function f is said to be topologically continuous if it is topologically continuous at each point of X.

Definition 1.5. [14] Let (X, d) be a generalized metric space. A subset C of X is called compact if every open cover of C has a finite subcover. A subset C of X is sequentially compact if, every sequence in C contains a convergent subsequence with limit in C.

Definition 1.6. A topological space X is sequentizlly compact if every sequence in X has a convergent subsequence.

Let X be a topological space and $A \subseteq X$. I have seen two definitions for A to be sequentially relatively compact in X:

1. the closure $\bar{A}$ of A in X is sequentially compact, which means that every sequence in $\bar{A}$ has a convergent subsequence (with limit in $\bar{A}$).

2. every sequence in A has a convergent subsequence with limit in $\bar{A}$.

Definition 1.7. [14] subset C of X is totally bounded if, for each $\epsilon \in \mathbb{R}^n_+$ with $\epsilon > 0$, there exists a finite number of elements $x_1, x_2, ..., x_p \in X$ such that

$$C \subseteq \bigcup_{i=1}^{p} B(x_i, \epsilon).$$

The set $x_1, ..., x_p$ is called a finite ϵ-net.

Theorem 1.2. *[14] If C is a subset of X, then the following affirmations hold:*

i) *C is compact if and only if C is sequentially compact if and only if, C is closed and totally bounded;*

ii) *C relatively compact, if and only if, C sequentially relatively compact, if and only if, C totally bounded.*

Definition 1.8. [14] Let (X, d) be a generalized metric space. If $A \subset X$ is a nonempty set, then the function

$$\sigma(A) = \sup \{d(x, y) : x; y \in A\}$$

is called the diameter of A. If $\sigma(A) < \infty$, then A is called a bounded set.

Theorem 1.3. *[14] Let (X, d) be a generalized metric space. For any compact set $A \subset X$ and for any closed set $B \subset X$ that is disjoint from A, there exist continuous functions $f : X \to [0, 1]$, $g : X \to [0, 1] \times [0, 1] \times \cdots \times [0,1] := [0, 1]^n$ such that*

1. *$f(x) = 0$ for all $x \in B$,*

2. *$f(x) = 1$ for all $x \in A$,*

3. *$g(x) = (1, \cdots, 1)$ for all $x \in B$,*

4. *$g(x) = (0, \cdots, 0)$ for all $x \in A$.*

Proof. Note that $d_i(x,B) = 0$ for any $x \in B$ and $d_i(x,A) = 0$ for any $x \in A$ and $d_i(x,A) > 0$ for any $x \in B$. Thus to get 1. and 2. Let $f : X \to [0,1]$ be defined by :

$$f(x) = \frac{\sum_{i=1}^{n} d_i(x,B)}{\sum_{i=1}^{n} d_i(x,A) + \sum_{i=1}^{n} d_i(x,B)}, x \in X$$

To prove that f is continuous, let $(x_m)_{m \in \mathbb{N}}$ be a sequence convergent to $x \in X$. Then

$$\begin{aligned}|f(x_m)-f(x)| &= \left|\frac{\sum_{i=1}^n d_i(x_m,B)}{\sum_{i=1}^n d_i(x_m,A)+\sum_{i=1}^n d_i(x_m,B)} - \frac{\sum_{i=1}^n d_i(x,B)}{\sum_{i=1}^n d_i(x,A)+\sum_{i=1}^n d_i(x,B)}\right| \\ &= \left|\frac{\sum_{i=1}^n d_i(x_m,B)\sum_{i=1}^n d_i(x,A) - \sum_{i=1}^n d_i(x_m,A)\sum_{i=1}^n d_i(x,B)}{(\sum_{i=1}^n d_i(x,A)+\sum_{i=1}^n d_i(x,B))(\sum_{i=1}^n d_i(x_m,A)+\sum_{i=1}^n d_i(x_m,B))}\right| \\ &\le \frac{\sum_{i=1}^n d_i(x,A)\sum_{i=1}^n |d_i(x_m,B)-d_i(x,B)|}{(\sum_{i=1}^n d_i(x,A)+\sum_{i=1}^n d_i(x,B))(\sum_{i=1}^n d_i(x_m,A)+\sum_{i=1}^n d_i(x_m,B))} \\ &+ \frac{\sum_{i=1}^n d_i(x,B)\sum_{i=1}^n |d_i(x_m,A)-d_i(x,A)|}{(\sum_{i=1}^n d_i(x,A)+\sum_{i=1}^n d_i(x,B))(\sum_{i=1}^n d_i(x_m,A)+\sum_{i=1}^n d_i(x_m,B))}\end{aligned}$$

Since for each $i = 1,\dots,m$, we have

$$|d_i(x_m,B)-d_i(x,B)| \to 0, |d_i(x_m,A)-d_i(x,A)| \to 0 \text{ as } m \to \infty$$

Therefore, as $m \to \infty$,

$$\frac{\sum_{i=1}^n d_i(x,A)\sum_{i=1}^n |d_i(x_m,B)-d_i(x,B)|}{(\sum_{i=1}^n d_i(x,A)+\sum_{i=1}^n d_i(x,B))(\sum_{i=1}^n d_i(x_m,A)+\sum_{i=1}^n d_i(x_m,B))} \to 0$$

and

$$\frac{\sum_{i=1}^n d_i(x,B)\sum_{i=1}^n |d_i(x_m,A)-d_i(x,A)|}{(\sum_{i=1}^n d_i(x,A)+\sum_{i=1}^n d_i(x,B))(\sum_{i=1}^n d_i(x_m,A)+\sum_{i=1}^n d_i(x_m,B))} \to 0$$

thus, we get

$$|f(x_m)-f(x)| \to 0 \text{ as } m \to \infty$$

We can easily prove that the following function $g : X \to [0,1]^n$ defined by

$$g(x) = \begin{pmatrix} \frac{d_1(x,A)}{d_1(x,A)+d_1(x,B)} \\ \cdots \\ \frac{d_n(x,A)}{d_n(x,A)+d_n(x,B)} \end{pmatrix}$$

is continuous and satisfies 3. and 4. □

Let (X,d) be a generalized metric space. We define the following metric spaces: Let $X_i = X, i = 1,\cdots,n$. Consider $\prod_{i=1}^n X_i$ with $\bar{d}$ defined by

$$\bar{d}((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^n d_i(x_i,y_i)$$

. The diagonal space of $\prod_{i=1}^n X_i$ is defined by

$$\tilde{X} = \{(x,\dots,x) \in \prod_{i=1}^n X_i : x \in X, i = 1,\cdots,n\}$$

which is a metric space with the following distance

$$d_*((x,\dots,x),(y,\dots,y)) = \sum_{i=1}^{n} d_i(x,y), \text{ for each x,y} \in X.$$

It is clear that $\tilde{X}$ is closed set in $\prod_{i=1}^{n} X_i$. Intuitively, X and $\tilde{X}$ are the same. This is shown in the following result.

Lemma 1.1. *[14] Let (X,d) be a generalized metric space. Then there exists a homeomorphism map $h : X \longrightarrow \tilde{X}$.*

Proof. Consider $h : X \to \tilde{X}$ defined by $h(x) = (x,\cdots,x)$ for all $x \in X$. Obviously h is bijective.

- To prove that h is a continuous map, let $x,y \in X$. Thus

  $$d_*(h(x),h(y)) \leq \sum_{i=1}^{n} d_i(x,y).$$

  For $\epsilon > 0$ we take $\delta = (\frac{\epsilon}{n},\cdots,\frac{\epsilon}{n})$, let $x_0 \in X$ be fixed and $B(x_0,\delta) = \{x \in X : d(x_0,x) < \delta\}$. Then for every $x \in B(x_0,\delta)$ we have

  $$d_*(h(x_0),h(x)) \leq \epsilon$$

  .

- Now , $h^{-1} : \tilde{X} \to X$ is a map defined by

  $$h^{-1}(x,\dots,x) = x, \ (x,\dots,x) \in \tilde{X}.$$

  To show that h^{-1} is continuous, let $(x,\dots,x),(y,\dots,y) \in \tilde{X}$. Then

  $$d(h^{-1}(x,\dots,x),h^{-1}(y,\dots,y)) = d(x,y).$$

  Let $\epsilon = (\epsilon_1,\dots,\epsilon_n) > 0$. We take $\delta = \frac{1}{n}(\min_{1\leq i\leq n} \epsilon_i)$ and we fix $(x_0,\dots,x_0) \in \tilde{X}$. Set

  $$B((x_0,\dots,x_0),\delta) = \{(x,\dots,x) \in \tilde{X} : d_*((x_0,\dots,x_0),(x,\dots,x)) < \delta\}$$

  . For $(x,\dots,x) \in B((x_0,\dots,x_0),\delta)$ we have

  $$d_*((x_0,\dots,x_0),(x,\dots,x)) < \delta \Rightarrow \sum_{i=1}^{n} d_i(x_0,x) < \frac{1}{n}(\min_{1\leq i\leq n} \epsilon_i).$$

Then

$$d_i(x_0, x) < \frac{1}{n}(\min_{1\leq i\leq n} \epsilon_i), i = 1, \ldots, n \Rightarrow d(x_0, x) < \epsilon.$$

Hence h^{-1} is continuous. □

Definition 1.9. A paracompact space is a topological space is which every open cover has an open refinement that is locally finite.

Every compact space is paracompact.

Every metric space is paracompact.

Theorem 1.4. *[14] Every generalized metric space is paracompact.*

Proof. Let X be a generalized metric space. By (1.1) there exists $\tilde{X}$, a metric space which is homeomorphic to X. Since every metric space is paracompact hence X is paracompact. □

Theorem 1.5. *[14] Let (X, d) be a generalized metric space. To any locally finite open covering $(U_i)_{i\in I}$ of X, we can associate a locally Lipschitzian partition of unity subordinated to it.*

Proof. From (1.4), X is paracompact, then there exists a family of locally finite open set, let us write,

$$\mathcal{V} = \{V_i | i \in I_*\}$$

covering of X such that

$$\overline{V}_i \subset U_i \text{ for every } i \in I_*.$$

Let us define for any $i \in I_*$ the function $f_i : X \to \mathbb{R}_+$ by

$$f_i(x) = \sum_{j=1}^{n} d_j(x, X \backslash V_i)$$

For each $x, y \in X$ we have

$$\left| \sum_{j=1}^{n} d_j(x, X\backslash V_i) - \sum_{j=1}^{n} d_j(y, X\backslash V_i) \right| \leq \sum_{j=1}^{n} d_j(x, y) \text{ for each } x, y \in X.$$

hence

$$\left|\sum_{j=1}^{n} d_j(x,X\backslash V_i) - \sum_{j=1}^{n} d_j(y,X\backslash V_i)\right| \leq Ad(x,y) \text{ for each } x,y \in X$$

where $A = (1,\dots,1) \in \mathcal{M}_{1\times n}(\mathbb{R}^+)$. Then for every $i \in I_*$, f_i is Lipschitzian and verifies

$$supp(f_i) = \overline{V_i} \subset U_i.$$

Let us introduce for any $i \in I_*$ the following function $\psi_i : X \to [0,1]$ defined by

$$\psi_i(x) = \frac{f_i(x)}{\sum_{i\in I_*} f_i(x)} \text{ for all } x \in X.$$

a) Firstly, we prove that ψ_i is locally Lipschitz on X. Indeed, let $x \in X$, then there exists a neighborhood V_x of x which meets only a finite number of $\{\overline{V_i}|i \in I_*\}$. That is there is $\{i_1,\dots,i_m\}$ such that

$$V_x \cap V_i = \emptyset \text{ for each } i \in I_*\backslash\{i_1,\dots,i_p\} \Rightarrow \sum_{i\in I_*} f_i(y) = \sum_{k=1}^{p} f_{i_k}(y) > 0, y \in V_x.$$

By the continuity of $\sum_{k=1}^{p} f_{i_k}$there exists a neighborhood $W_x \subset V_x$ of x and m, $\bar{M} > 0$ such that

$$m \leq \sum_{i\in I_*} f_i(y) = \sum_{k=1}^{p} f_{i_k}(y) \leq \bar{M} \text{ for any } y \in W_x.$$

Thus for $y, z \in W_x$, we get

$$\begin{aligned}
|\psi_i(z) - \psi_i(y)| &= \left|\frac{f_i(y)}{\sum_{i\in I_*} f_i(y)} - \frac{f_i(z)}{\sum_{i\in I_*} f_i(z)}\right| \\
&= \left|\frac{\sum_{k=1}^{p} f_{i_k}(z)f_i(y) - \sum_{k=1}^{p} f_{i_k}(y)f_i(z)}{\sum_{k=1}^{p} f_{i_k}(y)\sum_{k=1}^{p} f_{i_k}(z)}\right| \\
&\leq \frac{1}{m^2}\left|\sum_{k=1}^{p} f_{i_k}(z)f_i(y) - \sum_{k=1}^{p} f_{i_k}(y)f_i(z)\right| \\
&\leq \frac{1}{m^2}\sum_{k=1}^{p} |f_{i_k}(z)f_i(y) - f_{i_k}(y)f_i(z)| \\
&\leq \frac{1}{m^2}\sum_{k=1}^{p} |f_{i_k}(z) - f_{i_k}(y)|\,|f_i(y)| + \sum_{k=1}^{p} |f_{i_k}(y)|\,|f_i(y) - f_i(z)|\,.
\end{aligned}$$

Therefore

$$|\psi_i(z) - \psi_i(y)| \leq \frac{2\bar{M}p}{m^2}Ad(y,z) \text{ for any } y, z \in W_x$$

b) Now, we show that ψ_i is continuous. Let $x_0 \in X$. then there exists a neighborhood V_x of x which meets only a finite number of $\{\overline{V_i} | i \in I_*\}$. That is there is $\{i_1, \ldots, i_m\}$ such that

$$V_{x_0} \cap V_i = \emptyset \text{ for each } i \in I_* \backslash \{i_1, \ldots, i_p\}.$$

This implies that , for every $i \in I_* \backslash \{i_1, \ldots, i_p\}$ we have

$$V_{x0} \subset X \backslash V_i \Rightarrow f_i(V_{x_0}) = 0,$$

and

$$V_{x_0} \cap supp(f_i) = \emptyset \text{ for each } i \in I_* \backslash \{i_1, \ldots, i_p\}.$$

From a) we obtain

$$\sum_{i \in I_*} f_i(x) = \sum_{i=1}^{p} f_i(x) \text{ for each } x \in V_{x_0}.$$

Therefore,

$$\psi_i(x) = \frac{f_i(x)}{\sum_{k=1}^{p} f_{i_k}(x)} \text{ for every } x \in V_{x_0}$$

It is clear that $\sum_{k=1}^{p} f_{i_k}(x_0) \neq 0$, since for each $i \in I_*$, f_i is continuous function . Hence ψ_i is continuous on X.

□

Definition 1.10. [14] Let(X, d) be a generalized metric space. A subset Y of X is called dense whenever every $B(x, r) \cap Y \neq \emptyset$ for each $x \in X$ and $r \in \mathbb{R}^n_+, r = (r_1, ..., r_n), r_i > 0, i = 1, \ldots, n$.

We already have the following result

Corollary 1.1. *[14] Let Y be a subset of a generalized metric space (X, d). Then, Y is dense if and only if for every $x \in X$ there exists a sequence $(x_p)_{p \in \mathbb{N}}$ in Y satisfying*

$$x_p \to x \text{ as } p \to \infty.$$

Theorem 1.6. *[14](Cantor's intersection theorem). Let (X, d) be a cmplete generalized metric space. Let $(F_p)_{p \in \mathbb{N}}$ be a decreasing sequence of nonempty closed subsets of X such that*

$$\delta_{p \to \infty}(F_p) = 0 \in \mathbb{R}^n_+.$$

Then $\cap_{p \in \mathbb{N}} F_p$ contains exactly one point.

Proof. For all $p \in \mathbb{N}$ choose $x_p \in F_p$. Since $\delta(F_p) \to 0$ as $p \to \infty$, this implies that $(x_p)_{p\in\mathbb{N}}$ is Cauchy. Hence there exists $x \in X$ such that

$$x_p \to x \text{ as } p \to \infty.$$

We show that $x \in F_p$ for every $p \in N$. If $(x_p)_{p\in\mathbb{N}}$ is finite then $x_p = x$ for infinitely many p, so that $x \in F_p$ for infinitely many p. Since $F_{p+1} \subseteq F_p$ this implies $x \in F_p$ for each $p \in \mathbb{N}$. So suppose $(x_p)_{p\in\mathbb{N}}$ is infinite. For all $m \in \mathbb{N}$, $(x_m, x_{m+1}, \ldots, x_{m+k}, \ldots)$ is a sequence in F_m converging to x. Since $(x_p)_{p\geq m}$ is infinite, this implies that $x \in \overline{F_m}$. But F_m is closed, so $x \in F_m$. Therefore $x \in \cap_{p\in\mathbb{N}} F_p$. If $\cap_{p\in\mathbb{N}} F_p$ contains two points x and y then we have

$$d(x, y) \leq \delta(F_p) \to 0, \text{ as } p \to \infty \Rightarrow d(x, y) = 0.$$

Hence

$$\cap_{p\in\mathbb{N}} F_p = \{x\}$$

□

Theorem 1.7. *[14] The following are equivalent for a generalized metric space (X, d)*

1. *X is a complete space.*

2. *For any descending sequence $\{F_p\}$ of closed bounded subsets of X,*

$$\lim_{p\to\infty} \delta(F_p) = 0 \in \mathbb{R}^n_+.$$

Definition 1.11. [9] A Baire space is a topological space such that every intersection of a countable collection of open dense sets in the space is also dense.

Theorem 1.8. *[14] Every complete generalized metric space is a Baire space.*

1.1.2 Generalized Banach space

Now, we recall some definitions and properties of generalized Banach space.

Definition 1.12. [14] Let X be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on X we mean a map $\|.\| : X \to \mathbb{R}^n_+$ with the following properties:

i. $\|x\| \geq 0$ for all $x \in X$; if $\|x\| = 0$ then $x = 0$

ii. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$

iii. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a generalized normed space. If the generalized metric generated by $\|.\|$ (i.e $d(x, y) = \|x - y\|$) is complete then the space $(X, \|.\|)$ is called a generalized Banach space, where

$$\|x - y\| = \begin{pmatrix} \|x - y\|_1 \\ \cdots \\ \|x - y\|_n \end{pmatrix}$$

Notice that $\|.\|$ is a generalized normed space on X if and only if $\|.\|_i \,, i = 1, \ldots, n$ are norms on X.

Definition 1.13. [14] Let X and Y be two generalized normed spaces, $K \subset X$ and let $N : K \to Y$ be an operator. Then N is said to be:

i) compact, if for any bounded subset $A \subseteq K$ we have $N(A)$ is relatively compact, i.e. $\overline{N(A)}$ is compact;

ii) completely continuous, if N is continuous and compact;

iii) with relatively compact range, if N is continuous and $N(K)$ is relatively compact, i.e. $\overline{N(K)}$ is compact.

Definition 1.14. [14] Let $(X, \|.\|)$ be a generalized Banach space and $U \subset X$ an open subset such that $0 \in U$. The function $p_U : X \to \mathbb{R}_+$ defined by

$$p_U(x) = \inf\{\alpha > 0 : x \in \alpha U\},$$

is called the Minkowski functional of U.

Lemma 1.2. *[14] Let $(X, \|.\|)$ be a generalized Banach space and $U \subset X$ an open subset such that $0 \in U$,Then*

i) If $\lambda \geq 0$, then $p_U(\lambda x) = \lambda p_U(x)$.

ii) If U is convex we have

1. $p_U(x+y) \leq p_U(x) + p_U(y)$, *for every* $x, y \in U$.
2. $\{x \in X : p_U(x) < 1\} \subset U \subset \{x \in X : p_U(x) \leq 1\}$.
3. *if U is symmetric; then* $p_U(x) = p_U(-x)$.

iii) p_U *is continuous.*

Proof. i) Let $x \in X$ be arbitrary and $\lambda \geq 0$. We have

$$\begin{aligned} p_U(\lambda x) &= \inf\{\alpha > 0 : \lambda x \in \alpha U\} \\ &= \inf\{\alpha > 0 : x \in \lambda^{-1}\alpha U\} \\ &= \inf\{\lambda\beta > 0 : x \in \beta U\} \\ &= \lambda \inf\{\beta > 0 : x \in \beta U\} \\ &= \lambda p_U(x). \end{aligned}$$

ii) 1.) Let $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$x \in \alpha_1 U \text{ and } y \in \alpha_2 U.$$

Then

$$x + y \in \alpha_1 U + \alpha_2 U \Rightarrow \frac{x+y}{\alpha_1+\alpha_2} \in \frac{\alpha_1}{\alpha_1+\alpha_2}U + \frac{\alpha_2}{\alpha_1+\alpha_2}U.$$

because $\frac{\alpha_1}{\alpha_1+\alpha_2} + \frac{\alpha_2}{\alpha_1+\alpha_2} = 1$,and U convex Hence

$$x + y \in (\alpha_1 + \alpha_2)U. \tag{1.1}$$

For every $\epsilon > 0$ there exist $\alpha_\epsilon > 0$, $\beta_\epsilon > 0$ such that

$$\alpha_1 \leq \alpha_\epsilon \leq p_U(x) + \epsilon \text{ and } \alpha_2 \leq \beta\epsilon \leq p_U(y) + \epsilon$$

. From (1.1) we have

$$p_U(x+y) \leq \alpha_1 + \alpha_2 \Rightarrow p_U(x+y) \leq \alpha_1 + \alpha_2 \leq \alpha_\epsilon + \beta_\epsilon \leq p_U(x) + p_U(y) + 2\epsilon$$

Letting $\epsilon \to 0$ we obtain

$$p_U(x+y) \leq p_U(x) + p_U(y) \text{ for every } x, y \in U.$$

2.)Let $x \in X$ such that $p_U(x) < 1$, then there exists $\alpha \in (0,1)$ such that $p_U(x) \leq \alpha < 1$ and $x \in \alpha U \Rightarrow x = \alpha 1 + (1-\alpha)0 \in U$. because U convex. Therefore

$$\{x \in X : p_U(x) < 1\} \subset U.$$

For $x \in U$ we have

$$x = \alpha x \in U, \alpha = 1 \Rightarrow p_U(x) \leq 1.$$

then

$$\{x \in X : p_U(x) < 1\} \subset U \subset \{x \in X : p_U(x) \leq 1\}.$$

iii) Since $0 \in U$ then there exists $r > 0$ such that

$$B(0,r) = \{x \in X : \|x\| < r_*\} \subset U,$$

where

$$\|x\| = \begin{pmatrix} \|x\|_1 \\ \cdots \\ \|x\|_n \end{pmatrix}, r_* = \begin{pmatrix} r \\ \ldots \\ r \end{pmatrix}$$

Given $\epsilon > 0$, then $x + \epsilon B(0, r^*)$ is a neighborhood of x. For every $y \in x + \epsilon B(0, r^*)$ we have

$$\frac{x-y}{\epsilon} \in B(0, r_*) \subset U \Rightarrow p_U(\frac{x-y}{\epsilon}) \leq 1.$$

It is clear that

$$|p_U(x) - p_U(y)| \leq p_U(x-y) = \epsilon p_U(\frac{x-y}{\epsilon}) \leq \epsilon$$

Hence p_U is continuous.

□

Remark 1.1. In generalized metric space in the sense of Perov, the notions of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

1.1.3 Matrix convergence

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see [2, 22, 19]).

Definition 1.15. [22] A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers is said to be convergent to zero if

$$M^k \to 0 \text{ as } k \to \infty$$

Lemma 1.3. *[22] Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

i) M is convergent to zero;

ii) the matrix $I - M$ is non-singular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

iii) $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$;

iv) $(I - M)$ is non-singular and $(I - M)^{-1}$ has nonnegative elements.

Proof. $i) \Rightarrow ii)$ Assume that M is convergent to zero. We show that $I - M$ is non-singular; it suffices to prove that the linear system

$$(I - M)x = 0 \tag{1.2}$$

has only the null solution $(I - M)x = 0 \Rightarrow x = 0$ Let $x \in \mathbb{C}$ be a solution of the system (1.2), then

$$(I - M)x = 0 \Rightarrow x = Mx$$

,

$$M^2x = MMx = Mx = x$$

and

$$M^3x = MM^2x = Mx = x$$

hence

$$x = Mx = M^2x = M^3x = \ldots = M^kx$$

then $k \to \infty$; $M^k \to 0$ donc $x = 0$, hence $I - M$ is non singular. Furthermore, we have

$$I-(I-M)(I+M+\ldots+M^k) = I-I-M-M^2-\ldots-M^k+M+M^2+\ldots+M^k+M^{k+1} = M^{k+1} \to 0$$

as $k \to \infty$ This implies that

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots.$$

$ii) \Rightarrow iii)$ let (λ,v) eigenvectors and eigenvalues to M hence $Mv = \lambda v$, then $M^k v = \lambda^k v$ we have

$$\begin{aligned}(\lim_{k\to\infty} M^k)v &= \lim_{k\to\infty} M^k v \\ &= \lim_{k\to\infty} \lambda^k v \\ &= v \lim_{k\to\infty} \lambda^k\end{aligned}$$

and $M^k \to_{k\to\infty} 0$ hence $\lim_{k\to\infty} M^k = 0$, $0 = \lim_{k\to\infty} M^k v = v \lim_{k\to\infty} \lambda^k$ and , since by hypothesis $v \neq 0$,we must have $\lim_{k\to\infty} \lambda^k = 0$ which implice $|\lambda| < 1$.

$iii) \Rightarrow iv)$ assuming iii) ,it is then obvious that the matrix $A = I - M$ is non singular ,then $A^{-1} = (I - M)^{-1}$ and ,as A has a spectral radius less than unity by hypothesis ,applying ii) gives that

$$A^{-1} = I + M + M^2 + \ldots \tag{1.3}$$

, Since $M \geq 0$, So are all its powres ,thus $A^{-1} \geq 0$

$iv) \Rightarrow iii)$ assuming iv) ,let $x \geq 0$ be an eigenvectors of A with $Mx = \rho(M)x$ thus

$$(I - M)x = (1 - \rho(M))x \tag{1.4}$$

,Thus and as the matrix $(I - M)$ is non singular ,$1 - \rho(M) \neq 0$,Now

$$(I - M)^{-1}x = \frac{x}{1 - \rho(M)}$$

, and since $x \geq 0$ with $x \neq 0$ and since $(I - M)^{-1} \geq 0$ by hypothesis , it follows that $1 \geq \rho(M)$ .thus completing de proof □

Lemma 1.4. *[14] A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers is convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc.*

Lemma 1.5. *[14] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be convergent to zero. Then*

$$z \leq (I-M)^{-1}z \quad \text{for every} \quad z \in \mathbb{R}^n_+. \tag{1.5}$$

Proof. Since $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is convergent to zero, then from Lemma (1.3), $(I-M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and

$$(I-M)^{-1} = I + M + M^2 + \ldots$$

. Thus for every $z \in \mathbb{R}^n_+$ we have

$$(I-M)^{-1}z = \sum_{i=0}^{\infty} M^i z$$

and $\sum_{i=0}^{\infty} M^i z = z + Mz + M^2z + \ldots + M^kz + \ldots$ hence $\sum_{i=0}^{\infty} M^i z \geq z$ then $(I-M)^{-1}z \geq z$
□

Lemma 1.6. *[14] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be convergent to zero. Then*

$$P_M = \{z \in \mathbb{R}^n_+ : (I-M)z > 0\}$$

is nonempty and coincides with the set

$$\{(I-M)^{-1}z_0 : z_0 \in \mathbb{R}^n, z_0 > 0\}.$$

Proof. it is clear that $I - M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ by absurd to assume $(I-M)z < 0$ and let $z \in \mathbb{R}^n_+$, hence $\forall i$,$z_i > 0$

$$(I-M)z < 0 \Rightarrow z < Mz \Rightarrow Mz < M^2z$$

$$z < Mz < M^2z < \cdots < M^{n-2}z < M^{n-1}z < M^nz$$

Since that $M^n \longrightarrow 0$ so $M^nz \longrightarrow 0$ therefore z<0 (contradiction) . And is a singular matrix , then for every $z \in \mathbb{R}^n_+, z = (z_1, \cdots, z_n)$ with $z_i > 0, i = 1, \cdots, n$, we get $(I-M)z > 0$, This implies that $P_m \neq \emptyset$. Now we show that

$$P_m = \{(I-M)^{-1}z_0 : z_0 \in \mathbb{R}^n, z_0 > 0\}.$$

Indeed, if $z_0 \in \mathbb{R}^n$ and $z_0 > 0$, then

$$z := (I-M)^{-1}z_0 \geq z_0 \Rightarrow z > 0.$$

Hence $(I-M)z > 0$ and so $z \in P_M$. Conversely, if $z \in P_M$, then $z_0 := (I-M)z > 0$ and $z = (I-M)^{-1}z$. □

Definition 1.16. [14] We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero are the following:

1. A=$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$
2. A=$\begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b,c \in \mathbb{R}_+$ and $a + b < 1, c < 1$
3. A=$\begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b,c \in \mathbb{R}_+$ and $|a - b| < 1, a > 1, b > 0$.

Lemma 1.7. *[14] Let $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a triangular matrix with*

$$\max\{|a_{ii}|, i = 1, \ldots, n\} < \frac{1}{2}$$

Then the matrix $A = (I - M)^{-1}M$ is convergent to zero.

1.2 Gauge space and Generalized gauge space

In mathematics, a pseudo metric space is a generalization of a metric space in which the distance between two distinct points can be zero. In the same way as every normed space is a metric space, every semi normed space is a pseudo metric space. Because of this analogy the term semi metric space (which has a different meaning in topology) is sometimes used as a synonym, especially in functional analysis.

When a topology is generated using a family of pseudo metrics, the space is called a gauge space.

Definition 1.17. [9] A map $d : X \times X \to \mathbb{R}_+$ is said to be a pseudo-metric, or a gauge on the set X .if it has the following properties:

1. if $x = y$ then $d(x, y) = 0$.
2. $d(x, y) = d(y, x)$ for all $x, y, z \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 1.18. [16] A family $\mathcal{P} = \{d_\alpha\}_{\alpha\in\Lambda}$ of pseudo-metrics. on X (or a gauge structure on X) is said to be separating if for each pair of points $x, y \in X$ with $x \neq y$ there is a $d_\alpha \in \mathcal{P}$ such that $d_\alpha(x, y) \neq 0$. A pair $(X,\mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is called a gauge space.

Definition 1.19. [16] Let $(X,\mathcal{P})$ be a gauge space. Then, a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X is said to be Cauchy if, for every $\epsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n, x_{n+q}) \leq \epsilon$ for all $n \geq N$ and $q \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ is called convergent if there exists an $x^* \in X$ such that, for every $\epsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x^*, x_n) \leq \epsilon$ for all $n \geq N$. We write $x_n \to x^*$ as $n \to \infty$.

Definition 1.20. [13] Let $(X,\mathcal{P})$ be a gauge space with $\mathcal{P} = \{d_\alpha\}_{\alpha\in\Lambda}$. A map $T : D(T) \subset X \to X$ is a contraction if there exists a function $\varphi : \Lambda \to \Lambda$ and $a \in \mathbb{R}_+^\Lambda$, $a = a_{\alpha\alpha\in\Lambda}$ such that

$$d_\alpha(T(x), T(y)) \leq a_\alpha d_{\varphi(\alpha)}(x, y) \qquad \text{for all} \quad \alpha \in \Lambda \quad \text{and} \quad x, y \in D(T)$$

and

$$\sum_{i=1}^{\infty} a_\alpha a_{\varphi(\alpha)} a_{\varphi^2(\alpha)} \cdots a_{\varphi^{i-1}(\alpha)} d_{\varphi^i(\alpha)}(x, y) < \infty$$

for every $\alpha \in \Lambda$ and $x, y \in D(T)$. Here φ^i is the i^{th} iterate of φ.

Definition 1.21. [16] A gauge space is called sequentially complete if any Cauchy sequence is convergent. A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

Definition 1.22. [16] If $(X,\mathcal{P})$ is a gauge space, then $T : X \to X$ is continuous with respect to $\mathcal{P}$ if, for any sequence $(x_n)_{n\in\mathbb{N}}$ which converges (with respect to $\mathcal{P}$) to $x \in X$, we have that the sequence $(T(x_n))_{n\in\mathbb{N}}$ converges (with respect to $\mathcal{P}$) to $T(x)$.

Theorem 1.9. *(Gheorghiu [8]) Let $(X,\mathcal{P})$ be a complete gauge space and let $T : X \to X$ be a contraction. Then T has a unique fixed point which can be obtained by successive approximations starting from any element of X.*

we introduce the notions of a vector-valued pseudo-metric, generalized gauge space and generalized contraction.

Definition 1.23. [13] Let Z be a set. A vector-valued map $D : Z \times Z \to \mathbb{R}^n_+$ is said to be a vector-valued pseudo-metric, or a vector-valued gauge on Z, if it has the following properties: $D(u, u) = 0$; $D(u, v) = D(v, u)$; and $D(u, v) \leq D(u,w) + D(w, v)$ for all $u, v,w \in Z$.

Remark 1.2. A family $G = \{D_\alpha\}_{\alpha\in\Lambda}$ of vector-valued pseudo-metrics on Z (or a generalized gauge structure on Z) is said to be separating if for each pair of points $u, v \in Z$ with $u \neq v$, there is a $D_\alpha \in G$ such that $D_\alpha(u, v) \neq 0$. A pair (Z, G) of a nonempty set Z and a separating generalized gauge structure G on Z is called a generalized gauge space. For the generalized gauge spaces, the notions of a convergent sequence, Cauchy sequence , completeness and continuous are similar to those for usual gauge spaces. By analogy, we can introduce the vector version of Gheorghiu's notion of contraction.

Definition 1.24. [13] Let (Z, G) be a generalized gauge space with $G = \{D_\alpha\}_{\alpha\in\Lambda}$ A map $T : D(T) \subset Z \to Z$ is a generalized contraction if there exists a function $\varphi : \Lambda \to \Lambda$ and $M \in M_{n\times n}(\mathbb{R}_+)^\Lambda$, $M = \{D_\alpha\}_{\alpha\in\Lambda}$ such that

$$D_\alpha(T(u), T(v)) \leq M_\alpha D_{\varphi(\alpha)}(u, v) \quad \text{for all} \quad \alpha \in \Lambda \quad \text{and} \quad u, v \in D(T) \tag{1.6}$$

and

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \dots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u, v) < \infty \tag{1.7}$$

for every $\alpha \in \Lambda$ and $u, v \in D(T)$.

Chapter 2

Fixed Point Theorems in generalized Vector Metric and Banach Spaces

We present some new fixed point results for generalized contractions on spaces endowed with complete generalized metric space.

2.1 Banach principle theorem

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric spaces by Perov in 1964 [15]. For related results to Perov's fixed point theorem and for some generalizations and applications of it we refer to [7, 10, 17].

Theorem 2.1. *[15] Let (X, d) be a complete generalized metric space with $d : X \times X \longrightarrow \mathbb{R}^n$ and let $N : X \longrightarrow X$ be such that*

$$d(N(x),N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k \longrightarrow \infty$, then N has a unique fixed point $x_* \in X$,

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0),$$

for every $x_0 \in X$ and $k \geq 1$.

Proof. Let $x \in X$ and define the sequence $x_n = N^n(x)$, where $N^n = N \circ \ldots \circ N$. Using the fact that N is an M-contraction,we get

$$d(x_1,x_2) = d(N(x),N(x_1)) \leq Md(x,N(x))$$

and

$$d(x_2,x_3) = d(N(x_1),N(x_2)) \leq M^2 d(x,N(x))$$

, therefore

$$d(x_{k+1}, x_k) \leq M^k d(N(x), x)$$

and, as a consequence,

$$\begin{aligned} d(x_k, x_{k+m}) &\leq \sum_{i=0}^{m-1} d(x_{i+k}, x_{k+i+1}) \\ &\leq \sum_{i=0}^{m-1} M^{i+k} d(N(x),x) \\ &\leq (M^k + M^{k+1} + \ldots + M^{k+m-1}) d(N(x), x). \end{aligned}$$

From lemma1.3 we deduce that

$$d(x_k, x_{k+m}) \leq M^k (I - M)^{-1} d(N(x), x).$$

Hence (x_k) is a Cauchy sequence with respect to d and thus converges to some limit $x_* \in X$. The continuity of N guarantees that

$$x_* = N(x_*).$$

For uniqueness, let y_1 and y_2 be two fixed points of N, then

$$d(y_1, y_2) = d(N^k(y_1),N^k(y_2)) \leq M^k d(N(y_1),N(y_2)).$$

Since $M^k \to 0$ as $k \to \infty$, this implies $d(y_1, y_2) = 0$, so $y_1 = y_2$. □

Theorem 2.2. *[14] Let E be a Banach generalized space, $Y \subseteq E$ nonempty convex compact subset of E and $N : Y \longrightarrow Y$ be a single valued map. Assume that*

$$d(N(x),N(y)) \leq d(x, y) \text{ for all } x, y \in Y. \tag{2.1}$$

Then N has a fixed point.

Proof. For every $m \in \mathbb{N}$, we have $\frac{I}{2^m} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and

$$\frac{I}{2^{mk}} \to 0 \text{ as } k \to \infty.$$

Thus, for some $x_0 \in Y$ the mapping $N_m : Y \longrightarrow Y$ defined by

$$N_m(x) = (1 - \frac{1}{2^m})N(x) + \frac{1}{2^m}x_0 \in Y \text{ for all } x \in Y.$$

Hence, we get

$$d(N_m(x),N_m(y)) \leq (1 - \frac{1}{2^m})Id(x, y) \text{ for all } x, y \in Y.$$

since $\left|1 - \frac{1}{2^m}\right| \leq 1$ from lemma 1.3 we get

$$(1 - \frac{1}{2^{mk}})I \to 0 \text{ as } k \to \infty$$

From Theorem 2.1 there exists a unique $x_m \in Y$ such that

$$x_m = N_m(x_m), m \in \mathbb{N}.$$

Since Y is compact, then there exists a subsequence of $(x_m)_{m\in\mathbb{N}}$ converging to $x \in Y$. Now ,we show that $x = N(x)$.

$$\begin{aligned} d(x,N(x)) &= \begin{pmatrix} d_1(x,N(x)) \\ \dots \\ d_n(x,N(x)) \end{pmatrix} \\ &\leq d(x, x_m) + d(x_m,N(x_m)) + d(N(x_m),N(x)) \\ &\leq 2Id(x, x_m) + d(x_m,N(x_m)), \end{aligned}$$

and

$$\begin{aligned}
d(x_m,N(x_m)) &= \begin{pmatrix} d_1(x_m,N(x_m)) \\ \dots \\ d_n(x_m,N(x_m)) \end{pmatrix} \\
&= \begin{pmatrix} \|N(x_m) - x_m\|_1 \\ \dots \\ \|N(x_m) - x_m\|_n \end{pmatrix} \\
&= \begin{pmatrix} \left\|N(x_m) - (1 - \frac{1}{2^m})N(x_m) - \frac{1}{2^m}x_0\right\|_1 \\ \dots \\ \left\|N(x_m) - (1 - \frac{1}{2^m})N(x_m) - \frac{1}{2^m}x_0\right\|_n \end{pmatrix} \\
&= \begin{pmatrix} \left\|\frac{1}{2^m}N(x_m) - \frac{1}{2^m}x_0\right\|_1 \\ \dots \\ \left\|\frac{1}{2^m}N(x_m) - \frac{1}{2^m}x_0\right\|_n \end{pmatrix} \\
&\leq \frac{1}{2^m}d(N(x_m),N(x)) + \frac{1}{2^m}d(N(x),x) \\
&\overset{(2.1)}{\leq} \frac{1}{2^m}d(x_m,x) + \frac{1}{2^m}d(N(x),x)
\end{aligned}$$

Hence

$$d(x,N(x)) \leq (2 + \frac{1}{2^m})Id(x,x_m) + \frac{1}{2^m}d(N(x),x_0) \to 0 \text{ as } m \to \infty.$$

□

Theorem 2.3. *[14] Let E be a generalized Banach space, $\overline{B(0,r)}$ be the closed ball of radius $r \in \mathbb{R}^n_+$ and $N : \overline{B(0,r)} \longrightarrow E$ a contraction such that*

$$N(\partial\overline{B(0,r)}) \subset \overline{B(0,r)},$$

where

$$\partial\overline{B(0,r)} = \{x \in E : \sum_{i=1}^{n} \|x\|_i = \sum_{i=1}^{n} r_i, i = 1, \dots, n\}, r = (r_1, \dots, r_n).$$

Then N has a unique fixed point in $\overline{B(0,r)}$.

In 2010,Alexandru-Darius Filip and Adrian Petruşel [10] presented some extensions local and global fixed point results for generalized contractions on spaces endowed with vector-valued metrics.

Theorem 2.4. *[10] let (X,d) be a complete generalized metric space ,$x_0 \in X$, $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1, 2, \dots, m\}$ and let $f : \tilde{B}(x_0, r) \to X$ having the property that there exist $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that*

$$d(f(x),f(y)) \leq Ad(x,y) + Bd(y,f(x)) \tag{2.2}$$

for all $x, y \in \tilde{B}(x_0, r)$. We suppose that

1. *A is a matrix that converges to ward zero;*
2. *if $u \in \mathbb{R}_+^m$ is such that $u(I - A)^{-1} \leq (I - A)^{-1}r$, then $u \leq r$;*
3. *$d(x_0, f(x_0))(I - A)^{-1} \leq r$.*

Then $Fix(f) \neq \emptyset$.

In addition, if the matrix $A + B$ converges to zero, then $Fix(f) = \{x^*\}$.

Proof. We consider $(x_n)_{n\in\mathbb{N}}$ the sequence of successive approximations for the mapping f, defined by

$$\begin{cases} x_{n+1} = f(x_n), \forall n \in \mathbb{N} \\ x_0 \in X, \text{ be arbitrary} \end{cases} \tag{2.3}$$

using(3.)we have $d(x_0,x_1)(I - A)^{-1} = d(x_0,f(x_0))(I - A)^{-1} \leq r \leq (I - A)^{-1}r$ thus,by (2.) we get that $d(x_0,x_1) \leq r$ and hence ,$x_1 \in \tilde{B}(x_0, r)$

Similarly , $d(x_1,x_2)(I - A)^{-1} = d(f(x_0),f(x_1))(I - A)^{-1} \leq Ad(x_0,x_1)(I - A)^{-1} + Bd(x_1,f(x_0))(I - A)^{-1} \leq Ar + Bd(f(x_0),f(x_0))(I - A)^{-1} \leq Ar$ since $d(x_0,x_2) \leq d(x_0,x_1) + d(x_1,x_2)$, by (2.) we get $d(x_0,x_2)(I - A)^{-1} \leq d(x_0,x_1)(I - A)^{-1} + d(x_1,x_2)(I - A)^{-1} \leq Ir + Ar \leq (I + A + A^2 + \cdots) \leq (I - A)^{-1}r$ thus $d(x_0,x_2) \leq r$ and hence $x_2 \in \tilde{B}(x_0, r)$

Inductively, we construct the sequence $(x_n)_{n\in\mathbb{N}}$in $\tilde{B}(x_0, r)$, satisfying , for all $n \in \mathbb{N}$, the following conditions :

(i) $x_{n+1} = f(x_n)$

(ii) $d(x_0,x_n)(I-A)^{-1} \leq (I-A)^{-1}r$

(iii) $d(x_n,x_{n+1})(I-A)^{-1} \leq A^n r$

From (iii) we get, for all$n \in N$ and $p \in \mathbb{N}$, $p > 0$, that

$$\begin{aligned} d(x_n,x_{n+p})(I-A)^{-1} &= d(x_n,x_{n+1})(I-A)^{-1} + \cdots + d(x_{n+p-1},x_{n+p})(I-A)^{-1} \\ &\leq A^n r + A^{n+1}r + A^{n+2}r + \cdots + A^{n+p-1}r \\ &\leq A^n(r + Ar + \cdots + A^{p-1}r + \cdots) \\ &\leq A^n(I-A)^{-1}r \to 0, \quad \text{as } n \to \infty \end{aligned}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, using the fact $(\tilde{B}(x_0,r),d)$ is a complete metric space, we get that $(x_n)_{n\in\mathbb{N}}$ is convergent in the closed set $\tilde{B}(x_0,r)$, Thus, there exists $x^* \in \tilde{B}(x_0,r)$ such that $x^* = \lim_{n\to\infty} x_n$. Next, we show that $x^* \in Fix(f)$ Indeed, we have the following estimation :

$$\begin{aligned} d(x^*,f(x^*)) &\leq d(x^*,x_n) + d(x_n,f(x^*)) \\ &= d(x^*,x_n) + d(f(x_{n-1},f(x_*) \\ &\leq d(x^*,x_n) + Ad(x_{n-1},x^*) + Bd(x^*,x_n) \longrightarrow_{n\to\infty} 0 \end{aligned}$$

hence $x^* \in Fix(f)$,In addition, letting $p \to \infty$ in the estimation of $d(x_n,x_{n+p})$, we get

$$d(x_n,x^*) \leq A^n(I-A)^{-1}d(x_0,x_1) \tag{2.4}$$

We show now the uniqueness of the fixed point. Let $x^*, y^* \in Fix(f)$, with $x^* \neq y^*$. Then

$$\begin{aligned} d(x^*,y^*) &= d(f(x^*),f(y^*)) \\ &\leq Ad(x^*,y^*) + Bd(y^*,x^*) \\ &\leq (A+B)d(x^*,y^*) \end{aligned}$$

which implies $(I-A-B)d(x^*,y^*) \leq 0 \in \mathbb{R}^m$, Taking into account that I- A- B is non-singular and $(I-A-B)^{-1} \in M_{n\times n}(\mathbb{R}_+)$ we deduce That $d(x^*,y^*) \leq 0$ and thus $x^* = y^*$. □

Remark 2.1. By similitude to [6], a mapping $f : Y \subseteq X \to X$ satisfying the condition

$$d(f(x),f(y)) \leq Ad(x,y) + Bd(y,f(x)), \quad \forall x,y \in Y \tag{2.5}$$

for some matrices $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ with A a matrix that converges to ward zero, could be called an almost contraction of Perov type.

We have also a global version of Theorem (2.4), expressed by the following result

Corollary 2.1. *[10] Let (X, d) be a complete generalized metric space. Let $f : X \to X$ be a mapping having the property that there exist $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that*

$$d(f(x),f(y)) \leq Ad(x,y) + Bd(y,f(x)), \quad \forall x,y \in Y \tag{2.6}$$

If A is a matrix that converges to wards zero, then

1. *$Fix(f) \neq \emptyset$;*

2. *the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n := f_n(x_0)$ converges towards a fixed point of f, for all $x_0 \in X$;*

3. *one has the estimation*

$$d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1), \tag{2.7}$$

where $x^* \in Fix(f)$.
In addition, if the matrix $A + B$ converges to zero, then $Fix(f) = \{x^*\}$.

Theorem 2.5. *[10] Let $(X, |.|)$ be a Banach space and let $f_1, f_2 : X \times X \to X$ be two operators. Suppose that there exist $a_{ij}, b_{ij} \in \mathbb{R}_+$ $i, j \in 1, 2$ such that, for each $x := (x_1, x_2), y := (y_1, y_2) \in X \times X$, we has :*

1. $|f_1(x_1, x_2) - f_1(y_1, y_2)| \leq a_{11}|x_1 - y_1| + a_{12}|x_2 - y_2| + b_{11}|x_1 - f_1(y_1, y_2)| + b_{12}|x_2 - f_2(y_1, y_2)|,$

2. $|f_2(x_1, x_2) - f_2(y_1, y_2)| \leq a_{21}|x_1 - y_1| + a_{22}|x_2 - y_2| + b_{21}|x_1 - f_1(y_1, y_2)| + b_{22}|x_2 - f_2(y_1, y_2)|.$

In addition, assume that the matrix $A := \begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{pmatrix}$ converge to zero Then, the system

$$u_1 = f_1(u_1, u_2), \quad u_2 = f_1(u_1, u_2) \tag{2.8}$$

has at least one solution $x^* \in X \times X$. Moreover, if, in addition, the matrix $A + B$ converges to zero, then the above solution is unique.

Proof. Consider $E := X \times X$ and the operator $f : E \to \mathcal{P}_{cl}(E)$ given by the expression $f(x_1, x_2) := (f_1(x_1, x_2), f_2(x_1, x_2))$. Then our system is now represented as a fixed point equation of the following form :$x = f(x), x \in E$. Notice also that the conditions 1. + 2. can be jointly represented as follows:

$$\|f(x) - f(y)\| \leq A\,\|x - y\| + B\,\|x - f(y)\|\,, \qquad \text{for each } x, y \in E := X \times X. \tag{2.9}$$

Hence, Corollary (2.1) applies in (E, d), with $d(u, v) := \|u - v\| := \begin{pmatrix} |u_1 - v_1| \\ |u_2 - v_2| \end{pmatrix}$. □

Theorem 2.6. *[10] let X be a nonempty set and d,ρ be two generalized metrics on X let $f : X \to X$ be an operator, We assume that :*

1. *there exists $C \in M_{n\times n}(\mathbb{R}_+)$ such that $d(f(x),f(y)) \leq \rho(x,y).C$*
2. *(X,d) is a complete generalized metric space ;*
3. *$f : (X,d) \to (X,d)$ is continuous;*
4. *There exists $A,B \in M_{n\times n}(\mathbb{R}_+)$ such that for all $x,y \in X$,we has*

$$\rho(f(x),f(y)) \leq A\rho(x,y) + B\rho(y,f(y)) \tag{2.10}$$

if the metric A converge to wards zeros ,then $Fix(f) \neq 0$.

In addition if the metric A+B converge to zero then $Fix(f) = \{x^*\}$.

Proof. We consider the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$ defined recurrently by $x_{n+1} = f(x_n), x_0 \in X$ being arbitrary. The following statements hold:

$$\rho(x_1,x_2) = \rho(f(x_0),f(x_1)) \leq A\rho(x_0,x_1) + B\rho(x_1,f(x_0)) \leq A\rho(x_0,x_1)$$

$$\rho(x_2,x_3) = \rho(f(x_1),f(x_2)) \leq A\rho(x_1,x_2) + B\rho(x_2,f(x_1)) \leq A^2\rho(x_0,x_1)$$

$$\cdots$$

$$\rho(x_n,x_{n+1}) \leq A^n\rho(x_0,x_1), \forall n \in \mathbb{N}^*, n \geq 1.$$

Now, let$p \in \mathbb{N}, p > 0$. We estimate

$$\begin{aligned}
\rho(x_n,x_{n+p}) &\leq \rho(x_n,x_{n+1}) + \rho(x_{n+1},x_{n+2}) + \cdots + \rho(x_{n+p-1},x_{n+p}) \\
&\leq A^n \rho(x_0,x_1) + A^{n+1}\rho(x_0,x_1) + \cdots + A^{n+p-1}\rho(x_0,x_1) \\
&\leq A^n(I + A + \cdots + A^{p-1} + \cdots)\rho(x_0,x_1) \\
&= A^n(I - A)^{-1}\rho(x_0,x_1),
\end{aligned}$$

Letting $n \to \infty$ we obtain that $\rho(x_n, x_{n+p}) \to 0 \in \mathbb{R}^m$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect toρ. On the other hand, using the statement (1), we get

$$d(x_n,x_{n+p}) = d(f(x_{n-1}),f(x_{n+p-1})) \leq \rho(x_{n-1},x_{n+p-1}).C \leq A^{n-1}(I-A)^{-1}\rho(x_0,x_1).C \to_{n\to\infty} 0 \tag{2.11}$$

Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since (X, d) is complete, one obtains the existence of an element $x^* \in X$ such that $x^* = \lim_{n\to\infty} x_n$ with respect to d. We prove next that $x^* = f(x^*)$, that is $, Fix(f) \neq \emptyset$. Indeed, since $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$, letting $n \to \infty$ and taking into account that f is continuous with respect to d, we get that $x^* = f(x^*)$. The uniqueness of the fixed point x^* is proved below. Let $x^*, y^* \in Fix(f)$ such that $x^* \neq y^*$. We estimate

$$\rho(x^*,y^*) = \rho(f(x^*),f(y^*)) \leq A\rho(x^*,y^*) + B\rho(y^*,f(x^*)) \leq (A + B)\rho(x^*,y^*) \tag{2.12}$$

Thus, using the additional assumption on the matrix $A + B$, we have that

$$(I - A - B)\rho(x^*,y^*) \leq 0 \Rightarrow \rho(x^*,y^*) \leq 0 \Rightarrow x^* = y^* \tag{2.13}$$

□

2.2 Application

We consider the following system of differential equation with impulse effects:

$$x'(t) = f_1(t, x(t), y(t)), \quad y'(t) = f_2(t, x(t), y(t)), \text{ a.e. } t \in [0,1] \tag{2.14}$$

$$x\left(\tau^+\right) - x\left(\tau^-\right) = I_1(x(\tau), y(\tau)), \quad y\left(\tau^+\right) - y\left(\tau^-\right) = I_2(x(\tau), y(\tau)) \tag{2.15}$$

$$x(0) = x_0, \quad y(0) = y_0 \tag{2.16}$$

where $0 < \tau < 1, i = 1; 2$; J := [0; 1]; $f_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a function, $I_1; I_2 \in C(\mathbb{R} \times \mathbb{R}; \mathbb{R})$: The notations $x(\tau^+) = \lim_{h\to 0^+} x(\tau + h)$ and $x(\tau^-) = \lim_{h\to 0^+} x(\tau - h)$ stand for the right and the left limits of the function y at $t = \tau$, respectively. In order to de

ne a solutions for Problem(1), consider the space of piece- wise continuous functions:

$$PC([0,1], \mathbb{R}) = \{y : [0,1] \to \mathbb{R}, y \in C(J \backslash \{\tau\}, \mathbb{R});\ \text{such that}$$
$$y\left(\tau^-\right) \text{ and } y\left(\tau^+\right) \text{ exist and satisfy } \ y\left(\tau^-\right) = y(\tau)\Big\}$$

Endowed with the norm

$$\|y\|_{PC} = \sup\{|y(t)| : t \in J\},$$

PC is a Banach space. In the proof of the existence result for the problem we can easily proof the fol- lowing auxiliary lemma.

Lemma 2.1. *Let $f_1; f_2 \in L^1(J \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$. Then y is a solution of the impulsive system*

$$x'(t) = f_1(t, x(t), y(t)), \quad y'(t) = f_2(t, x(t), y(t)), \quad a, t \in [0,1] \tag{2.17}$$

$$x\left(\tau^+\right) - x\left(\tau^-\right) = I_1(x(\tau), y(\tau)), \quad y\left(\tau^+\right) - y\left(\tau^-\right) = I_2(x(\tau), y(\tau)) \tag{2.18}$$

$$x(0) = x_0, \quad y(0) = y_0 \tag{2.19}$$

if and only if y is a solution of the impulsive integral equation

$$\begin{cases} x(t) = x_0 + \int_0^t f_1(s, x(s), y(s))ds + I_1(x(\tau), y(\tau)), & \text{a.e. } t \in [0,1] \\ y(t) = y_0 + \int_0^t f_2(s, x(s), y(s))ds + I_2(x(\tau), y(\tau)), & \text{a.e. } t \in [0,1] \end{cases} \tag{2.20}$$

Assumption 2.2.1. *1. $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous,*

2. There exist a functions $l_i \in L^1(J; \mathbb{R}^+)$, $\quad i = 1,2,3,4$

$$|f_1(t,x,y) - f_1(t,\bar{x},\bar{y})| \leq l_1(t)|x - \bar{x}| + l_2(t)|y - \bar{y}|, \quad t \in J \text{ for all } x; \bar{x}; \bar{y}; y \in \mathbb{R}$$

and

$$|f_2(t,x,y) - f_2(t,\bar{x},\bar{y})| \leq l_3(t)|x - \bar{x}| + l_4(t)|y - \bar{y}|, \quad t \in J \text{ for all } x; \bar{x}; \bar{y}; y \in \mathbb{R}$$

3. There exist a constants $a_i \quad ; b_i \geq 0; i = 1; 2$ such that

$$|I_1(x, y) - I_1(\bar{x}, \bar{y})| \leq a_1|x - \bar{x}| + a_2|y - \bar{y}|, \text{ for all } x, \bar{x}, y, \bar{y} \in \mathbb{R}.$$

and

$$|I_2(x, y) - I_2(\bar{x}, \bar{y})| \leq b_1|x - \bar{x}| + b_2|y - \bar{y}|, \text{ for all } x, \bar{x}, y, \bar{y} \in \mathbb{R}.$$

Theorem 2.7. *Consider the system (2.14)-(2.16) suppose that the Assumption 2.2.1 is satisfied.and the matrix*

$$M = \begin{pmatrix} \|l_1\|_{L^1} + a_1 & \|l_2\|_{L^1} + a_2 \\ \|l_3\|_{L^1} + b_1 & \|l_4\|_{L^1} + b_2 \end{pmatrix}$$

converge to zero,Then there exists a unique solution of the integral equation(2.14)-(2.16).

Proof. Consider the operator $N : PC \times PC \to PC \times PC$ defined by

$$N(x,y) = \left\{ (N_1,N_2) \in PC \times PC : \left\{ \begin{array}{l} N_1(t) = x_0 + \int_0^t f_1(s,x(s),y(s))ds + I_1(x(\tau),y(\tau)) \\ N_2(t) = y_0 + \int_0^t f_2(s,x(s),y(s))ds + I_2(x(\tau),y(\tau)) \end{array} \right\} \right..$$

Clearly,fixed points of the operator N are solutions of System (2.14)-(2.16). Let

$$N_i(x,y) = \left\{ x_i + \int_0^t f_i(s,x(s),y(s))ds + I_i(x(\tau),y(\tau)), t \in J, i = 1,2 \right\}$$

where $x_1 = x_0; x_2 = y_0$; We show N satisfies the assumptions of corollary (2.1). Let us consider $x,y \in PC([0;1],\mathbb{R})$, then

$$\begin{aligned} |N_1(x(t),y(t)) - N_1(\bar{x}(t),\bar{y}(t))| &= |\int_0^1 f_1(t,x(s),y(s))ds + I_1(x(\tau),y(\tau)) - \int_0^1 f_1(t,\bar{x}(s),\bar{y}(s))ds \\ &\quad - I_1(\bar{x}(\tau),\bar{y}(\tau))| \\ &\le \int_0^1 |f_1(t,x(s),y(s)) - f_1(t,\bar{x}(s),\bar{y}(s))|ds \\ &\quad + |I_1(x(\tau),y(\tau)) - I_1(\bar{x}(\tau),\bar{y}(\tau))|, \end{aligned}$$

then

$$\begin{aligned} ||N_1(x(t),y(t)) - N_1(\bar{x}(t),\bar{y}(t))||_{PC} &\le ||f_1(t,x,y) - f_1(t,\bar{x},\bar{y})||_{PC} + ||I_1(x(\tau),y(\tau)) - I_1(\bar{x}(\tau),\bar{y}(\tau))||_{PC} \\ &\le \|l_1\|_{L^1} ||x - \bar{x}||_{PC} + \|l_2\|_{L^1} ||y - \bar{y}||_{PC} + a_1||x - \bar{x}||_{PC} \\ &\quad + a_2||y - \bar{y}||_{PC} \\ &\le (\|l_1\|_{L^1} + a_1) \|x - \bar{x}\|_{PC} + (\|l_2\|_{L^1} + a_2) \|y - \bar{y}\|_{PC} \end{aligned}$$

Similarly we have

$$\begin{aligned} |N_2(x(t),y(t)) - N_2(\bar{x}(t),\bar{y}(t))| &= |\int_0^1 f_2(t,x(s),y(s))ds + I_2(x(\tau),y(\tau)) \\ &\quad - \int_0^1 f_2(t,\bar{x}(s),\bar{y}(s))ds - I_2(\bar{x}(\tau),\bar{y}(\tau))| \\ &\le \int_0^1 |f_2(t,x(s),y(s)) - f_2(t,\bar{x}(s),\bar{y}(s))|ds \\ &\quad + |I_1(x(\tau),y(\tau)) - I_1(\bar{x}(\tau),\bar{y}(\tau))|, \end{aligned}$$

then

$$\begin{aligned}||N_1(x(t),y(t)) - N_1(\bar{x}(t),\bar{y}(t))||_{PC} \leq & ||f_1(t,x,y) - f_2(t,\bar{x},\bar{y})||_{PC} + ||I_1(x(\tau),y(\tau)) - I_1(\bar{x}(\tau),\bar{y}(\tau))||_{PC} \\ \leq & \|l_3\|_{L^1} ||x - \bar{x}||_{PC} + \|l_4\|_{L^1} ||y - \bar{y}||_{PC} + b_1||x - \bar{x}||_{PC} \\ & + b_2||y - \bar{y}||_{PC} \\ \leq & (\|l_3\|_{L^1} + b_1) \|x - \bar{x}\|_{PC} + (\|l_4\|_{L^1} + b_2) \|y - \bar{y}\|_{PC}\end{aligned}$$

Therefore

$$||N(x,y) - N(\bar{x},\bar{y})||_{PC} \leq M \begin{pmatrix} \|x - \bar{x}\|_{PC} \\ \|y - \bar{y}\|_{PC} \end{pmatrix} \text{ for all } (x;y);(\bar{x};\bar{y}) \in PC \times PC.$$

Hence, by theorem (2.1), the operator N has a unique fixed point which is solution of System (2.14)-(2.16).

□

Chapter 3

Fixed Point Theorems in generalized gauge Spaces

In 2000, Frigon [11] introduced the notion of generalized contraction in gauge spaces and proved that every generalized contraction on a complete gauge space (sequentially complete gauge space) has a unique fixed point. For a nice survey on the same subject see also Frigon [12]. In this chapter, The Gheorghiu's theorem is extended for generalized contractions on complete generalized gauge spaces. A second result is concerning with mappings which are contractive in Gheorghiu's sense only on one of its orbits. The results are Perov-Gheorghiu mixtures and have the advantages of both approaches.

3.1 Principle theorem

Theorem 3.1. *[13] Let (Z, G) be a complete generalized gauge space and let $T : Z \to Z$ be a generalized contraction. Then T has a unique fixed point which can be obtained by successive approximations starting from any element of Z.*

Proof. Let u_0 be an arbitrary element of Z. Define a sequence (u_k) by

$$u_{k+1} = T(u_k), k \in \mathbb{N}. \tag{3.1}$$

Then using (1.6) we have

$$\begin{aligned}
D_\alpha(u_k, u_{k+1}) &= D_\alpha(T(u_{k-1}), T(u_k)) \\
&\leq M_\alpha D_{\varphi(\alpha)}(u_{k-1}, u_k) \\
&= M_\alpha D_{\varphi(\alpha)}(T(u_{k-2}), T(u_{k-1})) \\
&\leq M_\alpha M_{\varphi(\alpha)} D_{\varphi^2(\alpha)}(u_{k-2}, u_{k-1}) \\
&\vdots \\
&\leq M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{k-1}(\alpha)} D_{\varphi^k(\alpha)}(u_0, u_1)
\end{aligned}$$

for every $\alpha \in \Lambda$ and $k = 1, 2, \ldots$ As a consequence we

$$\begin{aligned}
D_\alpha(u_k, u_{k+m}) &= D_\alpha(u_k, u_{k+1}) + \ldots + D_\alpha(u_{k+m-1}, u_{k+m}) \\
&\leq \sum_{n=0}^{m-1} M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{k+n-1}(\alpha)} D_{\varphi^{k+n}(\alpha)}(u_0, u_1) \\
&= \sum_{i=k}^{k+m-1} M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u_0, u_1)
\end{aligned}$$

Hence, according to (1.7), (u_k) is a Cauchy sequence. Let u^* be its limit. Then, letting $k \to \infty$ in (3.1) gives $u^* = T(u^*)$. For uniqueness, assume that u_1, u_2 are two fixed points of T. Then

$$\begin{aligned}
D_\alpha(u_1, u_2) &= D_\alpha(T(u_1), T(u_2)) \\
&\leq M_\alpha D_{\varphi(\alpha)}(u_1, u_2) \\
&\leq M_\alpha M_{\varphi(\alpha)} D_{\varphi^2(\alpha)}(u_1, u_2) \\
&\vdots \\
&\leq M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{k-1}(\alpha)} D_{\varphi^k(\alpha)}(u_1, u_2)
\end{aligned}$$

and using (1.7) we obtain that $D_\alpha(u_1, u_2) = 0$ for every $\alpha \in \Lambda$. Since family $\mathcal{G}$ is separating we deduce that $u_1 = u_2$. □

From the proof of Theorem 3.1 we immediately obtain the following result guaranteeing the existence of a fixed point as limit of the successive approximation sequence which starts from a given element of the space.

Theorem 3.2. *[13] Let $(Z, \mathcal{G})$ be a complete generalized gauge space with $\mathcal{G} = \{D_\alpha\}_{\alpha\in\Lambda}$ and let $T : Z \to Z$ be a mapping. Assume that there is $u_0 \in Z$, $C > 0$, $\varphi : \Lambda \to \Lambda$ and $M \in M_{n\times n}(\mathbb{R}_+)^\Lambda$, $M = \{M_\alpha\}_{\alpha\in\Lambda}$ such that the following conditions hold:*

$$D_\alpha(T(u), T(v)) \le M_\alpha D_{\varphi(\alpha)}(u, v) \quad \text{for all} \quad \alpha \in \Lambda \quad \text{and} \quad u, v \in Z; \tag{3.2}$$

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \cdots M_{\varphi^{i-1}(\alpha)} < \infty \quad \text{for every} \quad \alpha \in \Lambda; \tag{3.3}$$

$$D_\alpha(u_0, T(u_0)) \le C \quad \text{for all} \quad \alpha \in \Lambda$$

Then T has at least one fixed point which can be obtained by successive approximations starting from u_0.

Remark 3.1. [13] Here are some useful particular cases: If there is an integer $p \ge 2$ with $\varphi^p = \varphi$, then conditions (1.6) and (3.3) reduce to the assumption that

$M_{\varphi(\alpha)} \ldots M_{\varphi^{p-1}(\alpha)}$ is convergent to zero for every $\alpha \in \Lambda$.

Thus, if p = 2, that is $\varphi^2 = \varphi$ (Marinescu's situation), then hold if (1.6) and (3.3)

$M_{\varphi(\alpha)}$ is convergent to zero for every $\alpha \in \Lambda$.

In particular, if $\varphi = 1_\Lambda$ (Tarafdar's situation [21]), then (1.6) and (3.3) are satisfied provided that M_α is convergent to zero for every $\alpha \in \Lambda$.

Now we turn back to system (1.1). We assume that X is a complete gauge space with the family of pseudo-metrics $P = \{d_\alpha\}_{\alpha\in\Lambda}$. We denote $Z := X^2$, $T := (A,B)$ and $\mathcal{G} := \{D_\alpha\}_{\alpha\in\Lambda}$,where

$$D_\alpha(u, v) = \begin{bmatrix} d_\alpha(x, x_1) \\ d_\alpha(y, y_1) \end{bmatrix} \tag{3.4}$$

for every $u := (x, y)$, $v := (x_1, y_1) \in X^2$ and $\alpha \in \Lambda$. Then $(Z, \mathcal{G})$ is a complete generalized gauge space.

Specialized to this case, Theorems 3.1 and 3.2 yield the following results

Theorem 3.3. *[13] Assume that $(X,\mathcal{P})$ is a complete gauge space with $\mathcal{P} = \{d_\alpha\}_{\alpha\in\Lambda}$ and that there exists a function $\varphi : \Lambda \to \Lambda$ and nonnegative constants $a_\alpha, b_\alpha, \bar{a}_\alpha, \bar{b}_\alpha$ such that*

$$\begin{aligned} d_\alpha\left(A(x, y), A\left(x_1, y_1\right)\right) &\le a_\alpha d_{\varphi(\alpha)}\left(x, x_1\right) + b_\alpha d_{\varphi(\alpha)}\left(y, y_1\right), \\ d_\alpha\left(B(x, y), B\left(x_1, y_1\right)\right) &\le \bar{a}_\alpha d_{\varphi(\alpha)}\left(x, x_1\right) + \bar{b}_\alpha d_{\varphi(\alpha)}\left(y, y_1\right) \end{aligned} \tag{3.5}$$

for all $x, x_1, y, y_1 \in X$ and $\alpha \in \Lambda$. Let

$$M_\alpha := \begin{bmatrix} a_\alpha & b_\alpha \\ \bar{a}_\alpha & \bar{b}_\alpha \end{bmatrix}.$$

If

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \dots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u,v) < \infty \tag{3.6}$$

for all $u, v \in X^2$ and $\alpha \in \Lambda$, then system (1.1) has a unique solution. Moreover, the solution is the limit of the sequence of successive approximations

$$u_k = (x_k, y_k), \quad x_{k+1} = A(x_k, y_k), \quad y_{k+1} = B(x_k, y_k) \quad (k = 0, 1, \dots) \tag{3.7}$$

starting from any initial pair $(x_0, y_0) \in X^2$.

Proof. Clearly inequalities (3.5) can be written in the vector form

$$D_\alpha(T(u), T(v)) \leq M_\alpha D_{\varphi(\alpha)}(u, v).$$

The result is now a direct consequence of Theorem 3.1. □

Theorem 3.4. *[13] Under the assumptions of Theorem 3.3, if there is $u_0 = (x_0, y_0) \in X^2$ and $C > 0$ such that*

$$D_\alpha(u_0, T(u_0)) \leq C \tag{3.8}$$

and

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \dots M_{\varphi^{i-1}(\alpha)} < \infty \tag{3.9}$$

for every $\alpha \in \Lambda$ then system (1.1) has at least one solution which is the limit of sequence (3.7) starting from u_0.

Proof. The result is a direct consequence of Theorem 3.2. □

New,we will present some extension the fixed point and coupled fixed point theorems in spaces endowed with the case of complete generalized gauge spaces is discussed. We will point out first the framework of our study.

If $\mathcal{P} = \{p_\alpha\}_{\alpha \in \Lambda}$ and $\mathcal{Q} = \{q_\beta\}_{\beta \in \Gamma}$ are two separating (generalized) gauge structures on a set X (where Λ and Γ are directed sets), then for $r = \{r_\beta\}_{\beta\in\Gamma} \in (R^m_+)^\Gamma$ (with $0 \lhd r_\beta$ for every $\beta \in \Gamma$) and $x_0 \in X$ we will denote by $\overline{\tilde{B}_q(x_0;r)}^p$ the closure of $\tilde{B}_q(x_0;r)$ in $(X,\mathcal{P})$, where

$$\tilde{B}_q(x_0;r) = \{x \in X : q_\beta(x_0,x) \leq r_\beta, \forall\beta \in \Gamma\}. \tag{3.10}$$

In this case, the set $\tilde{B}_q(x_0,x)$ is sequentially closed in $(X,\mathcal{P})$. If there is just one separating (generalized) gauge structure $\mathcal{P} = \{p_\alpha\}_{\alpha\in\Lambda}$ on X then it is well known that $\tilde{B}_p(x_0;r)$ is sequentially closed in $(X,\mathcal{P})$. We can prove the following local fixed point theorems.

Theorem 3.5. *[16] Let X be a nonempty set endowed with two separating generalized gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in\Lambda}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in\Gamma}$ (where Λ and Γ are directed sets), $r = \{r_\beta\}_{\beta\in\Gamma} \in (R^m_+)^\Gamma$ (with $0 \lhd r_\beta$ for every $\beta \in \Gamma$), $x_0 \in X$ and $f : \overline{\tilde{B}_q(x_0;r)}^p \to X$ a continuous operator with respect to $\mathcal{P}$. One supposes that the following hold.*

1. *$(X,\mathcal{P})$ is a sequentially complete generalized gauge space.*
2. *There exist a function $\psi : \Lambda \to \Gamma$ and $C := \{C_\alpha\}_{\alpha\in\Gamma} \in (R^m_+)^\Gamma$ (with $0 \lhd C_\alpha$ for every $\alpha \in \Lambda$) such that*

$$p_\alpha(x,y) \leq C_\alpha \cdot q_{\psi(\alpha)}(x,y), \quad \text{for every } \alpha \in \Lambda, x, y \in \overline{\tilde{B}_q(x_0;r)}^p \tag{3.11}$$

3. *There exists a function $\varphi : \Gamma \to \Gamma$ and $M \in \mathcal{M}_{m\times m}(\mathbb{R}_+)^\Gamma$ with $M := \{M_\beta\}_{\beta\in\Gamma}$ such that, for every $\beta \in \Gamma$,the following implication holds:*

$$x, y \in \overline{\tilde{B}_q(x_0;r)}^p \implies q_\beta(f(x), f(y)) \leq M_\beta q_{\varphi(\beta)}(x,y) \tag{3.12}$$

4. *$\sum_{k=1}^\infty M_\beta M_{\varphi(\beta)} \cdots M_{\phi^{k-1}(\beta)} q_{\varphi^k(\beta)}(u,v) < \infty$, for each $\beta \in \Gamma$ and every $u, v \in \overline{\tilde{B}_q(x_0;r)}^p$*
5. *If $r^0 = \left\{r^0_\beta\right\}_{\beta\in\Gamma} \in \left(\mathbb{R}^m_+\right)^\Gamma$ $\left(\text{with } O \lhd r^0_\beta \text{ for every } \beta \in \Gamma\right)$ is given by the expression $r^0_\beta := (I - M_\beta)^{-1} q_\beta(x_0, f(x_0))$, then $r^0_\beta \leq r_\beta$, for each $\beta \in \Gamma$*

Then, f has a unique fixed point $x^* \in \overline{\tilde{B}_q(x_0;r^0)}^p$ and the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations of f converges to x^*, for any $x \in \overline{\tilde{B}_q(x_0;r^0)}^p$

Proof. Notice first that the set $\overline{\tilde{B}_q\left(x_0;r^0\right)}^p$ is invariant with respect to f; that is, $f : \overline{\tilde{B}_q\left(x_0;r^0\right)}^p \to \overline{\tilde{B}_q\left(x_0;r^0\right)}^p$ Indeed, Let $x \in \overline{\tilde{B}_q\left(x_0;r^0\right)}^p$ Then, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\tilde{B}_q\left(x_0;r^0\right)$ which converges (with respect to $\mathcal{P}$) to x. Since f is continuous with respect to $\mathcal{P}$, we get that the sequence $(f(u_n))_{n\in\mathbb{N}}$ converges (with respect to $\mathcal{P}$) to $f(x)$. So, we should show now that $(f(u_n)) \in \tilde{B}_q\left(x_0;r^0\right)$, for every $n \in \mathbb{N}$ Then, using the assumption (5), for each $\beta \in \Gamma$, we have

$$\begin{aligned} q_\beta\left(x_0, f\left(u_n\right)\right) \leq & q_\beta\left(f\left(u_n\right), f\left(x_0\right)\right) \\ & + q_\beta\left(x_0, f\left(x_0\right)\right) \\ \leq & M_\beta q_\beta\left(x_0, u_n\right) + q_\beta\left(x_0, f\left(x_0\right)\right) \\ \leq & M_\beta r^0_\beta + q_\beta\left(x_0, f\left(x_0\right)\right) = r^0_\beta \end{aligned} \tag{3.13}$$

Now, in a classical manner (see, e.g., Theorem 2.1 in Novac and Precup [24]), we get that, for any $x \in \overline{\tilde{B}_q\left(x_0;r^0\right)}^p$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy in $(X,\mathcal{Q})$. By assumption (2), the sequence is also Cauchy in $(X,\mathcal{P})$. Notice now that, since $(X,\mathcal{P})$ is a sequentially complete generalized gauge space, we have that $(\overline{\tilde{B}_q\left(x_0;r^0\right)}^p,\mathcal{P})$ is a sequentially complete generalized gauge space too. Thus, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent (with respect to) to a certain element $x^* \in \overline{\tilde{B}_q\left(x_0;r^0\right)}^p$. By the continuity of f with respect to $\mathcal{P}$, we get that $x^* = f(x^*)$. The uniqueness follows from assumptions (3) and (4). Indeed, if x^* and u^* are two distinct fixed points of f, then, for each $\beta \in \Gamma$, we have

$$\begin{aligned} q_\beta\left(x^*, u^*\right) &= q_\beta\left(f\left(x^*\right), f\left(u^*\right)\right) \leq M_\beta q_\beta\left(x^*, u^*\right) \\ &\leq \cdots \\ &\leq M_\beta M_{\varphi(\beta)} \cdots M_{\varphi^{k-1}(\beta)} q_{\varphi^k(\beta)}\left(x^*, u^*\right) \end{aligned} \tag{3.14}$$

Then, by (4), we get that $q_\beta(x^*, u^*) = 0$, for each $\beta \in \Gamma$. Since the family $\mathcal{Q}$ is separating, we obtain that $x^* = u^*$. □

In particular, from the above proof, we can obtain the following result.

Theorem 3.6. *[16] Let X be a non-empty set endowed with two separating generalized gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in\Lambda}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in\Gamma}$ (where Λ and Γ are directed sets), $r = \{r_\beta\}_{\beta\in\Gamma} \in (R^m_+)^\Gamma$ (with $0 \lhd r_\beta$ for every $\beta \in \Gamma$), $x_0 \in X$ and $f : \overline{\tilde{B}_q(x_0;r)}^p \to X$ a continuous operator with respect to $\mathcal{P}$. One supposes that the following hold.*

1. $(X,\mathcal{P})$ *is a sequentially complete generalized gauge space.*

2. *There exist a function* $\psi : \Lambda \to \Gamma$ *and* $C := \{C_\alpha\}_{\alpha\in\Gamma} \in (R_+^m)^\Gamma$ *(with* $0 \lhd C_\alpha$ *for every* $\alpha \in \Lambda$*) such that*

$$p_\alpha(x, y) \le C_\alpha \cdot q_{\psi(\alpha)}(x, y), \quad \text{for every } \alpha \in \Lambda, x, y \in \overline{\tilde{B}_q\left(x_0; r\right)}^p \tag{3.15}$$

3. *There exists a function* $\varphi : \Gamma \to \Gamma$ *and* $M \in \mathcal{M}_{m\times m}\left(\mathbb{R}_+\right)^\Gamma$ *with* $M := \{M_\beta\}_{\beta\in\Gamma}$ *such that, for every* $\beta \in \Gamma$ *,the following implication holds:*

$$x, y \in \overline{\tilde{B}_q\left(x_0; r\right)}^p \Longrightarrow q_\beta(f(x), f(y)) \le M_\beta q_{\varphi(\beta)}(x, y) \tag{3.16}$$

4. $\sum_{k=1}^{\infty} M_\beta M_{\varphi(\beta)} \cdots M_{\phi^{k-1}(\beta)} q_{\varphi^k(\beta)}(u, v) < \infty$, *for each* $\beta \in \Gamma$ *and every* $u, v \in \overline{\tilde{B}_q\left(x_0; r\right)}^p$

5. *If* $r^0 = \left\{r_\beta^0\right\}_{\beta\in\Gamma} \in \left(\mathbb{R}_+^m\right)^\Gamma$ *(with* $O \lhd r_\beta^0$ *for every* $\beta \in \Gamma\Big)$ *is given by the expression* $r_\beta^0 := \left(I - M_\beta\right)^{-1} q_\beta\left(x_0,\ f\left(x_0\right)\right),$ *then* $r_\beta^0 \le r_\beta,$ *for each* $\beta \in \Gamma$

6. *There exists* $S \in \mathbb{R}_+^m$ *(with* $0 \lhd S$ *), such that* $q_\beta(x_0,f(x_0)) \le S$ *for every* $\beta \in \Gamma$*,*

Then, f *has a unique fixed point* $x^* \in \overline{\tilde{B}_q\left(x_0; r^0\right)}^p$ *and the sequence* $\left(f^n(x)\right)_{n\in\mathbb{N}}$ *of successive approximations of* f *converges to* x^**, for any* $x \in \overline{\tilde{B}_q\left(x_0; r^0\right)}^p$

Theorem 3.7. *[16] Let* X *be a non-empty set endowed with two separating generalized gauge structures* $\mathcal{P} = \{p_\alpha\}_{\alpha\in\Lambda}$*,* $\mathcal{Q} = \{q_\beta\}_{\beta\in\Gamma}$ *(where* Λ *and* Γ *are directed sets), and* $f : X \to X$ *a continuous operator with respect to* $\mathcal{P}$ *. One supposes that the following hold.*

- $(X,\mathcal{P})$ *is a sequentially complete generalized gauge space.*

- *There exist a function* $\psi : \Lambda \to \Gamma$ *and* $C := \{C_\alpha\}_{\alpha\in\Gamma} \in (R_+^m)^\Gamma$ *(with* $0 \lhd C_\alpha$ *for every* $\alpha \in \Lambda$*) such that*

$$p_\alpha(x, y) \le C_\alpha \cdot q_{\psi(\alpha)}(x, y), \quad \text{for every } \alpha \in \Lambda, x, y \in \overline{\tilde{B}_q\left(x_0; r\right)}^p \tag{3.17}$$

- *There exists a function* $\varphi : \Gamma \to \Gamma$ *and* $M \in \mathcal{M}_{m\times m}(\mathbb{R}_+)^\Gamma$ *with* $M := \{M_\beta\}_{\beta\in\Gamma}$ *such that, for every* $\beta \in \Gamma$ *,the following implication holds:*

$$\begin{aligned} &x, y \in X \\ &\quad \Longrightarrow q_\beta(f(x), f(y)) \le M_\beta q_{\varphi(\beta)}(x, y) \end{aligned} \tag{3.18}$$

- $\sum_{k=1}^{\infty} M_\beta M_{\varphi(\beta)} \cdots M_{\phi^{k-1}(\beta)} q_{\varphi^k(\beta)}(u, v) < \infty$, *for each* $\beta \in \Gamma$ *and every* $u, v \in \overline{\tilde{B}_q(x_0; r)}^p$
- *If* $r^0 = \left\{r^0_\beta\right\}_{\beta\in\Gamma} \in \left(\mathbb{R}^m_+\right)^\Gamma$ *(with* $O \lhd r^0_\beta$ *for every* $\beta \in \Gamma$*) is given by the expression* $r^0_\beta := (I - M_\beta)^{-1} q_\beta(x_0, f(x_0))$, *then* $r^0_\beta \le r_\beta$, *for each* $\beta \in \Gamma$
- *There exists* $x_0 \in X$ *and* $S \in \mathbb{R}^m_+$ *(with* $0 \lhd S$ *), such that* $q_\beta(x_0, f(x_0)) \le S$ *for every* $\beta \in \Gamma$,

Then, f *has a unique fixed point* $x^* \in X$ *and the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *of successive approximations of* f *converges to* x^* .

Theorem 3.8. *[16] Let* X_1 *and* X_2 *be two nonempty sets endowed (resp.) with the separating generalized gauge structures and, respectively,* $\mathcal{P} = \{p_\alpha\}_{\alpha\in\Lambda}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in\Gamma}$ *(where* Λ *and* Γ *are directed sets), and denote* $\tilde{d}_{\alpha,\beta} : (X_1 \times X_2)^2 \to \mathbb{R}^2_+$

$$\tilde{d}_{\alpha,\beta}((x, y),(u, v)) := \begin{pmatrix} p_\alpha(x, u) \\ q_\beta(y, v) \end{pmatrix} \tag{3.19}$$

Let $f : X_1 \times X_2 \to X_1 \times X_2$ *be a continuous operator with respect to the product gauge structures* $\mathcal{P} \times \mathcal{Q}$ *We suppose that the following hold:*

i) $(X_1,\mathcal{P})$ *and* $(X_2,\mathcal{Q})$ *are sequentially complete generalized gauge spaces.*

ii) There exists a function $\varphi : \Lambda \times \Gamma \to \Lambda \times \Gamma$ *and* $M := \{M_{\alpha,\beta}\}_{\alpha\in\Lambda,\beta\in\Gamma} \in M_2(\mathbb{R}^+)^{\Lambda\times\Gamma}$ *such that, for every* $\alpha \in \Lambda$ *and every* $\beta \in \Gamma$ *the following holds :*

$$\begin{aligned} &\tilde{d}_{\alpha,\beta}(f(z), f(w)) \le M_{\alpha,\beta} \tilde{d}_{\varphi(\alpha),\varphi(\beta)}(z, w) \\ &\qquad\qquad\qquad\qquad \forall z, w \in X_1 \times X_2 \end{aligned} \tag{3.20}$$

iii) $\sum_{k=1}^{\infty} M_\beta M_{\varphi(\beta)} \cdots M_{\phi^{k-1}(\beta)} q_{\varphi^k(\beta)}(u, v) < \infty$ *for each* $\alpha \in \Lambda$ *and* $\beta \in \Gamma$.

iv) There exists $z_0 \in$ and $s^0 \in \mathbb{R}^2_+$ (with $0 \lhd s^0$), such that

$$\tilde{d}_{\alpha,\beta}\left(z_0, f\left(z_0\right)\right) \leq s^0 \tag{3.21}$$

Then, there exists a unique $z^* = (x^*, y^*) \in X_1 \times X_2$ such that $z^* = f(z^*)$ and the sequence $f^n(z_0) := (f_1^n(z_0), f_2^n(z_0))_{n\in\mathbb{N}}$ converges to z^*, where $f_1^0(x,y) = x$, $f_2^0(x,y) = y$ and

$$\begin{aligned} f_1^n(z) &:= f_1^{n-1}\left(f_1(z), f_2(z)\right) \\ f_2^n(z) &:= f_2^{n-1}\left(f_1(z), f_2(z)\right) \end{aligned} \tag{3.22}$$

for all $n \in \mathbb{N}^*$.
Moreover, for every $\alpha \in \Lambda$ and $\beta \in \Gamma$ we have the following estimation:

$$\tilde{d}_{\alpha,\beta}\left(f^n\left(z_0\right), z^*\right) \leq M^n(I-M)^{-1}\tilde{d}_{\alpha,\beta}\left(z_0, f\left(z_0\right)\right) \tag{3.23}$$

3.2 Application to integral systems

3.2.1 A system of integral equations with advanced argument

Consider the system of integral equations with advanced argument

$$\begin{cases} x(t) = \int_{t-1}^{t} f(s, x(s+2), y(s+2))\mathrm{d}s \\ y(t) = \int_{t-1}^{t} g(s, x(s+2), y(s+2))\mathrm{d}s \end{cases} \tag{3.24}$$

for $t \in [0,\infty)$. Assum that

$$\begin{aligned} |f(t,x,y) - f\left(t,x_1,y_1\right)| &\leq k_1(t)\left|x-x_1\right| + k_2(t)\left|y-y_1\right| \\ |g(t,x,y) - g\left(t,x_1,y_1\right)| &\leq k_3(t)\left|x-x_1\right| + k_4(t)\left|y-y_1\right| \end{aligned} \tag{3.25}$$

for every $x, x_1, y, y_1 \in \mathbb{R}, t \in [-1,\infty)$ and some $k_i \in L^1\left([-1,\infty), \mathbb{R}_+\right)$, i = 1, 2, 3, 4. For each $n \in \mathbb{N}$, let

$$\begin{aligned} a_n &= \int_{n-1}^{2n+1} k_1(t)\mathrm{d}t, \quad b_n = \int_{n-1}^{2n+1} k_2(t)\mathrm{d}t \\ \bar{a}_n &= \int_{n-1}^{2n+1} k_3(t)\mathrm{d}t, \quad \bar{b}_n = \int_{n-1}^{2n+1} k_4(t)\mathrm{d}t \end{aligned}$$

and consider the matrix

$$M_n = \begin{bmatrix} a_n & b_n \\ \bar{a}_n & \bar{b}_n \end{bmatrix}$$

Also define the matrix M_∞ by

$$M_\infty = \begin{bmatrix} |k_1|_{L^1([-1,\infty))} & |k_2|_{L^1([-1,\infty))} \\ |k_3|_{L^1([-1,\infty))} & |k_4|_{L^1([-1,\infty))} \end{bmatrix}$$

Our main result on system (3.24) is the following theorem.

Theorem 3.9. *[13] Let $f, g : [-1,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions and assume that inequalities (3.25) hold for some $k_i \in L^1([-1,\infty), \mathbb{R}_+)$,i=1,2,3,4.In addition assume that there is $u_0 = (x_0, y_0) \in C([0,\infty), \mathbb{R}^2)$ and $C > 0$ such that*

$$|T(u_0)(t) - u_0(t)| \leq C \quad \textit{for all} \quad t \in [0,\infty) \tag{3.26}$$

where $T = (A,B)$ is given bellow. If the matrix

$$M_\infty \textit{ is convergent to zero,} \tag{3.27}$$

then system (3.24) has at least one solution $(x, y) \in C([0,\infty), \mathbb{R}^2)$.

Proof. We shall use Theorem (3.4)Here $X = C[0,\infty)$,$\Lambda = \mathbb{N}$ and for $n \in \mathbb{N}$,$d_n : X \times X \to \mathbb{R}_+$ is given by

$$d_n(x,y) = \max_{t\in[n,2n+1]} |x(t) - y(t)|.$$

Let $A,B : C([0,\infty), \mathbb{R}^2) \to C[0,\infty)$ be defined by

$$A(x,y)(t) = \int_{t-1}^{t} f(s, x(s+2), y(s+2))\mathrm{d}s$$
$$B(x,y)(t) = \int_{t-1}^{t} g(s, x(s+2), y(s+2))\mathrm{d}s$$

First we prove the Lipschitz condition (3.5) with $\varphi : \mathbb{N} \to \mathbb{N}$ given by $\varphi(n) = n + 1$. Let $t \in [n, 2n+1]$. We have $t-1 \in [n-1, 2n]$, and when $s \in [t-1, t]$, then $s+2 \in [n+1, 2n+3]$.

It follows that

$$\begin{aligned}
|A(x,y)(t)-A\left(x_1,y_1\right)(t)| \leq & \int_{n-1}^{2n+1}\left|f(s,x(s+2),y(s+2))-f\left(s,x_1(s+2),y_1(s+2)\right)\right| \mathrm{ds} \\
\leq & \int_{n-1}^{2n+1} k_1(s)\left|x(s+2)-x_1(s+2)\right| \mathrm{ds} \\
& +\int_{n-1}^{2n+1} k_2(s)\left|y(s+2)-y_1(s+2)\right| \mathrm{ds} \\
\leq & \max_{s\in[n-1,2n+1]}\left|x(s+2)-x_1(s+2)\right| \int_{n-1}^{2n+1} k_1(s) \mathrm{ds} \\
& +\max_{s\in[n-1,2n+1]}\left|y(s+2)-y_1(s+2)\right| \int_{n-1}^{2n+1} k_2(s) \mathrm{ds} \\
\leq & \max_{\tau\in[n+1,2n+3]}\left|x(\tau)-x_1(\tau)\right| \int_{n-1}^{2n+1} k_1(s) \mathrm{ds} \\
& +\max_{\tau\in[n+1,2n+3]}\left|y(\tau)-y_1(\tau)\right| \int_{n-1}^{2n+1} k_2(s) \mathrm{ds} \\
= & a_n d_{n+1}(x,x_1)+b_n d_{n+1}(y,y_1).
\end{aligned}$$

Taking the maximum over $[n, 2n+1]$ yields

$$d_n(A(x,y),A(x_1,y_1)) \leq a_n d_{n+1}(x,x_1)+b_n d_{n+1}(y,y_1)$$

for every $(x,y),(x_1,y_1)\in X^2$. Similarly, for B,

$$d_n(B(x,y),B(x_1,y_1)) \leq a_n d_{\varphi(n)}(x,x_1)+b_n d_{\varphi(n)}(y,y_1) \tag{3.28}$$

for every $(x,y),(x_1,y_1)\in X^2$. Hence (3.5) holds. Furthermore, condition (3.8) is guaranteed by assumption (3.26). Also, for every $n\in\mathbb{N}$, $M_n \leq M_\infty$ and thus series (3.9) is dominated by

$$\sum_{k=0}^{\infty} M_\infty^k$$

which is convergent in view of assumption (3.27). Hence (3.9) is satisfied. Therefore Theorem 3.4 can be applied. □

3.2.2 An integral system without modification of the argument

Consider the system of integral equations

$$\begin{cases} x(t)=\int_{t-1}^{t} f(s,x(s),y(s))\mathrm{ds} \\ \\ y(t)=\int_{t-1}^{t} g(s,x(s),y(s))\mathrm{ds} \end{cases} \tag{3.29}$$

for $t \in [0,\infty)$, where

$$x(t) = \psi(t) \quad \text{and } y(t) = \phi(t) \quad \text{for } t \in [-1,0]$$

and ψ, ϕ are two given functions. We assume that inequalities (3.25) hold for every $x, x_1, y, y_1 \in \mathbb{R}, t \in [0,\infty)$ and some $k_i \in L^1_{locc}([0,\infty), \mathbb{R}_+)$, $i = 1{,}2{,}3{,}4$. For $n \in \mathbb{N}\backslash\{0\}$, we denote

$$a_n = \int_0^n k_1(t)\mathrm{d}t, \quad b_n = \int_0^n k_2(t)\mathrm{d}t$$

$$\bar{a}_n = \int_0^n k_3(t)\mathrm{d}t, \quad \bar{b}_n = \int_0^n k_4(t)\mathrm{d}t$$

and we consider the matrix

$$M_n = \begin{bmatrix} a_n & b_n \\ \bar{a}_n & \bar{b}_n \end{bmatrix}.$$

Theorem 3.10. *[13] Let $f, g : [-1,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions, $\phi, \psi \in C[-1,0]$ and assume that inequalities (3.25) hold for some $k_i \in L^1_{locc}([0,\infty), \mathbb{R}_+)$, $i =$ 1,2,3,4. If for every $n \in \mathbb{N}\backslash\{0\}$, matrix*

$$M_n \text{ is convergent to zero,} \tag{3.30}$$

then system (3.29) has a unique solution $(x, y) \in C([0,\infty),\mathbb{R}^2)$.

Proof. The result follows from Theorem 3.3 if we take into account Remark 1 about Tarafdar's situation. Here $X = C[0,\infty), \Lambda = \mathbb{N}\backslash\{0\}$, for each $n \in \mathbb{N}\backslash\{0\}$, $d_n : X \times X \to \mathbb{R}_+$ is given by

$$d_n(x, y) = \max_{t\in[0,n]} |x(t) - y(t)|$$

$\varphi : \mathbb{N}\backslash\{0\} \to \mathbb{N}\backslash\{0\}$, $\varphi(n) = n$, and $A,B : C([0,\infty),\mathbb{R}^2) \to C[0,\infty)$ are defined by

$$A(x, y)(t) = \int_{t-1}^t f(s, \tilde{x}(s), \tilde{y}(s))\mathrm{d}s$$

$$B(x, y)(t) = \int_{t-1}^t g(s, \tilde{x}(s), \tilde{y}(s))\mathrm{d}s$$

where

$$\tilde{x}(t) = \begin{cases} \psi(t) & \text{for } -1 \leqslant t < 0 \\ x(t) & \text{for } t \geqslant 0, \end{cases} \qquad \tilde{y}(t) = \begin{cases} \phi(t) & \text{for } -1 \leqslant t < 0 \\ y(t) & \text{for } t \geqslant 0 \end{cases}$$

□

Conclusion and perspective

The scientific basis of the fixed point theory was established in the 20th century. The fundamental result of this theory is the Banach contraction principle (from the 30s), which generated important lines of research and applications of the theory of functional equations, differential equations, integral equations .

In this work, We studied some fixed point theorems on some generalized spaces, and we saw how the contraction condition in Banach fixed point replaced by generalized contraction. Besides that we show some application of fixed point theorems in previous cases to find a solution for the existence and uniqueness of some systems.

We hope for the future, we will study new fixed point theorems in metric generalized space and generalized gauge space endowed with a graph.

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ملخص

الهدف من هذه المذكرة هو دراسة بعض نظريات النقطة الثابتة في فضاءات مترية معممة وكذلك في فضاءات القياس المعممة وقد ركزنا على نظرية برووف و امتداداتها ، كما استخ□نا بعض هذه النظريات لإثبات وجود حلول لمعادلات تفاضلية ذات نبض وأخرى تكاملية.

الكلمات المفتاحية :

فضاء متري معمم، تقارب المصفوفات، نظرية النقطة الثابتة ، نظرية بـرووف، فضاء القياس المعممة ، لمعادلات تفاضلية ذات نبض، معادلات تكاملية.

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*Abstrac**t***

The propose of this memoir is to study some fixed point theories in generalized metric spaces as well as in the generalized gauge spaces, We focused on perov's theorms and its extensions . Besides that, we applicable these results to prove the ..existence of solutions to differential equations with impulse and equations integrals.

Key Words .

Metric generalized space, matrix convergence , ,Fixed point theorem, theorem Perov, generalized gauge spaces, differential equations with impulse ,equations integrales .

Résumé

Le but de ce mémoire est d'étudier quelques théories du point fixe dans les espaces métriques généralisés ainsi que dans les espaces de gauge généralisés , nous sommes concentrés sur la théorie de Perov et ses extensions. Ensuite on applique certaines de ces théories pour montrer l'existence de solutions aux équations différentielles avec impulsion et intégrales.

Mots clés

Espace métrique généralisé , convergence matricielle, theorem de point fixe, theorem de Perov, espace de gauge généralisé, équation différentielle avec impulsion, équation integrale.