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This study

is dedicated to

my beloved parents,

who have been my source of

inspiration and give me strength

when I thought of giving up, who contin-

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and financial support. To my brothers, sisters, rela-

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course to all my

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NOURA.

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Contents

Symbol Description

The most frequently used notations, symbols, and abbreviations are listed below

$d(x,y)$	The distance between x and y.
(X,d)	The metric space X.
$(X,\|.\|)$	The linear normed space X.
$\mathcal{B}_X$	The family of all bounded sets in X.
$\mathcal{N}_X$	The family of all relatively compact sets in X.
$\overline{M}$	Closure of M with respect to the norm topology .
$\mathrm{co}(M)$	The convex hull of M.
$\overline{\mathrm{co}}(M)$	The closed convex hull of M.
$\mathrm{diam}(M)$	The diameter of the set A, where A is a subset of a metric space X.
$\mathfrak{B}_X$	The unit ball of X.
$\mathfrak{B}(x,r)$	The ball centred by x with radius r, $\mathfrak{B}_r$ if $x=0$.
MNC	Measure of noncompactness.
α	The Kuratowski MNC.
χ	The Hausdorff MNC.
X^*	Dual topology of X.
X^{**}	Bidual topology of X.
$\langle\cdot,\cdot\rangle_{X^*,X}$	The bilinear mapping from $X^* \times X$ into $\mathbb{R}$ (Duality bracket).
$\sigma(X,X^*)$	The weak topology.
$\sigma(X^*,X)$	The weak star topology.
$x_n \to x$	x_n converges strongly to x.
$x_n \rightharpoonup x$	x_n converges weakly to x.
$\overline{M}^w$	Closures of M with respect to the weak topology .
MWNC	Measure of weak noncompactness.
$\mathcal{W}_X$	The family of relatively weakly compact sets in X.
ω	The De Blasi MWNC.
$\mathcal{D}(T)$	The domaine definition of the operator T.
BOM	Block Operator Matrix.
$\mathrm{meas}(.)$	The Lebesgue measure.
L^1	The vector space of classes of functions whose absolutely integrable in the sense of Lebesgue.
$\mathcal{C}(J,X)$	The set of continuous functions from J into X.

Introduction

Fixed point theory is a powerful and fruitful tool in modern mathematics and may be considered as a core subject in nonlinear analysis. In the last 50 years, fixed point theory has been an active area of research with a wide range of applications in several fields. In fact, this theory constitutes an harmonious mixture of analysis (pure and applied), topology, and geometry. In particular, it has several important applications in various fields, such as physics, engineering, game theory, and biology (see [3]-[7]).

It has been observed that many, if not a majority, of equations can be modified to fit the following general scheme. We are given a set $\mathcal{M}$ and a transformation T which assigns to each $x \in \mathcal{M}$ a point $y = Tx \in \mathcal{M}$: The solutions we seek are represented by points invariant under T: These are the points satisfying

$$x = Tx. \tag{1}$$

A fixed point theorem is a result saying that a mapping T will have at least one fixed point under some conditions. Depending on the nature of this conditions, we can divide fixed point theory into two main branches[28] :

- In the first one, we may consider results which may be deduced from metric assumptions.
- In the second one, results are obtained using topological and geometrical properties of the set $\mathcal{M}$.

Perhaps, the most well-known result in the first kind is *Banach's contraction principle*. More precisely, in 1922, S. Banach formulated and proved a theorem which focused, under appropriate conditions, on the existence and uniqueness of a fixed point in a complete metric space [29]. Concerning the second branch, the main two results are *Brower's fixed point theorem* in 1910 and its infinite dimensional version, *Schauder's fixed point theorem* in 1930 [29]. In both theorems, notice that compactness plays an essential role.

Since in some Banach spaces we do not know the complete description of the family of all relatively compact sets in those spaces, in 1930 K. Kuratowski defined a function α on the family of all bounded set of metric space into the real half-axis, this function is the first *Measure of Noncompactness*. This latter became a very important branch of nonlinear functional analysis, are widely applied in fixed point theory, as like *Darbo's fixed point theorem* in 1955 [8] and *Sadoveskii's fixed point theorem* in 1967 [31] which generalized *Schauder's fixed point theorem*.

From a mathematical point of view, many problems arising from diverse areas of natural sciences involve the existence of solutions of nonlinear equations having the following form

$$Tx + Sx = x, \quad x \in \mathcal{M}; \tag{2}$$

where $\mathcal{M}$ is a nonempty, closed, and convex subset of a Banach space X, and where $T, S : \mathcal{M} \to X$ are two nonlinear mappings. In 1958, M. A. Krasnoselskii [29] combined between the two branches of the fixed point theory to solve equation (2); actually he combined the *Banach contraction principle* and the *Schauder fixed point theorem* where

(i) S is compact and continuous;

(ii) T is a contraction; and

(iii) $Tx + Sy \in \mathcal{M} \quad (\forall x,y \in \mathcal{M})$.

Then there exists $x \in \mathcal{M}$ such that $Sx + Tx = x$.

The loss of compactness of mappings in some problems oblige us to focus on fixed point results under the weak topology. The *Measure of Weak Noncompactness* is a very important tool used in this case. This measure was first introduced by F. S. De Blasi in 1977 [17], who proved the analogousness of *Sadovskii's fixed point theorem* for the weak topology.

This memoir is intended to study some generalisation versions of fixed point theorem of Schauder and Krasnoselskii on Banach spaces and product of two Banach spaces furnished with its norm and weak topology using the most useful technique of Measure of (Weak) Noncompactness to find a solution for some nonlinear integral equations.

The content of the memoir is organized in four chapters. Each chapter contains a number of section of theorems and applications. Chapter 1 is devoted to discuss *Measures of (Weak) Noncompactness* in the Banach space and give the main and most frequently used measures of (weak) noncompactness. In addition, we investigate some definitions and fundamental theorem in weak topology and basic fixed point theorem in metric space.

In Chapter 2, we present some generalisation of Schauder fixed point theorem that is Darbo's and Sadoveskii's fixed point theorems using the measure of noncompactness. Moreover, we study the sufficient conditions which ensure the invertibility of $(I - T)$ to find some user-friendly versions of fixed point theorems for the equation (2), in the setting that the involved operators are not necessarily compact and continuous. Next, with this fixed point results, we try to prove the existence and uniqueness in special case of solutions for some kind of Volterra-Hammerstein's integral equation

$$x(t) = g(t, x(t)) + \lambda \int_a^t \kappa(t, s) f(s, x(s)) \mathrm{d}s, \tag{3}$$

posed in Banach space $X = \mathcal{C}([a, b],\mathbb{R})$ with the usual supremum norm $\|x\|_\infty = \max\limits_{t\in[a,b]} |x(t)|$, by imposing some conditions on f, g and κ.

In Chapter 3, we show some generalized fixed point results of the Schauder-Tychonoff and Krasnoselskii type in the context that the involved operators are not weakly compact, invoking the technique of measures of weak noncompactness in Banach spaces. Finally, an application in Hammerstein's integral equation

$$x(t) = g(t, x(t)) + \lambda \int_0^1 \kappa(t, s) f(s, x(s)) \mathrm{d}s, \tag{4}$$

in $L^1(0,1)$, the space of Lebesgue integrable functions on (0,1) with values in $\mathbb{R}$. Here f, g and κ verify some conditions.

In the last Chapter , we study a coupled system of nonlinear functional integral equations in suitable Banach spaces. This system is reduced to a fixed point problem for a 2×2 block operator matrix with nonlinear inputs.

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{5}$$

We starting by given some assumptions on its entries under a strong topology and then a weak topology setting. Habitually, the last section is an application, we prove the existence of a solution of the following system of nonlinear integral:

$$\begin{cases} x(t) = \underbrace{f(t, x(t))}_{Ax(t)} + \underbrace{\left[\left(\int_0^{\sigma_1(t)} \kappa(t,s) f_1(s, y(\eta(s)))ds\right) \cdot u\right]}_{By(t)}, \\ y(t) = \underbrace{\left[\left(q(t) + \int_0^{\sigma_2(t)} p(t,s,x(s),x(\lambda s))ds\right) \cdot v\right]}_{Cx(t)} + \underbrace{g(t, y(t))}_{Dy(t)}. \end{cases} \tag{6}$$

where $u \in X\backslash\{0\}$ and $v \in X\backslash\{0\}$. We will seek the solutions of this system in the space $\mathcal{C}(J, X)$ endowed with the norm $\|\cdot\|_\infty$, X is a Banach space and f, σ_1, k, f_1, η, q, σ_2, p, g are given and verified some conditions.

Chapter 1

Basic Concepts

The study of fixed point need a lot of prerequisites from the general theories of topological notions and nonlinear operators. In this chapter we discuss some concepts needed for the results presented in this memoir.

1.1 Measure of Noncompactness

1.1.1 The General Notion of Measure of Noncompactness

The notion of measure of noncompactness was originally introduced in metric spaces. In this memoir we are going to use an useful axiomatic definition for Banach spaces, which was introduced in 1980 by Banaś and Goebel [9].

Definition 1.1. [9] Let X be a Banach space and $\mathcal{B}_X$ the family of all bounded subsets of X. A map

$$\mu : \mathcal{B}_X \longrightarrow [0; +\infty),$$

is called a measure of noncompactness (for short MNC) defined on X if it satisfies the following properties:

i) The family $\ker \mu(B) = \{B \in \mathcal{B}_X : \mu(B) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_X$.

ii) Monotonicity: $A \subset B \Rightarrow \mu(A) \leq \mu(B)$, for all A, $B \in \mathcal{B}_X$.

iii) Invariant under closure and convex hull: $\mu(B) = \mu(\overline{B}) = \mu(\mathrm{co}(B))$, for all $B \in \mathcal{B}_X$.

iv) Convexity: $\mu(\lambda A + (1-\lambda)B) \leqslant \lambda\mu(A) + (1-\lambda)\mu(B)$, for all $\lambda \in [0,1]$, and $A, B \in \mathcal{B}_X$.

v) Generalized Cantor intersection property: if $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty, closed and bounded subsets of X with $\lim\limits_{n\to\infty}\mu(B_n) = 0$, then the intersection B_∞ of all B_n is nonempty and compact.

Proposition 1.1. *[31] Let X be a Banach space and let $B \subset \mathcal{B}_X$,the map:*

$$\mathrm{diam}(B) = \begin{cases} 0, \textit{ if } B \textit{ is empty} \\ \sup\{\|x-y\| : x,y \in B\}, \textit{ else} \end{cases}$$

is MNC.

Proof. In fact:

i) The family $\ker \mathrm{diam}(B)$ is nonempty because for all B consists exactly one point $\mathrm{diam}(B) = 0$.

ii) Let A, B two bounded subsets such that $A \subset B$ so,
$\sup\{\|x-y\| : x,y \in A\} \leq \sup\{\|x-y\| x,y \in B\}$ which means $\mathrm{diam}(A) \leq \mathrm{diam}(B)$.

iii) a. We have $B \subset \overline{B}$ so, by monotonicity we get $\mathrm{diam}(B) \leq \mathrm{diam}(\overline{B})$.
On the other hand, let $\varepsilon > 0$ and $\overline{x}$, $\overline{y} \in \overline{B}$ so, there exist x, $y \in B$ such that $x \in \mathfrak{B}(\overline{x}, \frac{\varepsilon}{2})$, $y \in \mathfrak{B}(\overline{y}, \frac{\varepsilon}{2})$; which means $\|x - \overline{x}\| < \frac{\varepsilon}{2}$ and $\|y - \overline{y}\| < \frac{\varepsilon}{2}$. The use of triangular inequality give us:

$$\begin{aligned} \|\overline{x} - \overline{y}\| &= \|\overline{x} - \overline{y} + x - x + y - y\| \\ &\leq \|\overline{x} - x\| + \|x - y\| + \|\overline{y} - y\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \|x - y\| \\ &< \varepsilon + \|x - y\|, \quad \forall \varepsilon > 0, \end{aligned}$$

so,

$$\mathrm{diam}(\overline{B}) \leq \mathrm{diam}(B) + \varepsilon, \quad \forall \varepsilon > 0.$$

Now, let ε tends to 0 we get $\mathrm{diam}(\overline{B}) \leq \mathrm{diam}(B)$; we infer the equality.

b. Let x, $y \in \text{co}(B)$. Then $x = \sum_{i=1}^{n} t_i x_i$ and $y = \sum_{j=1}^{m} s_j y_j$ with $x_i, y_j \in B$, $\sum_{i=1}^{n} t_i = 1$ and $\sum_{j=1}^{m} s_j = 1$. Thus:

$$\begin{aligned}
\|x - y\| &= \left\| \sum_{i=1}^{n} t_i x_i - \sum_{j=1}^{m} s_j y_j \right\| \\
&= \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} t_i s_j x_i - \sum_{j=1}^{m} \sum_{i=1}^{n} s_j t_i y_j \right\| \\
&\leq \sum_{j=1}^{m} \sum_{i=1}^{n} s_j t_i \|x_i - y_j\| \\
&\leq \underbrace{\sum_{j=1}^{m} \sum_{i=1}^{n} s_j t_i}_{=1} \operatorname{diam}(B) \\
&= \operatorname{diam}(B).
\end{aligned}$$

and it follows that $\operatorname{diam}(\text{co}(B)) \leq \operatorname{diam}(B)$. Since the opposite inequality is obvious, we infer that the diameter is invariant under convex hull.

iv) It's clear that $\operatorname{diam}(A + B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$ $\forall A,\ B \in \mathcal{B}_X$. Moreover we have:

$$\begin{aligned}
\operatorname{diam}(\lambda B) &= \sup\{\|\lambda x - \lambda y\| : x, y \in B\} \\
&= |\lambda| \sup\{\|x - y\| : x, y \in B\} \\
&= |\lambda| \operatorname{diam}(B) \quad \forall B \in \mathcal{B}_X, \forall \lambda \in \mathbb{R}.
\end{aligned}$$

So, for all $\lambda \in [0,1]$:

$$\begin{aligned}
\operatorname{diam}(\lambda A + (1 - \lambda)B) &\leq \operatorname{diam}(\lambda A) + \operatorname{diam}((1 - \lambda)B) \\
&= \lambda \operatorname{diam}(A) + (1 - \lambda) \operatorname{diam}(B).
\end{aligned}$$

v) Cantor's intersection theorem: If $\{B_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n \to \infty} \operatorname{diam}(B_n) = 0$ so, it is either empty or consists of a single point. So it is sufficient to show that it is not empty. Pick an element x_n of B_n for each n. Since the diameter of B_n tends to zero and the B_n are decreasing, the $\{x_n\}_{n \in \mathbb{N}}$ form a Cauchy sequence. Since the Banach space is complete this Cauchy sequence converges to some point x. But each B_n is closed, and x is a limit of a sequence in B_n, x must lie in B_n. This is true for every n, and therefore the intersection of the B_n must contain x, then the intersection B_∞ of all B_n is nonempty and consists of exactly one point. □

Definition 1.2. [11] Let μ be a measure of noncompactness in the Banach space X. We will call the measure μ semi-homogeneous if:

vi) $\mu(\lambda B) = |\lambda|\mu(B)$ for $\lambda \in \mathbb{R}$.

If it satisfies the condition:

vii) $\mu(A + B) \leq \mu(A) + \mu(B)$

it is called sub-additive. The measure μ being both semi-homogeneous and sub-additive is said to be sub-linear.

Definition 1.3. [11] We say that a measure of noncompactness μ has the maximum property (or it is semi-additive) if:

viii) $\mu(A \cup B) = \max\{\mu(A),\mu(B)\}$.

The most important class of measures of noncompactness is described in the below given definition.

Definition 1.4. [11] A sub-linear measure of noncompactness μ which has the maximum property and is such that $\ker \mu = \mathcal{N}_X$ (fullness) is called regular measure of noncompactness.

Remark 1.1. Diameter is MNC non regular because it hasn't the maximum property and isn't full (i.e., $\ker \operatorname{diam} \subsetneq \mathcal{N}_X$).

1.1.2 The Kuratowski and Hausdorff Measures of Noncompactness and its Properties.

The most important examples of measures of noncompactness are

Definition 1.5. [11] Let (X,d) be a complete metric space. We define the function α of Kuratowski of the set $B \in \mathcal{B}_X$ the infimum of the numbers $\varepsilon > 0$ such that B admits a finite covering by sets of diameter smaller than ε, i.e.,

$$\alpha(B) = \inf\left\{\varepsilon > 0 : B \subset \bigcup_{i=1}^{n} S_i : S_i \subset X,\ \operatorname{diam}(S_i) < \varepsilon,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N}\right\}.$$

Definition 1.6. [11] Let (X, d) be a complete metric space. We define the function χ of Hausdorff of the set $B \in \mathcal{B}_X$ the infimum of the numbers $\varepsilon > 0$ such that B can be covered by finitely many balls of radius smaller than ε, i.e.,

$$\chi(B) = \inf\left\{\varepsilon > 0 : B \subset \bigcup_{i=1}^{n} \mathfrak{B}(x_i, r_i),\ x_i \in X,\ r_i < \varepsilon, i = 1, \ldots, n,\ n \in \mathbb{N}\right\}.$$

if X is a Banach space the definition of χ is equivalent to the following:

$$\begin{aligned}\chi(B) &= \inf\{\varepsilon > 0 : B \text{ has a finite } \varepsilon\text{-net}\}, \\ &= \inf\{\varepsilon > 0 : B \subset S + \varepsilon\mathfrak{B}_X, S \subset X,\ S \text{ is finite }\}.\end{aligned}$$

Proposition 1.2. *[11] The function α of Kuratowski is a non-singular regular measure of noncompactness in Banach space.*

Proof. **a) Fullness:** We want to prove that $\alpha(B) = 0 \Longleftrightarrow B$ is relatively compact:

$$\begin{aligned} B \text{ is relatively compact in a complet metric space} &\Longleftrightarrow B \text{ it is totally bounded} \\ &\Longleftrightarrow \forall\varepsilon > 0; \exists\ S_i; i = 1...\text{n such} \\ &\text{that } B \subseteq \bigcup_{i=1}^{n} S_i \text{ and } \operatorname{diam}(S_i) < \varepsilon \\ &\Longleftrightarrow \alpha(B) = 0. \end{aligned}$$

Which means that $\ker \alpha = \mathcal{N}_X$.

b) Monotonicity: Let A, $B \in \mathcal{B}_X$ with $A \subset B$ since any cover$\{B_i\}_{i=1}^{n}$ of B is a cover of A so,

$$\alpha(A) \leq \alpha(B).$$

c) Invariant under closure: By monotonicity of α we have clearly $\alpha(B) \leq \alpha(\overline{B})$. Let $\varepsilon > 0$, S_i be a bounded subset of X with $\operatorname{diam}(S_i) < \varepsilon + \alpha(B)$ for $i = 1, 2, \ldots, n$, and $B \subset \bigcup_{i=1}^{n} S_i$. Then $\overline{B} \subset \bigcup_{i=1}^{n} \overline{S_i}$. By proof of Proposition 1.1 "iii)-a." we get $\operatorname{diam}(S_i) = \operatorname{diam}(\overline{S_i})$ we conclude $\alpha(\overline{B}) \leq \alpha(B)$. So α invariant under closure.

d) Maximum property : By monotonicity we have $\alpha(A) \leq \alpha(A \cup B)$ and $\alpha(B) \leq \alpha(A \cup B)$ so,

$$\max\{\alpha(A),\alpha(B)\} \leq \alpha(A \cup B).$$

In the other hand, let $\max\{\alpha(A), \alpha(B)\} = s$ and $\varepsilon > 0$. By definition of α we know that A and B can be covered by a finite number of subsets of diameter smaller than $s + \varepsilon$. Obviously, the union of these covers is a finite cover of $A \cup B$. Hence we have $\alpha(A \cup B) \leq s + \varepsilon$, and now we obtain the semi-additivity of α.

e) Now we want to proof that α is **sub-linear:**

1. **Semi-homogeneity:**
Let S_i be a bounded subset of X with $\operatorname{diam}(S_i) < \alpha(B) + \varepsilon$ for $i = 1, 2, \ldots, n$ and $B \subset \bigcup_{i=1}^{n} S_i$. Then for any $\lambda, \lambda B \subset \bigcup_{i=1}^{n} \lambda S_i$ and we have proved that $\operatorname{diam}(\lambda S_i) = |\lambda| \operatorname{diam}(S_i)$. Hence it follows that $\alpha(\lambda B) \leq |\lambda|\alpha(B)$. if $\lambda = 0$ the claim is obvious, if not , analogously we have $\alpha(B) = \alpha(\lambda^{-1}(\lambda B)) \leq |\lambda^{-1}|\alpha(\lambda B)$, that is, $|\lambda|\alpha(B) \leq \alpha(\lambda B)$. This proves the semi-homogeneity.

2. **Algebraic sub-additivity:**
Let S_i be a bounded subset of X with $\operatorname{diam}(S_i) < \alpha(A) + \varepsilon$ for each $i = 1, 2, \ldots, n$ and $A \subseteq \bigcup_{i=1}^{n} S_i$. Furthermore, let G_i be a bounded subset of X with $\operatorname{diam}(G_j) < \alpha(B) + \varepsilon$ for each $j = 1, \ldots, m$ and $B \subseteq \bigcup_{j=1}^{m} G_j$. Then $A + B \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (S_i + G_j)$ and
$$\operatorname{diam}(S_i + G_j) < \alpha(A) + \alpha(B) + 2\varepsilon, \quad \text{for all } \varepsilon > 0.$$
Let ε tends to 0, and this shows the inequality.

f) Invariant under convex hull: Clearly $\alpha(B) \leq \alpha(\operatorname{co}(B))$, and it suffices to show $\alpha(\operatorname{co}(B)) \leq \alpha(B)$. Let $\{S_i\}_{i=1}^{n}$ be a bounded subset of X with $\operatorname{diam}(S_i) < d$ for each $i = 1, \ldots, n$ and $B = \bigcup_{i=1}^{n} S_i$. We can assume that every S_i is a convex set since $\operatorname{diam}(\operatorname{co}(S_i)) = \operatorname{diam}(S_i)$. By definition of the convex hull we obtain:

$$\operatorname{co}(B) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in S_i, 1 \leq i \leq n \right\}. \tag{1.1}$$

Let $\epsilon > 0$ and $S = \left\{(\lambda_1, \ldots, \lambda_n) : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0,\ i = 1, \ldots, n\right\}$. Then S is a compact subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|(\lambda_1, \ldots, \lambda_n)\|_\infty = \max_{1\leq i\leq n} |\lambda_i|$.

We put

$$M = \sup\left\{\|x\| : x \in \bigcup_{i=1}^{n} S_i\right\}.$$

And let

$$T = \{(t_{j,1}, \ldots, t_{j,n}) : j = 1, \ldots, m\} \subset S$$

be a finite $\frac{\epsilon}{(Mn)}$-net for S, with respect to the $\|.\|_\infty$-norm.

Hence if $\sum_{i=1}^{n} \lambda_i x_i$ is a convex combination of elements of B, where we suppose that $x_i \in S_i$ for $i = 1, \ldots, n$, then there exists $(t_{j,1}, \ldots, t_{j,n}) \in T$ such that

$$\|(\lambda_1, \ldots, \lambda_n) - (t_{j,1}, \ldots, t_{j,n})\|_\infty < \frac{\epsilon}{Mn}, \tag{1.2}$$

since

$$\sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} t_{j,i} x_i + \sum_{i=1}^{n} (\lambda_i - t_{j,i})\, x_i, \tag{1.3}$$

it follows from (1.1), (1.2), and (1.3) that

$$\mathrm{co}(B) \subset \bigcup_{j=1}^{m} \left\{\sum_{i=1}^{n} t_{j,i} S_i\right\} + \frac{\epsilon}{Mn} \sum_{i=1}^{n} A_i,$$

where $A_i = \{x \in X : \|x\| \leq M\}$ for $i = 1,2, \ldots, n$. Now we have since α is monotone, sub-linear and has the maximum property:

$$\begin{aligned}
\alpha(\mathrm{co}(B)) &\leq \alpha\left(\bigcup_{j=1}^{m} \left\{\sum_{i=1}^{n} t_{j,i} S_i\right\} + \frac{\epsilon}{Mn} \sum_{i=1}^{n} A_i\right) \\
&\leq \alpha\left(\bigcup_{j=1}^{m} \left\{\sum_{i=1}^{n} t_{j,i} S_i\right\}\right) + \alpha\left(\frac{\epsilon}{Mn} \sum_{i=1}^{n} A_i\right) \\
&\leq \max_{1\leq j\leq m} \alpha\left(\sum_{i=1}^{n} t_{j,i} S_i\right) + \frac{\epsilon}{Mn} \sum_{i=1}^{n} \alpha\left(A_i\right) \\
&\leq \max_{1\leq j\leq m} \sum_{i=1}^{n} t_{j,i} \alpha\left(S_i\right) + \frac{\epsilon}{Mn} 2nM \\
&< d \max_{1\leq j\leq m} \sum_{i=1}^{n} t_{j,i} + 2\epsilon < d + 2\epsilon.
\end{aligned}$$

So α is invariant under convex hull.

g) Non-singularity:

Let $B = \{x_i\}_{i=1}^{n}$ and we know that any singleton $\{x_i\}$ is a compact subset so, $\alpha(\{x_i\}) = 0 \ \forall i = 1 \ldots n$, and using the semi-additivity we obtain

$$\alpha(B) = \alpha(\bigcup_{i=1}^{n} x_i) = \max\{\alpha(x_i) : i = 1,...,n\} = \max\{0,0,...,0\} = 0.$$

h) Verify the generalization Cantor's intersection theorem: Let $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(B_n) = 0$ and let $\{x_i\}_{i=1}^{\infty}$ be a sequence such that $x_n \in B_n \ \forall \ n \in \mathbb{N}$ and $C_n = \{x_i\}_{i\geq n}$, so, C_n is also a decreasing sequence verifies $C_n \subset B_n \ \forall n \in \mathbb{N}$ and we have :

$$\alpha(C_0) = \alpha(x_0 \cup C_1) = \max\{\alpha(x_0),\alpha(C_1)\} = \alpha(C_1).$$

So; for all $n \in \mathbb{N}$ $\alpha(C_0) = \alpha(C_n)$ and by monotonicity we infer $\alpha(C_0) = \alpha(C_n) \leq \alpha(B_n) \xrightarrow{n\longrightarrow\infty} 0$ which imply that C_0 is relatively compact set. Thus the sequence (x_n) has a convergent subsequence $\{x_{n_k}\}$ with $x = \lim x_{n_k} \in X$. Since B_n is closed in X, we get $x \in B_n$ for all $n \geqslant 0$, that is, $x \in B_\infty$, so, B_∞ is nonempty.

Moreover, as

$$\alpha(B_\infty) \leqslant \alpha(B_n) \xrightarrow{n\longrightarrow\infty} 0$$

which means that B_∞ is relatively compact,but $\{B_n\}_{n\geq 0}$ is closed so $B_\infty = \bigcap_{n=0}^{\infty} B_n$ is also closed, from all that we infer B_∞ is compact. $\square$

Theorem 1.1. *[31] Let $\mathfrak{B}_X$ be the unit ball in X. Then $\alpha(\mathfrak{B}_X) = \chi(\mathfrak{B}_X) = 0$ if X is finite-dimensional, and $\alpha(\mathfrak{B}_X) = 2$, $\chi(\mathfrak{B}_X) = 1$ in the opposite case.*

The next theorem shows that the functions α and χ are in some sense equivalent.

Theorem 1.2. *[31] Let X be a Banach space and B be a bounded subset of X. Then:*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite-dimensional spaces these inequalities are sharp.

1.2 Basics of weak topology

There are two classes of topologies that by and large include everything of interest. The first and most familiar is the class of topologies that are generated by a metric. The second class is the class of weak topologies. To present some fixed point theory in a Banach spaces we need to understand this latter (which is different from the first topology and weaker than it). The "weak topologies" arise naturally in this setting which is the subject of the present section.

1.2.1 Weak and Weak* Topologies

Definition 1.7. Let X be a Banach space and X^* its dual. The weak topology denoted $\sigma(X, X^*)$, is the most economical topology in X, in the sense that it has the fewest open sets (ie., the coarsest topology) such that, each map $f : X \to \mathbb{R}$, $f \in X^*$, is continuous.

Proposition 1.3. *[15] The weak topology $\sigma(X, X^*)$ is Hausdorff.*

Proposition 1.4. *[15] Let (x_n) be a sequence in X. Then,*

i) $[x_n \rightharpoonup x$ *weakly in* $\sigma(X, X^*)] \Leftrightarrow [\langle f, x_n\rangle \to \langle f, x\rangle \quad , \forall f \in X^*]$,

ii) *If $x_n \to x$ strongly, then $x_n \rightharpoonup x$ weakly in $\sigma(X, X^*)$.*

Remark 1.2. When X is finite-dimensional, the weak topology $\sigma(X, X^*)$ and the usual topology are the same.

Definition 1.8. Let X be a Banach space, we recall that

- A sequence $\{x_n\}_{n\in\mathbb{N}}$ is weakly Cauchy if for every $x^* \in X^*$, the sequence $\{x^*(x_n)\}_{n\in\mathbb{N}}$ is Cauchy in the scalar filed.
- X is sequentially weakly complete[1] if any weakly Cauchy sequence in X is weakly convergent.

[1] Each reflexive Banach space is sequentially weakly complete, but there are nonreflexive spaces enjoying this property. For instance, $L^1(\Omega)$, with Ω measurable in $\mathbb{R}^n$, although non-reflexive, is sequentially weakly complete.

Definition 1.9. Let X, Y be tow Banach spaces and $T : X \to Y$ be a mapping. We say that:

- T is said to be weakly continuous if it is continuous from X weak $\sigma(X, X^*)$ into Y weak $\sigma(Y, Y^*)$.
- T is said to be sequentially weakly continuous, if for every sequence $x_n \subset X$ and $x \in X$ such that $x_n \rightharpoonup x$ we have that $Tx_n \rightharpoonup Tx$.
- T is said to be sequentially weakly-strongly continuous on X, if for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have $Tx_n \to Tx$.
- T is said to have sequentially closed graph if its graph $G(T)$ is sequentially closed in $X \times Y$.

The dual space X^* we may endow it with the weak topology, the weakest one such that all linear forms in X^{**} are continuous, on other words

Definition 1.10. The weak* topology of the dual space of a normed space X is the coarsest topology for X^* such that, for each x in X, the linear functional $x^* \mapsto \langle x^*, x\rangle$ on X^* is continuous.

Remark 1.3.

- All properties which are stated for weak topology of X can be adapted and hold for the weak* topology of X^*.
- Neither the weak topology on X nor the weak* topology on X^* is metrisable (except for the case in which X is finite dimensional).

Definition 1.11. Let X be a Banach space and let $J : X \to X^{**}$ be the canonical injection from X into X^{**} defined as follows: given $x \in X$ the map $f \longmapsto \langle f, x\rangle$ is a continuous linear functional on X^*, thus it is an element of X^{**}, which we denote by J_x. We have

$$\langle J_x, f\rangle_{X^{**}, X^*} = \langle f, x\rangle_{X^*, X}, \quad \forall x \in X,\ \forall f \in X^*.$$

The space X is said to be reflexive if J is surjective, i.e., $J(X) = X^{**}$.

1.2.2 Closedness and Compactness in Weak Topology

Theorem 1.3. *(Mazur's Theorem, [15]) The weak closure of every convex subset in a Banach space coincides with its strong closure.*

Definition 1.12. The set M is weakly compact, if it is compact in the topology $\sigma(X, X^*)$.

Theorem 1.4. *(Kakutani's Theorem, [15]) A Banach space X is reflexive if and only if its closed unit ball $\overline{\mathfrak{B}}_X$ is weakly compact.*

Corollary 1.1. *[4] Every closed, bounded, convex subset of a reflexive Banach space X is weakly compact.*

Proof. Let X be a Banach space. According to Kakutani's Theorem 1.4, the closed unit ball of X is weakly compact. Hence so, is any closed ball. According to Mazur's Theorem 1.3, every closed, convex subset of X is weakly closed. Therefore any closed, convex, bounded subset of X is a weakly closed subset of a weakly compact set and hence must be weakly compact. □

Now, we present the Eberlein–Šmulian's criteria for weak compactness of subsets of a Banach space.

Theorem 1.5. *(Eberlein-Šmulian's Theorem, [4]) Let A a subset of Banach space X the following assertions are equivalent*

(i) The set A is relatively weakly compact, i.e., A's closure is weakly compact,

(ii) The set A is relatively weakly sequentially compact, i.e., every sequence of members of A contains a subsequence weakly converging in X.

Some consequence of the Eberlein Šmulian Theorem is the following result.

Theorem 1.6. *(Krein-Šmulian's Theorem,[19] p. 434) The closed, convex hull of weakly compact subset of a Banach space is weakly compact.*

The following three theorems are fundamental tool for the proofs of the existence of solutions for several functional integral equations (see Chapter 4). The proof of those theorems can be found in [18] , [32] and [32] again respectively.

Theorem 1.7. *(Dobrakov's Theorem) Let K be a compact Hausdorff space and let X be a Banach space. Let $(f_n)_n$ be a bounded sequence in $\mathcal{C}(K, X)$, and $f \in \mathcal{C}(K, X)$. Then, $(f_n)_n$ is weakly convergent to f if, and only if, $(f_n(t))_n$ is weakly convergent to $f(t)$ for each $t \in K$.*

Theorem 1.8. *(Arzéla-Ascoli Theorem S) Let X be a Banach space. A subset F in $\mathcal{C}([a, b],X)$ is relatively compact if and only if.*

(i) F is equicontinuous on $[a, b]$, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in F$ whenever $|x - y| < \delta$, and $x, y \in [a, b]$,

(ii) There exists a dense subset D in $[a, b]$ such that, for each $t \in D$, $F(t) = \{f(t)|f \in F\}$ is relatively compact in X.

Theorem 1.9. *(Arzéla-Ascoli Theorem W) Let X be a sequentially weakly complete Banach space. A family F in the space $\mathcal{C}([a,b]; X)$, endowed with the uniform weak convergence topology, is sequentially relatively compact if and only if:*

(i) F is weakly equicontinuous on $[a, b]$, that's $\varphi(F)$ is equicontinuous for all $\varphi \in X^*$.

(ii) there exists a dense subset D in $[a, b]$ such that, for each $t \in D$, the section $F(t) = \{f(t)|\ f \in F\}$ is sequentially weakly relatively compact in X.

Lemma 1.1. *([19], p. 414) Let A be weakly closed, K is weakly compact subsets of a Banach space, then $A + K$ is weakly closed.*

1.3 Measure of Weak Noncompactness

1.3.1 The Axiomatic Measure of Weak Noncompactness

Following [12], we will adopt the following axiomatic approach of measure of weak noncompactness in several Banach spaces.

Definition 1.13. Let X be a Banach space and $\mathcal{B}_X$ the family of bounded subsets of X. A map

$$\psi : \mathcal{B}_X \longrightarrow [0, +\infty)$$

is called a measure of weak noncompactness (for short MWNC) defined on X if it satisfies the following conditions:

i) The family $\ker\psi(B) = \{B \in \mathcal{B}_X : \psi(B) = 0\}$ is nonempty and $\ker\psi \subset \mathcal{W}_X$.

ii) Monotonicity: $A \subset B \Rightarrow \psi(A) \leq \psi(B)$, for all A, $B \in \mathcal{B}_X$.

iii) Invariant under weak closure and convex hull $\psi(B) = \psi(\overline{B}^w) = \psi(\text{co}(B))$, for all$B \in \mathcal{B}_X$.

iv) Convexity: $\psi(\lambda A+(1-\lambda)B) \leqslant \lambda\psi(A)+(1-\lambda)\psi(B)$, for all$A$, $B \in \mathcal{B}_X$ and $\lambda \in [0, 1]$.

v) Generalized Cantor intersection property: If $(B_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with B_1 bounded and $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \ldots$ and such that $\lim_{n\to+\infty} \psi(B_n) = 0$, then the set $B_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty and weakly compact.

Remark 1.4. A measure of weak noncompactness is said to be regular if it has properties (i)-(v), full and has the maximum property. (see Definition 1.4)

1.3.2 The De Blasi Measure of Weak Noncompactness

Recall that the notion of the measure of weak noncompactness introduced by De Blasi [17]; this is the map defined as follows:

Definition 1.14. Let B be a non void bounded subset of Banach space X. The Blasi measure $\omega(B)$ of noncompactness of B in the weak topology is defined by:

$$\omega(B) = \inf\{t > 0 : \text{there exists } C \in \mathcal{W}_X \text{ such that } B \subset C + t\mathfrak{B}_X\}.$$

Proposition 1.5. *The Blasi measure is a regular measure of weak noncompactness.*

Proof. **i) Fullness:** The proof of the part “only if” is given in [21], while the “if” part is trivial.

ii) Monotonicity: let B_1, B_2 tow bounded sets such that $B_1 \subset B_2$. By definition of ω there exists $C \in \mathcal{W}_X$ and $t > \omega(B_2)$ such that :

$$B_1 \subset B_2 \subset C + t\mathfrak{B}_X$$

so,

$$\omega(B_1) < t,$$

let $t \longrightarrow \omega(B_2)$ we get

$$\omega(B_1) \leq \omega(B_2).$$

iii) Homogeneity: let B be a bounded set. By definition of ω there exists $C \in \mathcal{W}_X$ and $t > \omega(B)$ such that:

$$B \subset C + t\mathfrak{B}_X, \tag{1.4}$$

homogeneity is obvious if $\lambda = 0$. So the multiplication (1.4) by $\lambda \in \mathbb{R}^*$ we get:

$$\lambda B \subset \lambda C + \lambda t\mathfrak{B}_X$$

and since λC is also weakly compact so, $\omega(\lambda B) < \lambda t$, and let t tends to $\omega(B)$ we will get

$$\omega(\lambda B) \leq \lambda\omega(B). \tag{1.5}$$

And we have

$$\omega(B) = \omega(\lambda\lambda^{-1}B) \leq \lambda^{-1}\omega(\lambda B);\ \forall\lambda \in \mathbb{R}^*$$

which means

$$\lambda\omega(B) \leq \omega(\lambda B). \tag{1.6}$$

By (1.5) and (1.6) ω is homogeneous.

iv) Sub-additivity: Let B_1, B_2 tow bounded sets such that. By monotonicity we have:

$$\begin{cases} B_1 \subset B_1 \cup B_2 \\ B_2 \subset B_1 \cup B_2 \end{cases} \Rightarrow \begin{cases} \omega(B_1) \leq \omega(B_1 \cup B_2) \\ \omega(B_2) \leq \omega(B_1 \cup B_2) \end{cases} \Rightarrow \max\{\omega(B_1),\omega(B_2)\} \leq \omega(B_1 \cup B_2).$$

In the other hand, using the definition of ω; there exist $C_1, C_2 \in \mathcal{W}_X$, and $t_1 > \omega(B_1)$, $t_2 > \omega(B_2)$ such that :

$$\begin{cases} B_1 \subset C_1 + t_1\mathfrak{B}_X \\ B_2 \subset C_2 + t_2\mathfrak{B}_X \end{cases} \Rightarrow B_1 \cup B_2 \subset (C_1 + t_1\mathfrak{B}_X) \cup (C_2 + t_2\mathfrak{B}_X)$$

$$\Rightarrow B_1 \cup B_2 \subset C_1 \cup C_2 + \max\{t_1,t_2\}\mathfrak{B}_X,$$

and since C_1, C_2 are weakly compacts sets so, $C_1 \cup C_2$ is also weakly compact set so,

$$\omega(B_1 \cup B_2) < \max\{t_1,t_2\}\mathfrak{B}_X$$

and let $\{t_1, t_2\}$ tend to $\{\omega(B_1), \omega(B_2)\}$ we will have

$$\omega(B_1 \cup B_2) \leq \max\{\omega(B_1),\omega(B_2)\}\mathfrak{B}_X.$$

And sub-additivity of ω is hold.

v) Invariance under passage to the weak closure[17]: From the definition of $\omega(B)$ there exist $C \in \mathcal{W}_X$ and $t > 0$ such that $B \subset C + t\mathfrak{B}_X$. Then $B \subset \overline{\text{co}}(C) + t\mathfrak{B}_X$ and $\overline{\text{co}}(C)$ is weakly compact, by the Krein-Šmulian Theorem 1.6. Then $\overline{\text{co}}(C) + t\mathfrak{B}_X$ being the sum of $\overline{\text{co}}(C)$, which is weakly compact, and $t\mathfrak{B}_X$, which is weakly closed, by Lemma 1.1 is weakly closed. Thus $\overline{B}^w \subset \overline{\text{co}}(C) + t\mathfrak{B}_X$ implies $\omega(\overline{B}^w) < t$.
Let t tends to $\omega(B)$ we will have:

$$\omega(\overline{B}^w) \leq \omega(B).$$

The reverse inequality is obvious.

vi) Invariance under passage to the convex hull[10]: Since $B \subset \text{co}(B)$ then $\omega(B) \leq \omega(\text{co}(B))$.
Conversely, we show that $\omega(\text{co}(B)) \leq \omega(B)$. To see this, let $t > \omega(B)$. From the definition of the De Blasi measure of weak noncompactness it follows that there exists a weakly compact set C such that

$$B \subset C + t\mathfrak{B}_X. \tag{1.7}$$

We claim that $\text{co}(B) \subset \text{co}(C) + t\mathfrak{B}_X$. Indeed, let $x \in \text{co}(B)$. Then there exist $(x_1, \ldots, x_n) \in B^n$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $x = \sum_{i=1}^{n} \lambda_i x_i$. From (1.7) it follows that for each x_i there is $c_i \in C$ such that $\|x_i - c_i\| \leq t$. and we have $\sum_{i=1}^{n} \lambda_i c_i \in \text{co}(C)$ so:

$$\begin{aligned}\|x - \sum_{i=1}^{n} \lambda_i c_i\| &= \|\sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i c_i\| \\ &\leq \underbrace{\sum_{i=1}^{n} \lambda_i}_{=1} \|x_i - c_i\| \\ &\leq t.\end{aligned}$$

This implies that $\mathrm{co}(B) \subset \mathrm{co}(C) + t\mathfrak{B}_X \subset \overline{\mathrm{co}}(C) + t\mathfrak{B}_X$. By the Krein–Šmulian Theorem 1.6 we know that $\overline{\mathrm{co}}(C)$ is weakly compact and so $\omega(\mathrm{co}(B)) \leq t$. Letting t goes to $\omega(B)$ we get the desired result that is $\omega(\mathrm{co}(B)) \leq \omega(B)$.

vii) Cantor's intersection property[10]: Obviously this measure is non-singular(i.e., $\omega(B \cup \{x\}) = \omega(B), \forall B \in \mathcal{B}_X, \forall x \in X$). Now choose $x_n \in B_n; n = 1, 2, \ldots$. Then:

$$\omega\left(\bigcup_{n=1}^{\infty}\{x_n\}\right) = \omega\left(\bigcup_{n=k}^{\infty}\{x_n\}\right) \leq \omega(B_k).$$

Since $\lim_{n\to\infty} \omega(B_n) = 0$ we have $\omega(\bigcup_{n=1}^{\infty}\{x_n\}) = 0$, and therefore $\left(\bigcup_{n=1}^{\infty}\{x_n\}\right)$ is relatively weakly compact. By the Eberlein–Šmulian Theorem 1.5 $\{x_n\}_{n\in\mathbb{N}}$ is weakly sequentially compact which means it contains a subsequence which converges weakly to some point $x \in X$. Since all the sets are weakly closed then $x \in B_n$ for $n = 1, 2, \ldots$. Thus, $x \in \bigcap_{n=1}^{\infty} B_n$ so, B_∞ is nonempty Moreover, as

$$\omega(B_\infty) \leqslant \omega(B_n) \xrightarrow{n\longrightarrow\infty} 0,$$

which means that B_∞ is weakly compact and this completes the proof. □

Theorem 1.10. *[17] Let X be a Banach space. Then:*

a) If X is reflexive, $\omega(\mathfrak{B}_X) = 0$,

b) $\omega(\mathfrak{B}_X) = 1$, otherwise.

To proof this theorem we need to state the following known result due to Rådström.

Lemma 1.2. *[27] Let M, N and L be given subsets of a Banach space X. Suppose that N is convex and closed, L is bounded and $M + L \subset N + L$. Then $M \subset N$.*

Proof. (**Theorem** 1.10)

a) By using Kakutani's Theorem 1.4; it is known that X is reflexive if and only if $\mathfrak{B}_X$ is weakly compact and hence the first statement is obvious.

b) Now, let X be non reflexive Since $\mathfrak{B}_X \subset \{0\} + 1\mathfrak{B}_X$ then $\omega(\mathfrak{B}_X) \leq 1$. Suppose $\omega(\mathfrak{B}_X) < 1$. From the definition of ω there exist $C \in \mathcal{W}_X$ and t, $\omega(\mathfrak{B}_X) \leq t < 1$, such that $\mathfrak{B}_X \subset C + t\mathfrak{B}_X$. Thus

$$\mathfrak{B}_X \subset \overline{\text{co}}(C) + t\mathfrak{B}_X$$

and

$$(1-t)\mathfrak{B}_X + t\mathfrak{B}_X \subset \overline{\text{co}}(C) + t\mathfrak{B}_X$$

Since $\overline{\text{co}}(C)$ is strongly closed and convex, $t\mathfrak{B}_X$ is bounded, Lemma 1.2 implies $(1-t)\mathfrak{B}_X \subset \overline{\text{co}}(C)$ By the Krein–Šmulian theorem $\overline{\text{co}}(C)$ is weakly compact. But $(1-t)\mathfrak{B}_X$ is closed, convex thus it is weakly closed in weakly compact so, it is too. Since

$$x \to (1-t)^{-1}x$$

is weakly continuous, then $\mathfrak{B}_X$ is weakly compact and therefore X is reflexive. This is a contradiction. Accordingly, $\omega(\mathfrak{B}_X) = 1$ and the proof is complete. □

1.4 Definitions and fundamental theorems

Definition 1.15. Let T be a map from a set X into Y.

i) T is said to be one-to-one (injective) if $T(x_1) = T(x_2)$ implies $x_1 = x_2$ for $x_1, x_2 \in X$,

ii) T is said to be onto (surjective) if for each $y \in Y$ there exists an $x \in X$ such that $T(x) = y$,

iii) T is said to be bijective if it is both one-to-one and onto.

Definition 1.16. Let (X, d) be a metric space and $T : X \to X$ a map. If there exists a constant $k > 0$ such that

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X$$

then we say that T is k-Lipschitz. In particular, we call T is contractive if $k < 1$.

In 1922 S.Banach stated one of the most important fixed point theorem in metric spaces which is well known as Banach contraction principle (in short BCP) as follows:

Theorem 1.11. *(Banach contraction principle [29]) Let (X, d) be a complete metric space and $T : X \to X$ be a contraction. Then T has a unique fixed point $x^* \in X$.*

Lemma 1.3. *[29] Let $(X, \|.\|)$ be a linear normed space, $\mathcal{M} \subset X$. Assume that the mapping $T : \mathcal{M} \to X$ is contractive with constant $\gamma < 1$; then the inverse of $F := I - T : \mathcal{M} \to (I - T)(\mathcal{M})$ exists, and*

$$\left\|F^{-1}x - F^{-1}y\right\| \leq \frac{1}{1-\gamma}\|x - y\|, \quad x, y \in F(\mathcal{M}). \tag{1.8}$$

An important generalization of contraction is the so-called nonlinear contraction mapping.

Definition 1.17. Let $\mathcal{M}$ be a subset of X. The mapping $T : \mathcal{M} \to X$ is called a nonlinear contraction, if there exists continuous and non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\phi(r) < r$ for $r > 0$, such that

$$\|Tx - Ty\| \leq \phi(\|x - y\|), \quad \forall x, y \in \mathcal{M}. \tag{1.9}$$

In [14], Boyd and Wong obtained a general result of BCP, we will recall it with its important result in such a way:

Lemma 1.4. *Let X be a Banach space, and $T : X \to X$ be a nonlinear contraction so, T has a unique fixed point.*

Lemma 1.5. *If an operator $T : X \to X$ is ϕ-nonlinear contractive, then $F := I - T$ is a homeomorphism of X onto X.*

In 1968, Bryant [16] extended BCP as follows:

Theorem 1.12. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that for some positive integer n, T^n is contraction on X. Then, T has a unique fixed point.*

Proof. By BCP, T^n has a unique fixed point, say $x \in X$ with $T^n(x) = x$. Since

$$T(x) = T(T^n(x)) = T^n(T(x)),$$

it follows that $T(x)$ is a fixed point of T^n, and thus, by the uniqueness of x, we have $T(x) = x$, that is, T has a fixed point. Since, the fixed point of T is necessarily a fixed point of T^n so, is unique. $\square$

Definition 1.18. Let (X, d) be a metric space and $\mathcal{M}$ be a subset of x and $T : \mathcal{M} \to X$ a map. If there exists a constant $h > 0$ such that $d(Tx, Ty) \geq hd(x, y), \quad \forall x, y \in \mathcal{M}$ then we say that T is weakly expansive. In particular, we call T is expansive if $h > 1$.

Remark 1.5. We note that an (weakly) expansive map $T : \mathcal{M} \to X$ may not be continuous. If $T : \mathcal{M} \subset X \to X$ is a weakly expansive map, we will denote by

$$\text{lip}(T) = \max\{h \geq 0 : d(Tx, Ty) \geq hd(x, y), x, y \in \mathcal{M}\}.$$

As usual, Lip(T) denotes the Lipschitz constant for T if T is a Lipschitz map.

Lemma 1.6. *[33] Let X be a complete metric space and $\mathcal{M}$ a closed subset of X. Assume that the mapping $T : \mathcal{M} \to X$ is expansive and $T(\mathcal{M}) \supset \mathcal{M}$. Then T admits a unique fixed point in $\mathcal{M}$, that is, there is a unique $x^* \in \mathcal{M}$ such that $Tx^* = x^*$.*

Corollary 1.2. *In Theorem 1.12, if we are replaced the contraction condition of T^n by T^n is expansive and onto so, the result is hold.*

Lemma 1.7. *[33] Let $(X, \|.\|)$ be a linear normed space, $\mathcal{M} \subset X$. Assume that the mapping $T : \mathcal{M} \to X$ is expansive with constant $h > 1$.*
Then the inverse of $F := I - T : \mathcal{M} \to (I - T)(\mathcal{M})$ exists and

$$\left\|F^{-1}x - F^{-1}y\right\| \leq \frac{1}{h-1}\|x - y\|, \quad x, y \in F(\mathcal{M}).$$

Theorem 1.13. *(Dominated convergence theorem S, Lebesgue, [15]) Let Ω a nonempty set and (f_n) be a sequence of functions in L^1 that satisfy*

(i) $f_n(x) \to f(x)$ a.e.on Ω,

(ii) *there is a function* $g \in L^1$ *such that for all* $n, |f_n(x)| \leq g(x)$ *a.e.on* Ω.

Then $f \in L^1$ *and* $\| f_n - f \|_1 \to 0$. *where* $\| f \|_1 = \int\limits_{\Omega} |f(x)| dx$.

Definition 1.19. Let $f : X \to Y$, where (X, Σ, μ) is a measure space and Y is a topological vector space. We say that f is pettis integrable if

$$\varphi \circ f \in L^1(X,\Sigma,\mu) \text{ for all } \varphi \in Y^*.$$

And there exist a vector $e \in Y$ so, that

$$\forall \varphi \in Y^* : \ \langle \varphi, e \rangle = \int_X \langle \varphi, f(x) \rangle d\mu(x).$$

Theorem 1.14. *(Dominated convergence theorem W, [25]) Let* f *be a function from* $\mathcal{M}$ *into* X *satisfying the following two conditions:*

(a) *There exists a sequence of Pettis integrable functions*

   $f_n : \mathcal{M} \to X$, $n \in \mathbb{N}$, *such that* $\lim\limits_n x^* f_n = x^* f$ *in measure, for each* $x^* \in X^*$,

(b) *There exists a Pettis integrable function* $g : \mathcal{M} \to X$ *such that* $|x^* f_n| \leq |x^* g|$ μ*.a.e for each* $x^* \in X^*$ *and* $n \in \mathbb{N}$.

Then f *is Pettis integrable and* $\lim\limits_n \int\limits_E f_n d\mu = \int\limits_E f d\mu$ *weakly for all* $E \in \Sigma$.

Definition 1.20. Let X be a Banach space. An operator $f : X \to X$ is said to be:

- Compact [2] if $f(B)$ is relatively compact for every bounded subset $B \subset X$.
- Weakly compact if $f(B)$ is relatively weakly compact whenever B is a bounded subset of X.

[2] There is some books which add the continuity condition.

Chapter 2

Fixed Point Theorems Under Strong Topology Using MNC

The aim of this chapter is to give some generalisation of Schausder's and Krasnoselskii's fixed point using the measure of noncompactness.

2.1 Generalisation of Schauder's Fixed Point Theorem Using Measure of Noncompactness

At first, let us recall the well-known Schauder fixed point theorem that will be used and generalised later.

Theorem 2.1. *(Schauder, 1930 [29]) Let $\mathcal{M}$ be a nonempty, convex and compact subset of a Banach space X. Then, every continuous mapping $T : \mathcal{M} \to \mathcal{M}$ has at least one fixed point.*

2.1.1 Darbo's Fixed Point Theorem

In 1955, Darbo extended the Schauder fixed point theorem to the setting of non-compact operators, introducing the notion of (k, μ)-set contraction as follows:

Definition 2.1. Let X be a Banach space, and let μ a measure of noncompactness. A self-mapping $T : X \longrightarrow X$ is said to be a (k, μ)-set contraction if T is bounded, and there exists some constant $k \geq 0$ such that

$$\mu(TB) \leq k\mu(B),$$

for every nonempty bounded subset B of X.

Example 2.1. Any compact mapping is a $(0, \mu)$-set contraction.

Theorem 2.2. *(Darbo 1955 [8]) Let $\mathcal{M}$ be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : \mathcal{M} \to \mathcal{M}$ be a continuous operator. If T is a (k, μ)-set contraction, where μ is an arbitrary measure of noncompactness and $k \in (0, 1[$, then T has at least one fixed point, and the set of fixed points of T belongs to* $\ker \mu$*; i.e.,*

$$\mu(\mathrm{Fix}(T)) = \mu(\{x \in \mathcal{M} : Tx = x\}) = 0.$$

Proof. Consider the sequence of sets

$$M_n = \begin{cases} M_0 = \mathcal{M} & \text{if } n = 0, \\ \overline{\mathrm{co}}(TM_{n-1}) & \text{otherwise.} \end{cases}$$

Then:

$$\mu(M_{n+1}) = \mu(\overline{\mathrm{co}}(TM_n)) = \mu(TM_n) \leq k\mu(M_n) \leq \cdots \leq k^{n+1}\mu(M_0),$$

and consequently:

$$\lim_{n\to\infty} \mu(M_n) = 0.$$

Because $M_{n+1} \subset M_n$ and which means that $(M_n)_{n\in\mathbb{N}}$ is decreasing sequence, and since μ generalise the Contor's intersection theorem, we infer that $M_\infty = \bigcap_{n=0}^{\infty} M_n$ is a convex closed compact set, and we have $TM_\infty \subset M_\infty$, then by Schauder fixed point theorem T has at least one fixed point.
And we have:

$$\mu(\mathrm{Fix}(T)) = \mu(\{x \in \mathcal{M} : Tx = x\}) = \mu(\{x \in M_\infty : Tx = x\}) \leq \mu(M_\infty) = 0.$$

Which give us $\mathrm{Fix}(T) \in \ker \mu$. □

2.1.2 Sadoveskii's Fixed Point Theorem

In 1967, Sadoveskii gave a fixed point result more general than the Darbo theorem using the concept of condensing operator which is define in such a way:

Definition 2.2. Let X be a Banach space, and let μ a measure of noncompactness. A self-mapping $T : X \longrightarrow X$ is said to be a μ-condensing if T is bounded, and the inequality holds:

$$\mu(TB) < \mu(B),$$

for every nonempty bounded subset B of X.

Theorem 2.3. *(Sadoveskii 1967 [31]) Let $\mathcal{M}$ be a nonempty, bounded, closed and convex subset of a Banach space X, and let $T : \mathcal{M} \to \mathcal{M}$ be a continuous operator. If T is a μ-condensing, where μ is a measure of noncompactness which has the maximum property, then T has at least one fixed point, and:*

$$\mu(\mathrm{Fix}(T)) = 0.$$

Proof. Let us choose a point $m \in \mathcal{M}$ and denote by Σ the class of all closed and convex subsets K of C such that $m \in K$ and $T(K) \subset K$; that is:

$$\Sigma = \{K \subset \mathcal{M}; K \text{ is closed, convex and } T\text{-invariant with } m \in K\}.$$

Also set

$$B = \bigcap_{K \in \Sigma} K,$$

and

$$C = \overline{\mathrm{co}}(T(B) \cup \{m\}).$$

Obviously $\Sigma \neq \emptyset$ as $m \in \Sigma$ and $B \neq \emptyset$ as $m \in B$.
Furthermore, we have:

$$T(B) = T(\bigcap_{K \in \Sigma} K) \subset \bigcap_{K \in \Sigma} T(K) \subset \bigcap_{K \in \Sigma} K = B,$$

and so, we have

$$T : B \to B.$$

Now we want to show that $B = C$. Indeed, since $m \in B$, $T(B) \subset B$ and B is closed and convex; it follows that $C = \overline{\text{co}}(T(B) \cup \{m\}) \subset B$. This implies $T(C) \subset T(B) \subset C$ and so, $C \in \Sigma$, and hence $B \subset C$. Therefore the properties of μ now imply that

$$\mu(B) = \mu(C) = \mu(T(B) \cup \{m\}) = \max\{\mu(T(B)), \mu(\{m\})\} = \mu(T(B)).$$

Since T is μ-condensing, it follows that $\mu(B) = 0$ so, B is compact. Obviously B is also convex. Thus from Schauder fixed point theorem there is a fixed point for the mapping $T : \mathcal{M} \to \mathcal{M}$.

And since we have $T(\text{Fix}(T)) = \text{Fix}(T)$, and μ is condensing then $\mu(\text{Fix}(T)) = 0$. □

2.2 Non-compact Type Krasnoselskii Fixed Point

In 2015, Xiang and Georgiev gave some user-friendly forms of Krasnoselskii fixed point theorems, where they showed the sufficient conditions which ensure the Lipschitz invertibility of $(I - T)$. The following two subsections explain this.

2.2.1 The Expansive Case

Lemma 2.1. *[33] Let $T : X \to X$ be Lipschitzian with constant $\beta > 0$. Assume that for each $y \in X$, the map $T_y : X \to X$ defined by $T_y x = Tx + y$ satisfies that T_y^p is expansive and onto for some $p \in \mathbb{N}$. Then $(I - T)$ maps X onto X, the inverse of $F := I - T : X \to X$ exists, and*

$$\left\| F^{-1}x - F^{-1}y \right\| \leq \gamma_p \|x - y\|, x, y \in X, \tag{2.1}$$

where

$$\gamma_p = \frac{\beta^p - 1}{(\beta - 1)\left[\text{lip}\left(T^p\right) - 1\right]}.$$

Proof. Let $y \in X$ be an arbitrary point. Because T_y^p is expansive, it follows that

$$\left\| T_y^p x - T_y^p z \right\| \geq \text{lip}\left(T_y^p\right) \|x - z\|, \quad \forall x, z \in X.$$

Now, we claim that both $(I - T)$ and $(I - T^p)$ map X onto X. Indeed, notice that T_y^p is onto; thus, Lemma 1.6 ensures the existence of a unique $x^* \in X$ such that $T_y^p x^* = x^*$. It then follows by Corollary 1.2 that x^* is the unique fixed point of T_y. Hence, we have

$$(I-T)x^* = y,$$

which gives that $I-T : X \to X$ is onto. Observe that T^p is expansive and onto. Then an application of Lemma 1.6 to $\tilde{T}_y x = T^p x + y$ shows there is a unique x^* so, that $\tilde{T}_y x^* = x^*$, implying $I - T^p : X \to X$ is onto. So, the claim is proved. Next, for each $x, y \in X$, by expansiveness of T^p, one easily obtains that

$$\begin{aligned}
\|(I-T^p)\,x-(I-T^p)\,y\| &= \|(x-y)-(T^p(x)-T^p(y))\| \\
&\geq |\,\|(T^p(x)-T^p(y))\|-\|(x-y)\|\,| \\
&\geq \|(T^p(x)-T^p(y))\|-\|(x-y)\| \\
&\geq [\mathrm{lip}\,(T^p)-1]\,\|x-y\| \geq 0,
\end{aligned}$$

which shows that $(I-T^p)$ is one-to-one. Summing up the aforementioned arguments, we derive that $(I-T^p)^{-1}$ exists on X. Therefore, we infer that $(I-T)^{-1}$ exists on X, because

$$(I-T)^{-1} = \overbrace{(I-T^p)^{-1}}^{\text{exists}} \underbrace{\sum_{k=0}^{p-1} T^k}_{\text{exists}}. \tag{2.2}$$

From the previous, Then Lemma 1.7 entails that

$$\left\|(I-T^p)^{-1}x-(I-T^p)^{-1}y\right\| \leq \frac{1}{\mathrm{lip}\,(T^p)-1}\|x-y\|, \quad \forall x,y \in (I-T^p)\,(X).$$

By definition,

$$\mathrm{Lip}\left((I-T^p)^{-1}\right) \leq \frac{1}{\mathrm{lip}\,(T^p)-1}. \tag{2.3}$$

A series of induction calculations yields that

$$\begin{aligned}
\left\|T^k x - T^k y\right\| &\leq \beta\left\|T^{k-1}x-T^{k-1}y\right\| \\
&\;\;\vdots \\
&\leq \beta^k\|x-y\|, \quad \forall x,y \in X \text{ and } k \in \mathbb{N},
\end{aligned} \tag{2.4}$$

and for $k=p$, and using the expensiveness of T^p we get

$$\mathrm{lip}\,(T^p)\,\|x-y\| \leq \|T^p x - T^p y\| \leq \beta^p\|x-y\|, \quad \forall x,y \in X.$$

Recalling $\mathrm{lip}\,(T^p) > 1$, we obtain $\beta > 1$. So, we conclude from (2.2), (2.3) and (2.4) that

$$\begin{aligned}
\operatorname{Lip}\left((I-T)^{-1}\right) &\leq \operatorname{Lip}\left((I-T^p)^{-1}\right)\sum_{k=0}^{p-1}\operatorname{Lip}\left(T^k\right)\\
&\leq \frac{1}{\operatorname{lip}\left(T^p\right)-1}\sum_{k=0}^{p-1}\beta^k\\
&= \frac{\beta^p-1}{(\beta-1)\left(\operatorname{lip}\left(T^p\right)-1\right)},
\end{aligned}$$

which proves the lemma. □

Corollary 2.1. *[33]*

Let $T : X \to X$ be a bounded linear operator. Assume that T^p is expansive and onto for some $p \in \mathbb{N}$. Then the conclusion of Lemma 2.1 holds. In such case, $\operatorname{Lip}(T) = \|T\|$.

Proof. Because T is a bounded linear operator, $\operatorname{Lip}(T) = \|T\|$. Let $y \in X$ be fixed. By induction, one deduces that

$$\begin{aligned}
&T_y x = Tx + y\\
&T_y^2 x = T_y(Tx+y) = T(Tx+y) + y = T^2x + Ty + y\\
&\quad\vdots\\
&T_y^k x = T^k x + T^{k-1}y + \cdots + Ty + y, \text{ for all } k \in \mathbb{N}.
\end{aligned}$$

This shows

$$\left\|T^k x - T^k z\right\| = \left\|T_y^k x - T_y^k z\right\|, \text{ for all } k \in \mathbb{N} \text{ and for all } x, z \in X.$$

Particularly, for $k = p$

$$\operatorname{lip}\left(T^p\right)\|x-y\| \leq \|T^p x - T^p z\| = \left\|T_y^p x - T_y^p z\right\|, \text{ for all } x, z \in X.$$

Consequently, T_y^p is expansive and onto, and so, Lemma 2.1 works. □

Lemma 2.2. *[33] Let $\mathcal{M}$ be a subset of X. Assume that $T : \mathcal{M} \to X$ is k-Lipschitz map, that is,*

$$\|Tx - Ty\| \leq k\|x-y\|, \quad x, y \in \mathcal{M}.$$

Then for each bounded subset Ω of $\mathcal{M}$, we have

$$\alpha(T(\Omega)) \leq k\alpha(\Omega).$$

Now, we are ready to state and prove the first result of this section.

Theorem 2.4. *[33] Let $\mathcal{K} \subset X$ be a nonempty, bounded, closed and convex subset suppose that $T : X \to X$ and $S : \mathcal{K} \to X$ such that*

(i) T fulfils the conditions of Lemma 2.1;

(ii) S is a strictly γ_p^{-1}-set contractive map (or a γ-set contractive map with $\gamma < \gamma_p^{-1}$);

(iii) $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$.

Then there exists a point $x^* \in \mathcal{K}$ with $Sx^* + Tx^* = x^*$.

Proof. Because $T : X \to X$ satisfies all conditions of Lemma 2.1 $(I - T)$ maps X onto X. Because $S : \mathcal{K} \to X$, it follows that for every $x \in \mathcal{K}$, there exists $y \in X$ such that

$$y - Ty = Sx \quad \Longleftrightarrow \quad (I - T)y = Sx.$$

By Lemma 2.1 again, there exists $(I - T)^{-1}$, and thus, from (iii) and the previous equality, we obtain $y = (I - T)^{-1}Sx \in \mathcal{K}$. Now, let A be a subset of $\mathcal{K}$. From (2.1), $(I - T)^{-1}$ is γ_p-Lipschitz so, by the Lemma 2.2 we infer

$$\alpha\left(\left((I - T)^{-1}S\right)(A)\right) \leq \gamma_p \alpha(S(A)),$$

which, together with (ii), implies that $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is a condensing map. Applying Sadoveskii's fixed point Theorem 2.3, we obtain that there exists an $x^* \in \mathcal{K}$ such that $(I - T)^{-1}Sx^* = x^*$, which is the same as $Sx^* + Tx^* = x^*$. The proof of the theorem is thus complete. □

An easy consequence of Corollary 2.1 and Theorem 2.4 is the following.

Corollary 2.2. *[33] In Theorem 2.4 if only (i) is replaced by that*

(i') $T : X \to X$ is a linear and bounded operator, and T^p is expansive and onto for some $p \in \mathbb{N}$.

Then there exists a point $x^* \in \mathcal{K}$ with $Sx^* + Tx^* = x^*$.

2.2.2 Contraction case

The following result is analogous to Lemma 2.1 and a basic tool to consider the case when T_y^p (for some $p \in \mathbb{N}$) is a contractive map.

Lemma 2.3. *[33] Let $T : X \to X$ be Lipschitz with constant $\beta \geq 0$ Assume that for each $y \in X$, the map $T_y : X \to X$ defined by $T_y x = Tx + y$ satisfies that T_y^p is contractive for some $p \in \mathbb{N}$. Then $(I - T)$ maps X onto X, the inverse of $F := I - T : X \to X$ exists and*

$$\left\| F^{-1}x - F^{-1}y \right\| \leq \rho_p \|x - y\|, \quad x, y \in X, \tag{2.5}$$

where

$$\rho_p = \begin{cases} \frac{p}{1-\text{Lip}(T^p)}, & \text{if } \beta = 1, \\ \frac{1}{1-\beta}, & \text{if } \beta < 1, \\ \frac{\beta^p - 1}{(\beta-1)[1-\text{Lip}(T^p)]}, & \text{if } \beta > 1. \end{cases}$$

Proof. Since T_y^p is contraction, using Lemma 1.3, and using the similar arguments to Lemma 2.1 we obtain that $(I - T^p)^{-1}$ exists.

The contraction of T_y^p is also give us the existence and uniqueness of x^* such that $T_y^p x^* = x^*$, now using Theorem 1.12 we deduce that $T_y x^* = x^*$ for all $y \in X$ which means that $(I - T)$ is onto.
And consequently, $(I - T)^{-1}$ exists on X because of (2.2). Going back to (2.2) and (2.4), one concludes that

(1) if $\beta = 1$, $\sum_{k=0}^{p-1} \beta^k = p$, and

$$\text{Lip}\left((I - T)^{-1}\right) \leq \frac{1}{1 - \text{Lip}(T^p)} \sum_{k=0}^{p-1} \beta^k = \frac{p}{(1 - \text{Lip}(T^p))}.$$

(2) if $\beta < 1$, $\sum_{k=0}^{p-1} \beta^k = \frac{1 - \beta^p}{1 - \beta}$, and

$$\text{Lip}\left((I - T)^{-1}\right) \leq \frac{1}{1 - \text{Lip}(T^p)} \cdot \sum_{k=0}^{p-1} \beta^k = \frac{1}{(1 - \text{Lip}(T^p))} \cdot \frac{1 - \beta^p}{1 - \beta},$$

since T is β-Lipschitz with $\beta < 1$ in this case, we infer that $\text{Lip}(T^p) = \beta^p$, so

$$\text{Lip}\left((I - T)^{-1}\right) \leq \frac{1}{1 - \beta}.$$

(3) if $\beta > 1$, $\sum_{k=0}^{p-1} \beta^k = \frac{1-\beta^p}{1-\beta}$, and

$$\mathrm{Lip}\left((I-T)^{-1}\right) \leq \frac{1-\beta^p}{1-\mathrm{Lip}(T^p)(1-\beta)}.$$

In conclusion,

$$(I-T)^{-1} \leq \begin{cases} \frac{p}{1-\mathrm{Lip}(T^p)}, & \text{if } \beta = 1, \\ \frac{1}{1-\beta}, & \text{if } \beta < 1, \\ \frac{\beta^p - 1}{(\beta-1)[1-\mathrm{Lip}(T^p)]}, & \text{if } \beta > 1. \end{cases}$$

This proves the desired estimate(2.5). □

Corollary 2.3. *[33] Let $T : X \to X$ be a linear and bounded operator. Assume that T^p is contractive for some $p \in \mathbb{N}$. Then the conclusions of Lemma 2.3 hold.*

Proposition 2.1. *[33] Let T be the same as Corollary 2.3. Then T has a unique fixed point in X, and T is a $\|T\|$-set contractive map. Obviously, the number $\|T\|$ may be larger than 1.*

Combining Lemmas 2.3 and 2.2 and the ideas used to prove Theorem 2.4, one can easily derive the following Krasnoselskii fixed point result.

Theorem 2.5. *[33] Let $\mathcal{K} \subset X$ be a nonempty, bounded, closed and convex subset. Suppose that $T : X \to X$ and $S : \mathcal{K} \to X$ such that*

(i) T satisfies the conditions of Lemma 2.3;

(ii) S is a strictly ρ_p^{-1}-set contractive map (or a ρ-set contractive map with $\rho < \rho_p^{-1}$); and

(iii) $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$.

Then the sum $S + T$ possesses at least one fixed point in $\mathcal{K}$.

Corollary 2.4. *[33] In Theorem 2.5, if only (i) is replaced by that*

(i') $T : X \to X$ is a linear and bounded operator, and T^p is contractive for some $p \in \mathbb{N}$,

then $S+T$ has at least one fixed point in $\mathcal{K}$.

Inspired by the proofs of Theorems 2.4 and 2.5, we now can formulate an abstract fixed point theorem that summarizes Theorems 2.4 and 2.5.

Theorem 2.6. *[33] Let $\mathcal{K} \subset X$ be a nonempty, bounded, closed and convex subset. Suppose that $T : X \longrightarrow X$ and $S : \mathcal{K} \longrightarrow X$ such that*

(i) $(I-T)^{-1}$ is Lipschitz invertible with constant $\gamma > 0$;

(ii) S is a strictly γ^{-1}-set contractive map (or a ρ-set contractive map with $\rho < \gamma^{-1}$); and

(iii) $S(\mathcal{K}) \subset (I-T)(X)$ and $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$.

Then the equation $Sx + Tx = x$ has at least one solution in $\mathcal{K}$.

2.2.3 An general case

In this subsection, we study modifications of Krasnoselskii fixed point theorem given by Xiang and Georgiev in the case where $(I-T)$ is one-to-one. To do so, the following notation will be employed.

Notation: Let $\mathcal{M}$ and $\mathcal{K}$ be two subsets of X; $T : \mathcal{M} \to X$ and $S : \mathcal{K} \to X$ two mappings. We shall denote by $\mathcal{F} = \mathcal{F}(\mathcal{M}, \mathcal{K}; T, S)$ the following set

$$\mathcal{F} = \{x \in \mathcal{M} : x = Tx + Sy \text{ for some } y \in \mathcal{K}\}.$$

Theorem 2.7. *[33] Let $\mathcal{K}$ be a nonempty, bounded, closed and convex subset of X with $\mathcal{K} \subset \mathcal{D}(T) \subset X$ and $T : \mathcal{D}(T) \to X$ a map. Suppose that $S : \mathcal{K} \to X$ is continuous such that*

(i) $(I-T)$ is one-to-one;

(ii) $\alpha(T(A) + S(A)) < \alpha(A)$ for all $A \subset \mathcal{K}$ with $\alpha(A) > 0$;

(iii) if $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\mathcal{D}(T), \mathcal{K}; T, S)$ with $x_n \to x$ and $Tx_n \to y$, then $x \in \mathcal{D}(T)$ and $y = Tx$; and

(iv) $S(\mathcal{K}) \subset (I-T)(\mathcal{D}(T))$ *and* $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$.

Then the sum $S+T$ *has at least one fixed point in* $\mathcal{K}$.

Proof. Because $(I-T) : \mathcal{D}(T) \to X$ is one-to-one, the inverse of $(I-T)^{-1}$ exists on its range $(I-T)(\mathcal{D}(T))$. From $S : \mathcal{K} \to X$ and $S(\mathcal{K}) \subset (I-T)(\mathcal{D}(T))$, we conclude that the operator $N = (I-T)^{-1}S : \mathcal{K} \to \mathcal{D}(T)$ is well defined and that the set $\mathcal{F}$ is nonempty. For each $x \in \mathcal{F}$, by the definition of $\mathcal{F}$, there exists a $y \in \mathcal{K}$ such that $x = Tx + Sy$, that is, $x = Ny$. This shows $\mathcal{F} \subset N(\mathcal{K})$. On the other hand, if $x \in N(\mathcal{K})$, then there exists a $y \in \mathcal{K}$ so, that $Ny = x$ or equivalently $x = (I-T)^{-1}Sy$ or $(I-T)x = Sy$. Consequently, $x \in \mathcal{F}$, from which it follows $N(\mathcal{K}) \subset \mathcal{F}$ and then $\mathcal{F} = N(\mathcal{K})$.
Let $x \in \mathcal{F}$. Then there exists a $y \in \mathcal{K}$ such that $x = Tx + Sy$. The second part of (iv) then gives $x \in \mathcal{K}$. Therefore, $\mathcal{F} \subset \mathcal{K}$ and thus N maps $\mathcal{K}$ into itself.
Now, let $x_0 \in \mathcal{K}$ and

$$\mathcal{A} = \{A : x_0 \in A \subset \mathcal{K},\ A \text{ is a closed, convex set and } N(A) \subset A\}.$$

Because $x_0 \in \mathcal{K}$, $\mathcal{K}$ is closed, convex and $\mathcal{F} = N(\mathcal{K}) \subset \mathcal{K}$, we obtain that $\mathcal{K} \in \mathcal{A}$, that is, $\mathcal{A} \neq \varnothing$. Moreover, for any $A \in \mathcal{A}$, we have

$$(I-T)^{-1}S(A) = (I-T+T)(I-T)^{-1}S(A) = S(A) + T(I-T)^{-1}S(A).$$

The definition of $\mathcal{A}$ gives $(I-T)^{-1}S(A) = N(A) \subset A$, and so, we obtain from the previous equality

$$(I-T)^{-1}S(A) \subset T(I-T)^{-1}S(A) + S(A) \subset T(A) + S(A).$$

This fact, together with (ii), yields that

$$\alpha(N(A)) \leq \alpha(T(A) + S(A)) < \alpha(A) \text{ for all } A \in \mathcal{A} \text{ with } \alpha(A) > 0. \tag{2.6}$$

Put $A_0 = \bigcap\limits_{A \in \mathcal{A}} A$. Then $x_0 \in A_0 \subset \mathcal{K}$, A_0 is a closed, convex set and $N(A_0) \subset A_0$, and therefore $A_0 \in \mathcal{A}$. Notice that $\overline{\text{co}}\{N(A_0), x_0\} \subset A_0$.Hence, we have

$$N(\overline{\text{co}}\{N(A_0), x_0\}) \subset N(A_0) \subset \overline{\text{co}}\{N(A_0), x_0\},$$

which implies that $\overline{\text{co}}\{N(A_0), x_0\} \in \mathcal{A}$. The definition of A_0 then yields $\overline{\text{co}}\{N(A_0), x_0\} = A_0$. Thus, by the properties of α we obtain

$$\alpha(A_0) = \alpha(\overline{\text{co}}\{N(A_0), x_0\}) = \alpha(\{N(A_0), x_0\}) = \alpha(N(A_0)). \tag{2.7}$$

Recalling that $A_0 \in \mathcal{A}$, we then deduce from (2.6) and (2.7) that $\alpha(A_0) = 0$, thus entails that A_0 is a nonempty compact convex subset of $\mathcal{K}$ and $N(A_0) \subset A_0$.
We next verify that $N : A_0 \to A_0$ is continuous. Indeed, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in A_0 with $x_n \to x$. Set $y_n = (I-T)^{-1}Sx_n$ and $y = (I-T)^{-1}Sx$ (this is well-defined because $x \in A_0 \subset \mathcal{K}$). Then $(I-T)y_n = Sx_n$ and $(I-T)y = Sx$. Hence, $y_n, y \in A_0 \cap \mathcal{F}$, and so, $\{y_n\}_{n\in\mathbb{N}}$ has a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ converging to some $y_0 \in A_0$. Evidently, by the continuity of S,

$$Ty_{n_k} = y_{n_k} - (I-T)y_{n_k} \to y_0 - Sx = y_0 - (I-T)y. \tag{2.8}$$

It follows from (2.8) and (iii) that $y_0 - (I-T)y = Ty_0$, and thus, $y_0 = y = (I-T)^{-1}Sx$ because $(I-T)$ is injective. Summing up the previous arguments, we have derived

$$(I-T)^{-1}Sx_{n_k} \to (I-T)^{-1}Sx.$$

We next claim that

$$(I-T)^{-1}Sx_n \to (I-T)^{-1}Sx.$$

Suppose the contrary; then there exists a neighbourhood U of $(I-T)^{-1}Sx$ and a subsequence $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $(I-T)^{-1}Sx_{n_j} \notin U$ for all $j \geq 1$. Naturally, $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ converges to x; then reasoning as before, we may extract a subsequence $\{x_{n_{j_k}}\}_{k\in\mathbb{N}}$ of $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ so, that $(I-T)^{-1}Sx_{n_{j_k}} \to (I-T)^{-1}Sx$. But this is a contradiction, because $(I-T)^{-1}Sx_{n_j} \notin U$ for all $j \geq 1$. The claim is hence confirmed, and finally, $(I-T)^{-1}S : A_0 \to A_0$ is continuous.
Now, the celebrated Schauder fixed point theorem guarantees that $N = (I-T)^{-1}S$ has at least one fixed point in A_0. This finishes the proof of the theorem. □

Corollary 2.5. *[33] Let $\mathcal{K}$ be a nonempty, bounded, closed and convex subset of X and $T : \mathcal{D}(T) \subset X \to X$ a map. Suppose that $S : \mathcal{K} \to X$ is continuous such that*

(i) $(I-T)$ is continuously invertible;

(ii) $\alpha\left((I-T)^{-1}S(A)\right) < \alpha(A)$ for all $A \subset \mathcal{K}$ with $\alpha(A) > 0$; and

(iii) $S(\mathcal{K}) \subset (I-T)(\mathcal{D}(T))$ and $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$.

Then the sum $S + T$ admits one fixed point in $\mathcal{K}$.

In the following theorem we are replaced the boundedness of $\mathcal{K}$ and the requirement of $\mathcal{K} \subset \mathcal{D}(T)$ by a compactness condition on the set $\mathcal{F}(\mathcal{D}(T), \mathcal{K}; T, S)$.

Theorem 2.8. *[33] Let $\mathcal{K} \subset X$ be a nonempty, closed and convex subset and $T : \mathcal{D}(T) \subset X \to X$ a mapping. Suppose that $S : \mathcal{K} \to X$ is continuous such that*

(i) $(I - T)$is one-to-one;

(ii) the set $\mathcal{F}(\mathcal{D}(T), \mathcal{K}; T, S)$ is relatively compact;

(iii) if $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ for which $x_n \to x$ and $Tx_n \to y$, then $x \in \mathcal{D}(T)$ and $y = Tx$; and

(iv) $S(\mathcal{K}) \subset (I - T)(\mathcal{D}(T))$ and $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$.

Then the sum $S + T$ has one fixed point in $\mathcal{K}$.

Proof. It is sufficient to show that the operator $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is compact and continuous. Thanks to the fact $\mathcal{F} = (I - T)^{-1}S(\mathcal{K})$ and (ii), we obtain that $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is compact. For the continuity, let $y_n, y \in \mathcal{K}$ with $y_n \to y$, and let $x_n = (I - T)^{-1}Sy_n$ and $x = (I - T)^{-1}Sy$. The definition of $\mathcal{F}$ implies that $x_n \in \mathcal{F}$ and $(I - T)x_n \to Sy$ by the continuity of S. In view of $x_n \in \mathcal{F}$ where $\mathcal{F}$ is relatively compact, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converging to some x_0. Accordingly, $Tx_{n_k} \to x_0 - Sy$. The closedness of T in $\mathcal{F}$ (condition (iii)) therefore tells us that $x_0 - Sy = Tx_0$, that is, $x_0 = (I - T)^{-1}Sy$. Because $I - T$ is injective, it follows $x_0 = x$.
The same argument as performed at the end proof of Theorem 2.7 shows $x_n \to x$, and consequently, $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is continuous.

Hence Schauder's fixed point theorem works. □

2.3 Application

Let $X = \mathcal{C}([a, b],\mathbb{R})$ with the usual supremum norm $\|x\| = \max_{t\in[a,b]} |x(t)|$. In this present section, our main objective is to prove some existence and unique (in a special case) results for the following Volterra-Hammerstein's integral equation

$$x(t) = g(t, x(t)) + \lambda \int_a^t \kappa(t, s)f(s, x(s))\mathrm{d}s, \tag{2.9}$$

where κ defined on $\Delta = \{(t, s) : a \leq t \leq b \ , a \leq s \leq t\}$ is essentially bounded and measurable and $f, g : [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous with f is β-Lipschitz with $\beta > 1$. We mention that the case where $f = I$ in equation (2.9) was studying in [33], we add some conditions in this study to find the existence where f is non linear. To perform such task, we shall use an explicit formula for the Hausdorff MNC in the space $\mathcal{C}([a, b]; \mathbb{R})$, which was introduced and studied in [11], this MNC defines as follows:

$$\chi(B) = \frac{1}{2} \lim_{\delta \to 0} \left\{ \sup_{x \in B} \left[\max_{0 \leq r \leq \delta} \|x - x_r\| \right] \right\},$$

where $B \in \mathcal{B}_X$ and x_r denotes the r-translate of the function x, i.e.,

$$x_r(t) = \begin{cases} x(t + r), & a \leq t \leq b - r, \\ x(b), & b - r \leq t \leq b. \end{cases}$$

Let us now introduce the operators $T, S : X \to X$ as follows:

$$(Tx)(t) = \lambda \int_a^t \kappa(t, s) f(s, x(s)) \mathrm{d}s, \tag{2.10}$$

and

$$(Sy)(t) = g(t, y(t)). \tag{2.11}$$

Then one can easily know that $S : X \to X$ is continuous and bounded because g is continuous. Let $z \in X$ we define the mapping $T_z : X \to X$ by $T_z x = Tx + z$. For each $x, y \in X$, one readily derives from (2.10) that

$$\begin{aligned} |(T_z x)(t) - (T_z y)(t)| &\leq \lambda \int_a^t |\kappa(t, s)||f(s, x(s)) - f(s, y(s))| ds \\ &\leq \lambda \int_a^t |\kappa(t, s)|\beta|x(s) - y(s)| ds \\ &\leq c(t - a)\|x - y\|, \end{aligned} \tag{2.12}$$

where $c = \lambda.\beta. \operatorname{ess} \sup_{(t,s) \in \Delta} |\kappa(t, s)| < \infty$.

Again we have

$$\begin{aligned}|(T_z^2x)(t)-(T_z^2y)(t)| &\leq \lambda\int_a^t|\kappa(t,s)|\left|f\left(s,\lambda\int_a^s\kappa(s,\sigma)f(\sigma,x(\sigma))\mathrm{d}\sigma+z(s)\right)\right.\\ &\left.-f\left(s,\lambda\int_a^s\kappa(s,\sigma)f(\sigma,y(\sigma))\mathrm{d}\sigma+z(s)\right)\right|\mathrm{d}s\\ &\leq \lambda\int_a^t|\kappa(t,s)|\beta\left|\lambda\int_a^s\kappa(s,\sigma)f(\sigma,x(\sigma))\mathrm{d}\sigma-\lambda\int_a^s\kappa(s,\sigma)f(\sigma,y(\sigma))\mathrm{d}\sigma\right|\mathrm{d}s\\ &\leq c\lambda\beta\int_a^t|\kappa(t,s)||(s-a)|x(s)-y(s)|\mathrm{d}s\\ &\leq \frac{c^2(t-a)^2}{2!}\|x-y\|.\end{aligned} \tag{2.13}$$

By induction, one can deduce from (2.13) and (2.12) that

$$|(T_z^nx)(t)-(T_z^ny)(t)|\leq\frac{[c(t-a)]^n}{n!}\|x-y\|.$$

Hence,

$$\|T^nx-T^ny\|\leq\frac{[c(b-a)]^n}{n!}\|x-y\|. \tag{2.14}$$

Notice that

$$\lim_{n\to\infty}\frac{[c(b-a)]^n}{n!}=0.$$

Then

$$p=\min\left\{n\in\mathbb{N}:\frac{[c(b-a)]^n}{n!}<1\right\}.$$

is finite, and T_z^p is a contraction, thanks to (2.14). On the other hand, one can also easily deduce from (2.12) that T is M-Lipschitz with $M=c(b-a)$ so, the use of Lemma 2.3, we obtain that $(I-T)$ maps X onto X, the inverse of $I-T:X\to X$ exists, and

$$\left\|(I-T)^{-1}x-(I-T)^{-1}y\right\|\leq\rho_p\|x-y\|,\forall x,y\in X \tag{2.15}$$

where

$$\rho_p=\begin{cases}\frac{p}{1-\mathrm{Lip}(T^p)}, & \text{if } M=1,\\ \frac{1}{1-M}, & \text{if } M<1,\\ \frac{M^p-1}{(M-1)[1-\mathrm{Lip}(T^p)]}, & \text{if } M>1.\end{cases}$$

we shall study Equation (2.9) by considering three cases: $M<1$, $M=1$, and $M>1$. Our strategy is to apply Theorem 2.5 to find a fixed point for the operator $S+T$ in X. The proof will be broken up into several steps. In order to do so, assume that the functions involved in Equation 2.9 fulfill the following conditions:

($\mathcal{H}1$) κ is non-negative on Δ.

($\mathcal{H}2$) There are two constants $B > A \geq 0$ such that

$$(1-m)A \leq g(t,x) \leq (1-m)B, \quad \forall (t,x) \in [a,b] \times [A,B].$$

where $m = \lambda \min_{a \leq t \leq b} \int_a^t \kappa(t,s)ds.$

($\mathcal{H}3$) For each fixed $t \in [a,b], x,y \in [A,B]$ with $x \neq y$, we have

$$f(t,0) \equiv 0, \quad x(t) \leq f(t,x(t)),$$

$$g(t,0) \equiv 0, \quad |g(.,x) - g(.,y)| \leq \phi(|x-y|),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and continuous function satisfying the growth condition $\phi(r) < (1-M)r$ for all $r > 0$.

Theorem 2.9. *Suppose that the conditions ($\mathcal{H}1$) –($\mathcal{H}3$) hold. Then Equation (2.9) has one and only one positive solution $x \in \mathcal{C}([a,b],\mathbb{R})$ satisfying $A \leq x(t) \leq B$ for all $t \in [a,b]$.*

Proof.

Claim 1: We want to prove the third condition in theorem 2.5. To see this, we define

$$\mathcal{K} = \{x \in X : A \leq x(t) \leq B, \quad t \in [a,b]\}.$$

Then $\mathcal{K}$ is a closed, convex, and bounded subset of X. Let $x, y \in \mathcal{K}$. We have from (2.10), (2.11) and ($\mathcal{H}3$) that

$$\begin{aligned}(Tx)(t) + (Sy)(t) &= \lambda \int_a^t \kappa(t,s)f(s,x(s)) - f(s,0)ds + g(t,y(t)) - g(t,0) \\ &\leq M|x(t) - 0| + (1-M)|y(t) - 0| \\ &\leq MB + (1-M)B = B.\end{aligned} \tag{2.16}$$

On the other hand,

$$\begin{aligned}(Tx)(t) + (Sy)(t) &= \lambda \int_a^t \kappa(t,s)f(s,x(s))ds + g(t,y(t)) \\ &\geq mA + (1-m)A = A.\end{aligned} \tag{2.17}$$

It follows from (2.16) and (2.17) that $Tx + Sy \in \mathcal{K}$ for all $x, y \in \mathcal{K}$. Hence, the condition (iii) of Theorem 2.5 is satisfied.

Recall $M < 1$, and then T : $X \to X$ is a contraction; that is, the assumption (i) of Theorem 2.5 is fulfilled and follows from (2.15) that $(I-T)^{-1}$ is Lipschitz invertible with constant $(1-M)^{-1}$.

Claim 2: We show that S is a strictly $(1-M)$-set contractive map. To this end, let B be a subset of $\mathcal{K}$ and $x \in B$. Looking that

$$(Sx)_r(t) = g(\zeta, x_r(t)),$$

where

$$\zeta = \begin{cases} t+r, & a \le t \le b-r, \\ b, & b-r \le t \le b. \end{cases}$$

And we have

$$\begin{aligned} |(Sx)(t) - (Sx)_r(t)| &\le |g(t,x(t)) - g(t,x_r(t))| + |g(t,x_r(t)) - g(\zeta,x_r(t))| \\ &\le \phi(|x(t) - x_r(t)|) + |g(t,x_r(t)) - g(\zeta,x_r(t))|. \end{aligned} \tag{2.18}$$

Notice that ϕ is continuous and nondecreasing and the function g is uniformly continuous on $[a{,}b] \times [A{,}B]$, thus, it follows from (2.18) that

$$\begin{aligned} \lim_{\delta\to0}\left\{\sup_{x\in B}\left[\max_{0\le r\le\delta}\|(Sx)-(Sx)_r\|\right]\right\} &\le \phi\left(\lim_{\delta\to0}\left\{\sup_{x\in B}\left[\max_{0\le r\le\delta}\|x-x_r\|\right]\right\}\right) \\ &< (1-M)\left(\lim_{\delta\to0}\left\{\sup_{x\in B}\left[\max_{0\le r\le\delta}\|x-x_r\|\right]\right\}\right), \end{aligned}$$

which means that

$$\chi(S(B)) \le (1-M)\chi(B),$$

which illustrates that S is a strictly $(1-M)$-set contractive map. Now, invoking Theorem 2.5, we obtain that Equation (2.9) has at least one solution in $\mathcal{K}$.

Finally, let $x, y \in \mathcal{K}$ be any two solutions of Equation (2.9). Again we have

$$\begin{aligned} |x(t) - y(t)| &\le \left|\int_a^t \kappa(t,s)\beta|x(s)-y(s)|ds\right| + |g(t,x(t)) - g(t,y(t))| \\ &\le M\|x-y\| + |g(t,x(t)) - g(t,y(t))|. \end{aligned}$$

Suppose now there exists $t_0 \in [a,b]$ such that $x(t_0) \neq y(t_0)$. Then $(\mathcal{H}3)$ give

$$\begin{aligned}\|x-y\| &\leq M\|x-y\| + \phi(\|x-y\|) \\ &< M\|x-y\| + (1-M)\|x-y\|,\end{aligned}$$

which is a contradiction. This completes the proof. □

Next, let us investigate the supercritical case when $M \geq 1$. In this case, we rewrite ρ_p using the definition of p as

$$\rho_p = \begin{cases} \frac{pp!}{p!-[c(b-a)]^p}, & \text{if } M = 1, \\ \frac{(M^p-1)p!}{(M-1)\{p!-[c(b-a)]^p\}}, & \text{if } M > 1. \end{cases}$$

We now assume that the functions concerning Equation (2.9) satisfy the following hypotheses:

($\mathcal{H}4$) There exists an $R > 0$ such that $\rho_p g_R \leq R$, where

$$g_R = \sup\{|g(t,y)| : (t,y) \in [a,b] \times [-R,R]\}.$$

($\mathcal{H}5$) For each fixed $t \in [a,b]$, we have

$$|g(t,x) - g(t,y)| \leq \phi_p(|x-y|), \forall x,y \in [-R,R],$$

where $\phi_p : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing continuous function satisfying $\phi_p(r) < \rho_p^{-1} r$, for all $r > 0$.

($\mathcal{H}6$) For each fixed $t \in [a,b]$ we have

$$f(t,0) \equiv 0$$

By invoking Theorem 2.5, we derive the following result.

Theorem 2.10. *Suppose that the conditions* ($\mathcal{H}4$) , ($\mathcal{H}5$) *and* ($\mathcal{H}6$) *hold. Then Equation* (2.9) *has at least one solution in* $\mathcal{C}([a,b],\mathbb{R})$.

Proof. The use of condition ($\mathcal{H}6$) give us

$$(T0)(t) = 0 \quad \forall t \in [a,\, b]$$

which means that

$$((I-T)0)(t) = 0 \quad \forall t \in [a,\, b].$$

by the invertibility of $(I - T)$ we infer that

$$((I - T)^{-1}0)(t) = 0 \quad \forall t \in [a, b]. \tag{2.19}$$

Now, if $x = Tx + Sy$ with $y \in \mathfrak{B}_R$, then one has

$$x(t) = (I - T)^{-1} g(t, y(t)).$$

Now, by (2.19) and $(\mathcal{H}4)$ we can easily deduce hat

$$\begin{aligned}
\|x\| &= \sup_{t \in [a, b]} \left|(I - T)^{-1} g(t, y(t))\right| \\
&= \sup_{t \in [a, b]} \left|(I - T)^{-1} g(t, y(t)) - ((I - T)^{-1}(0))(t)\right| \\
&\leq \sup_{t \in [a, b]} \rho_p \left|g(t, y(t)) - 0\right| \\
&\leq \rho_p g_R \leq R
\end{aligned}$$

that is, $x \in \mathfrak{B}_R$. The remaining argument is similar to that of Theorem 2.9 and therefore is omitted. □

Chapter 3

Fixed Point Theorems Under Weak Topology Using MWNC

In this chapter, we study some generalisation of Schauder's and Krasnoselskii's fixed point theorems invoking the technique of measures of weak noncompactness in Banach spaces, in the context that the involved operators are not weakly compact.

3.1 Generalisation of Tychonoff's Fixed Point Theorem Using Measure of Weak Noncompactness

Let's us show some fixed point theorems. The following Tychonoff's Theorem, stated for Banach spaces endowed with it weak topology.

Theorem 3.1. *(Tychonoff [20]) Let X be a Banach space and let $\mathcal{M}$ be a weakly compact, convex subset of X. Then, each weakly continuous mapping T, $T : \mathcal{M} \to \mathcal{M}$, has a fixed point.*

Definition 3.1. Let X be a Banach space. A map $T : \mathcal{M} \subset X \to X$ is called (k, ψ)-weakly set contractive (resp. ψ-weakly condensing), if it is (k, ψ)-set contractive (resp. ψ-condensing) for some measure of weak noncompactness ψ.

Presently, we want to state Darbo's fixed point theorem under weak topology.

Theorem 3.2. *[20] Let $\mathcal{M}$ is non void, bounded, closed, convex subset of X. Each weakly continuous mapping T, $T : \mathcal{M} \to \mathcal{M} \subset X$, which is a (β,ω)-weakly set contractive mapping, has a fixed point.*

Proof. Let $M_1 = \mathcal{M}$ and $M_{n+1} = \overline{\text{co}}\,(TM_n)$. It is clear that the sequence $(M_n)_{n\in\mathbb{N}}$ consists of nonempty closed convex decreasing subsets of $\mathcal{M}$. Since T is (β,ω)-weakly set contractive, then we have

$$\omega\left(M_2\right) = \omega\left(\overline{\text{co}}\left(TM_1\right)\right) = \omega\left(TM_1\right) \leqslant \beta\omega\left(M_1\right).$$

Proceeding by induction we get:

$$\omega\left(M_{n+1}\right) \leqslant \beta^n \omega(\mathcal{M})$$

and therefore $\lim\limits_{n\to\infty} \omega\left(M_n\right) = 0$. Using Contor's property (vi) of $\omega(\cdot)$ we infer that $\mathcal{N} := \bigcap\limits_{n=1}^{\infty} M_n$ is a nonempty closed convex weakly compact subset of $\mathcal{M}$. Moreover, it is easily seen that $T\mathcal{N} \subset \mathcal{N}$. Now, the use of Theorem 3.1 concludes the proof. □

Let N be a nonlinear operator from X into itself. In some cases, we deal with operators which are continuous and weakly compact. Since neither the continuity implies the weak continuity nor the weak compactness implies the strong compactness, we can't use Schauder or Tychonoff's fixed point theorems, for this reason Latrach, Taoudi, and Zeghal[24] used the following conditions:

$$(\mathcal{A}1)\quad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } X, \text{ then} \\ (Nx_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } X. \end{cases}$$

$$(\mathcal{A}2)\quad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } X, \text{ then} \\ (Nx_n)_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } X. \end{cases}$$

Remark 3.1.
1) Operators satisfying $(\mathcal{A}1)$ or $(\mathcal{A}2)$ are not necessarily weakly continuous.

2) Continuous mappings satisfying $(\mathcal{A}1)$ are called ws-compact mappings and continuous mappings satisfying $(\mathcal{A}2)$ are called ww-compact mappings.

3) Every (β,ω)-weakly set contractive map satisfies ($\mathcal{A}2$).

4) A map N satisfies ($\mathcal{A}2$) if and only if it maps relatively weakly compact sets into relatively weakly compact ones.

5) A map N satisfies ($\mathcal{A}1$) if and only if it maps relatively weakly compact sets into relatively compact ones.

6) The condition ($\mathcal{A}2$) holds true for every bounded linear operator.

7) In reflexive Banach spaces, a mapping N verifying ($\mathcal{A}1$) is compact. This follows from the fact that bounded sets in reflexive Banach spaces are relatively weakly compact.

Theorem 3.3. *[24] Let $\mathcal{M}$ be a nonempty closed convex subset of a Banach space X. Assume that $T : \mathcal{M} \to \mathcal{M}$ is a continuous map which verifies ($\mathcal{A}1$). If $T(\mathcal{M})$ is relatively weakly compact, then there exists $x \in \mathcal{M}$ such that $Tx = x$.*

Proof. Let $\mathcal{C} = \overline{\text{co}}(T\mathcal{M})$ the closed convex hull of $T\mathcal{M}$. Since $\mathcal{M}$ is a closed convex subset of X satisfying $T(\mathcal{M}) \subseteq \mathcal{M}$, then $\mathcal{C} \subseteq \mathcal{M}$ and therefore $T\mathcal{C} \subseteq T\mathcal{M} \subseteq \overline{\text{co}}(T\mathcal{M})$. This shows that T maps $\mathcal{C}$ into itself. By hypothesis, $T\mathcal{M}$ is relatively weakly compact, so applying the Krein–Šmulian theorem one sees that $\mathcal{C}$ is weakly compact too. Let $\{\theta_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$, then it has a weakly convergent subsequence, say $\{\theta_{n_k}\}_{k\in\mathbb{N}}$. By hypothesis $\{T\theta_k\}_{k\in\mathbb{N}}$ has a strongly convergent subsequence and therefore $T\mathcal{C}$ is relatively compact. Now the use of the Schauder fixed point theorem concludes the proof. □

Theorem 3.4. *[24] Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a Banach space X. Assume that $T : \mathcal{M} \to \mathcal{M}$ is a continuous map satisfying ($\mathcal{A}1$). If T is (β,ω)-weakly set contractive, then there exists $x \in \mathcal{M}$ such that $Tx = x$.*

Proof. Let $M_1 = \mathcal{M}$ and $M_{n+1} = \overline{\text{co}}\,(TM_n)$. It is clear that the sequence $(M_n)_{n\in\mathbb{N}}$ consists of nonempty closed convex decreasing subsets of $\mathcal{M}$. Since T is (β,ω) -weakly set contractive, then we have

$$\omega\,(M_2) = \omega\,(\overline{\text{co}}\,(TM_1)) = \omega\,(TM_1) \leqslant \beta\omega\,(M_1)\,.$$

Proceeding by induction we get:

$$\omega\left(M_{n+1}\right) \leqslant \beta^{n} \omega(\mathcal{M})$$

and therefore $\lim_{n \rightarrow \infty} \omega\left(M_{n}\right)=0$. Using Contor's property (vi) of $\omega(.)$ we infer that $\mathcal{N}:=\bigcap_{n=1}^{\infty} M_{n}$ is a nonempty closed convex weakly compact subset of $\mathcal{M}$. Moreover, it is easily seen that $T\mathcal{N} \subset \mathcal{N}$. Accordingly, $T\mathcal{N}$ is relatively weakly compact. Now, the use of Theorem 3.3 concludes the proof. □

We can generalise this last theorem by weakly condensing hypothesis as follows:

Theorem 3.5. *[30] Let $\mathcal{M} \subset X$ be a nonempty closed, convex and bounded subset. Suppose that $T: \mathcal{M} \rightarrow \mathcal{M}$ is a continuous map satisfying the hypothesis $(\mathcal{A}1)$. If T is weakly condensing, then there exists $x^{*} \in \mathcal{M}$ such that $T x^{*}=x^{*}$.*

Arino, S. Gautier, and J.P. Penot in 1984, [6] established the following interesting fixed point theorem for weakly sequentially continuous mappings between Banach spaces.

Theorem 3.6. *(Arino, Gautier, Penot) Let $\mathcal{M}$ be a nonempty weakly compact convex subset of a Banach space X. Then each sequentially weakly continuous map $T: \mathcal{M} \rightarrow \mathcal{M}$ has a fixed point in $\mathcal{M}$.*

Theorem 3.7. *[30] Let $\mathcal{M} \subset X$ be a nonempty, bounded, closed and convex subset. Suppose that $T: \mathcal{M} \rightarrow \mathcal{M}$ is weakly sequentially continuous and weakly condensing map. Then T has at least one fixed point in $\mathcal{M}$.*

Lemma 3.1. *[2] Let $\mathcal{M}$ be a subset of X and $T: \mathcal{M} \rightarrow X$ is k-Lipschizian map. Assume that T satisfies the hypothesis $(\mathcal{A}2)$ (or is a sequentially weakly continuous map). Then :*

$$\omega(T(A)) \leq k \omega(A)$$

for each bounded subset A of $\mathcal{M}$; here, $\omega(\cdot)$ stands for the De Blasi measure of weak noncompactness.

Proof. Let A be a bounded subset of $\mathcal{M}$ and $r>\omega(A)$. There exist $0 \leq r_{0}<r$ and a weakly compact subset K of X such that $A \subseteq K+\mathfrak{B}_{r_{0}}$. Now we show that

$$T A \subseteq T K+\mathfrak{B}_{k r_{0}} \subseteq \overline{T K}^{w}+\mathfrak{B}_{k r_{0}}. \tag{3.1}$$

To see this let $x \in A$. Then there is a $y \in K$ such that $\|x - y\| \leq r_0$. Since T is k-Lipschitzian, then $\|Tx - Ty\| \leq k\|x - y\| \leq kr_0$. This proves (3.1). Further, since T satisfies ($\mathcal{A}2$), then the Eberlein-Šmulian theorem implies that $\overline{TK}^w$ is weakly compact. Consequently:

$$\omega(TA) \leq kr_0 \leq kr.$$

Letting $r \to \omega(A)$ we get

$$\omega(TA) \leq k\omega(A).$$

□

Corollary 3.1. *[30] Let $T : X \to X$ be a bounded linear operator. Then $\omega(T(A)) \leq \|T\|\omega(A)$ for each bounded subset A of X.*

Proof. We know that any bounded operator T is $\|T\|$-Lipschizian, thus by the previous lemma we get the conclusion. □

The next lemma proves that the same property holds for nonlinear contraction mapping provided that it is ww-compact.

Lemma 3.2. *[1] Let T be a ww-compact nonlinear contraction mapping on a Banach X. Then for each bounded subset A of X one has*

$$\omega(T(A)) \leq \phi(\omega(A)).$$

3.2 Weakly Non-compact Fixed Point Results of the Krasnoselskii Type

In 2013, Taoudi and Xiang used the techniques of measures of weak noncompactness to obtain some new generalized fixed point results of Krasnoselskii type, where they replaced the weak continuity of the involved operators by the hypotheses ($\mathcal{A}1$) and ($\mathcal{A}2$) and interchange the weak compactness of S by a k-weakly set contractive map with $k < 1$ or not. The following theorems illustrate this.

Notation: In what follows, we are denote by ψ to a regular measure of weak noncompactness.

Theorem 3.8. *[30] Let X be a Banach space and ψ a measure of weak noncompactness on X. Let $\mathcal{K} \subset X$ be a nonempty, bounded, closed convex subset and $T : X \to X$ be a map. Suppose that and $S : \mathcal{K} \to X$ is sequentially weakly continuous such that:*

(i) $(I - T)$ is invertible, $S(\mathcal{K}) \subset (I - T)(X)$ and $(I - T)^{-1}S$ is ψ-weakly condensing;

(ii) $[x = Tx + Sy, y \in \mathcal{K}] \Rightarrow x \in \mathcal{K}$ (or $S(\mathcal{K}) \subset (I - T)(\mathcal{K})$);

(iii) if (x_n) is a sequence in $\mathcal{F}(X, \mathcal{K}; T, S)$ with $x_n \rightharpoonup x$ for some $x \in \mathcal{K}$ then $Tx_n \rightharpoonup Tx$.

Then the sum $S + T$ possesses at least one fixed point in $\mathcal{K}$.

Proof. For each $y \in \mathcal{K}$, by the second part of (i), there exists $x \in X$ with $x - Tx = Sy$ and by (ii) we have $x = (I - T)^{-1}Sy := Ny \in \mathcal{K}$. Therefore, one obtains $N(\mathcal{K}) \subset \mathcal{K}$. Let $x_0 \in \mathcal{K}$ and

$$\mathcal{A} = \{A : x_0 \in A \subset \mathcal{K}, A \text{ is a closed convex set and } \mathrm{N}(A) \subset A\}.$$

Clearly, $\mathcal{A} \neq \emptyset$ since $\mathcal{K} \in \mathcal{A}$. Put $A_0 = \bigcap_{A \in \mathcal{A}} A$. Then $x_0 \in A_0 \subset \mathcal{K}$, A_0 is also a closed convex set and $N(A_0) \subset A_0$. Notice that $\overline{\mathrm{co}}\{N(A_0), x_0\} \subset A_0$. We thus have

$$N(\overline{\mathrm{co}}\{N(A_0), x_0\}) \subset N(A_0) \subset \overline{\mathrm{co}}\{N(A_0), x_0\},$$

which shows that $\overline{\mathrm{co}}\{N(A_0), x_0\} \in \mathcal{A}$. It then follows that $\overline{\mathrm{co}}\{N(A_0), x_0\} = A_0$. Using the properties of measures of weak noncompactness we get:

$$\psi(A_0) = \psi(\overline{\mathrm{co}}\{N(A_0), x_0\}) = \psi(\{N(A_0), x_0\}) = \psi(N(A_0)).$$

By the ψ-condensibility of N, we obtain $\psi(A_0) = 0$, and therefore, A_0 is a nonempty weakly compact convex set. Now we show that $N : A_0 \to A_0$ is weakly sequentially continuous. To see this, let $z, z_n \in A_0$ such that $z_n \rightharpoonup z$ and set $M = \{z_n : n \in \mathbb{N}\}$. Clearly, $N(M)$ is relatively weakly compact. Thus, there is a subsequence (z_{n_k}) of (z_n) such that

$$Nz_{n_k} \rightharpoonup u.$$

Using the equality

$$Nz_{n_k} = Sz_{n_k} + TNz_{n_k},$$

together with the weak sequential continuity of S and assumption (iii), we deduce $u = Tu + Sz$ and thus $u = (I-T)^{-1}Sz = Nz$. Accordingly

$$Nz_{n_k} \rightharpoonup Nz.$$

Now a standard argument shows that

$$Nz_n \rightharpoonup Nz.$$

Suppose the contrary, then there would exist a weak neighbourhood V^w of Nz and a subsequence $\left(z_{n_j}\right)$ of (z_n) such that $Nz_{n_j} \notin V^w$ for all $j \geq 1$. Naturally, $\left(z_{n_j}\right)$ converges weakly to z, then arguing as before we may extract a subsequence $\left(z_{n_{jk}}\right)$ of $\left(z_{n_j}\right)$ such that $Nz_{n_{j_k}} \rightharpoonup Nz$, which is absurd, since $Nz_{n_{j_k}} \notin V^w$ for all $k \geq 1$. Finally, N is weakly sequentially continuous. Now an application of Theorem 3.7 yields a point $x^* \in A_0$ with $x^* = Nx^* = (I-T)^{-1}Sx^*$, that is, $Tx^* + Sx^* = x^*$. □

Corollary 3.2. *[30] Let $\mathcal{K} \subset X$ be a nonempty, bounded, closed and convex subset. Suppose that T maps X into X and $S : \mathcal{K} \to X$ is sequentially weakly continuous such that:*

(i) $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $(I-T)^{-1}$ verifies $(\mathcal{A}2)$;

(ii) S is a (ρ,ω)-weakly set contractive map with $\rho < \gamma^{-1}$;

(iii) $S(\mathcal{K}) \subset (I-T)(X)$ and $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$ (or $S(\mathcal{K}) \subset (I-T)(\mathcal{K})$);

(iv) if (x_n) is a sequence in $\mathcal{F}(X,\mathcal{K};T,S)$ with $x_n \rightharpoonup x$ for some $x \in \mathcal{K}$ then $Tx_n \rightharpoonup Tx$.

Then the sum $S + T$ has at least one fixed point in $\mathcal{K}$.

Proof. Let $A \subset \mathcal{K}$ be bounded. We deduce from (i) and Lemma 3.1 that

$$\omega\left(\left((I-T)^{-1}S\right)(A)\right) \leq \gamma\omega(S(A)) \leq \rho\gamma\omega(A) < \omega(A).$$

This implies that $(I-T)^{-1}S$ is ω-weakly condensing. The result now follows from Theorem 3.7. □

Lemma 3.3. *[30] Let $\mathcal{M}, \mathcal{K}$ be two subsets of a linear normed space X, $S : \mathcal{K} \to X$ and $T : \mathcal{M} \to X$ be two maps. Assume that the condition (iii) of Theorem 3.8 holds. In addition, suppose further that the following conditions are satisfied*

(i) for all $z \in S(\mathcal{K})$, the map $T_z : \mathcal{M} \to X$ defined by $T_z x = Tx + z$ has a unique fixed point in $\mathcal{M}$; and

(ii) the set $\mathcal{F}(\mathcal{M}, \mathcal{K}; T, S)$ is relatively weakly compact.

Then $S : \mathcal{K} \to X$ is weakly compact and $S(\mathcal{K}) \subset (I - T)(\mathcal{M})$.

Proof. For each $x \in \mathcal{K}$, one can easily know from (i) that the equation

$$Ty + Sx = y, \tag{3.2}$$

has a unique solution $y = \tau Sx \in \mathcal{M}$. Thus, the mapping $\tau S : \mathcal{K} \to \mathcal{M}$ given by $x \to \tau Sx$ is well defined. In addition, we observe that $\tau Sx \in \mathcal{F}$ and hence $\tau S(\mathcal{K}) \subset \mathcal{F} \subset \mathcal{M}$. It follows from (3.2) that $Sx = (I - T)\tau Sx$ for all $x \in \mathcal{K}$. Therefore, we infer that

$$S(\mathcal{K}) = (I - T)\tau S(\mathcal{K}) \subset (I - T)(\mathcal{F}) \subset (I - T)(\mathcal{M}). \tag{3.3}$$

It is easy to see from (iii) that $(I - T)$ is sequentially weakly continuous on $\mathcal{F}$. Notice that $\mathcal{F}$ is relatively weakly compact. Thus, one can readily conclude from (3.3) that $S : \mathcal{K} \to X$ is weakly compact. This proves the lemma. □

Corollary 3.3. *[13] Let $\mathcal{K}, S$, (ii), (iii) be the same as Theorem 3.8. In addition, assume that*

(i') $T : X \to X$ is a contraction with constant $\alpha \in [0,1)$;

(i") the set $\mathcal{F}(\mathcal{M}, \mathcal{K}; T, S)$ is relatively weakly compact.

Then the conclusion of Theorem 3.8 holds.

We next see another version of Theorem 3.8.

Theorem 3.9. *[30] Let X be a Banach space and ψ be a measure of weak noncompactness on X. Let $\mathcal{K} \subset X$ be a nonempty, bounded, closed convex subset and $T : X \to X$ be a map. Suppose that and $S : \mathcal{K} \to X$ is sequentially weakly continuous such that* *(i)* $(I-T)$ *is injective;*

(ii) $\psi(T(A)+S(A))<\psi(A)$ *for all* $A\subset\mathcal{K}$ *with* $\psi(A)>0$*;*

(iii) $S(\mathcal{K})\subset(I-T)(X)$ *and* $[x=Tx+Sy, y\in\mathcal{K}]\Longrightarrow x\in\mathcal{K}$ *(or* $S(\mathcal{K})\subset(I-T)(\mathcal{K})$*);*

(iv) if (x_n) *is a sequence in* $\mathcal{F}(X,\mathcal{K};T,S)$ *with* $x_n\rightharpoonup x$ *for some* $x\in\mathcal{K}$ *then* $Tx_n\rightharpoonup Tx$*.*

Then the sum $S+T$ *possesses at least one fixed point in* $\mathcal{K}$*.*

Proof. For each $y\in\mathcal{K}$, by the first part of (iii), there exists $x\in X$ with

$$x-Tx=Sy.$$

Again by the second part of (iii) and (i), we have $x=(I-T)^{-1}Sy:=Ny\in\mathcal{K}$. Therefore, one obtains $N(\mathcal{K})\subset\mathcal{K}$. Let $x_0\in\mathcal{K}$ and

$$\mathcal{A}=\{A : x_0\in A\subset\mathcal{K}, A \text{ is a closed convex set and } N(A)\subset A\}.$$

Then $\mathcal{A}$ is nonempty. Furthermore, for any $A\in\mathcal{A}$ one has that

$$(I-T)^{-1}S(A)\subset T(I-T)^{-1}S(A)+S(A)\subset T(A)+S(A).$$

This fact couples with (ii) to yield

$$\psi(N(A))<\psi(A) \text{ for all } A\in\mathcal{A} \text{ with } \psi(A)>0. \tag{3.4}$$

Put $A_0=\bigcap_{A\in\mathcal{A}}A$. Keeping in mind (3.4), then imitating the proof of Theorem 3.8 one derives that A_0 is a nonempty weakly compact convex subset of $\mathcal{K}$ and $N(A_0)\subset A_0$. The reasoning in Theorem 3.8 shows that $N: A_0\to A_0$ is sequentially weakly continuous. The result then follows from Theorem 3.8. □

And we have this consequence of Theorem 3.9.

Corollary 3.4. *[30] Let* $\mathcal{K}$ *be a nonempty, bounded, closed and convex subset of a Banach space* X*. Suppose that* $S:\mathcal{K}\to X$ *and* $T: X\to X$ *are two weakly sequentially continuous mappings satisfying:*

(i) $S(\mathcal{K})$ *is relatively weakly compact;*

(ii) T *is a strict contraction with constant* τ*;*

(iii) $(x = Tx + Sy, y \in \mathcal{K})$ *implies* $x \in \mathcal{K}$*.*

Then $T + S$ *has at least one fixed point in* $\mathcal{K}$*.*

Proof. Since T is a strict contraction with constant $\tau \in [0,1)$, then by Lemma 1.3 the mapping $I - T$ is a homeomorphism from X into $(I - T)X$. Next, let y be fixed in $\mathcal{K}$. The map which assigns to each $x \in X$ the value $Tx + Sy$ defines a strict contraction from X into X. So, by the BCP, the equation $x = Tx + Sy$ has a unique solution $x \in X$. By hypothesis (iii) we have $x \in \mathcal{K}$. Hence, $S(\mathcal{K}) \subseteq (I - T)(\mathcal{K})$. Moreover, taking into account the sub-additivity of the De Blasi measure of weak noncompactness and using Lemma 3.1 we get for any bounded subset A of $\mathcal{K}$.

$$\omega(T(A) + S(A)) \leq \omega(T(A)) + \omega(S(A)) = \omega(T(A)) \leq \tau\omega(A) < \omega(A). \tag{3.5}$$

The result follows from Theorem 3.9. □

Together with Lemmas 1.7-3.1 and Theorem 3.5, some new forms of Krasnoseskii's fixed point theorem can be derived.

Theorem 3.10. *[30] Let* $\mathcal{K} \subset X$ *be a nonempty closed, convex and bounded subset. Suppose that* T *and* S *map* $\mathcal{K}$ *into* X *such that*

(i) T *is an expansive mapping with constant* $h > 1$*;*

(ii) $(I - T)^{-1}$ *satisfies the hypothesis* $(\mathcal{A}2)$*;*

(iii) S *is ws-compact on* $\overline{\text{co}}(\mathcal{F}(\mathcal{K}, \mathcal{K}; T, S))$*;*

(iv) $z \in S(\mathcal{K})$ *implies* $T(\mathcal{K}) + z \supset \mathcal{K}$*, where* $T(\mathcal{K}) + z = \{y + z |\ y \in T(\mathcal{K})\}$*;*

(v) S *is a strictly* $((h-1), \omega)$*-weakly set contractive map (or a* (k, ω)*-weakly set contractive map with* $k < h - 1$*).*

Then there exists a point $x^* \in \mathcal{K}$*, with* $Sx^* + Tx^* = x^*$*.*

Remark 3.2. It is worthy of pointing out that $\overline{\text{co}}(\mathcal{F}(\mathcal{K}, \mathcal{K}; T, S)) \subset \mathcal{K}$ and T may not be continuous since T is assumed only to be expansive.

Proof. From (i) and (iv), for each $y \in \mathcal{K}$, we know that the mapping $T + Sy : \mathcal{K} \to X$ satisfies the assumptions of Lemma 1.6. Hence the equation $Tx + Sy = x$ has a unique solution $x = \tau y \in \mathcal{K}$, so that the mapping $\tau : \mathcal{K} \to \mathcal{K}$ is well defined. In view of Lemma 1.7, we obtain that $\tau y = (I - T)^{-1}Sy$ for all $y \in \mathcal{K}$. Let $A \subset \mathcal{K}$ be bounded. From (ii), Lemmas 3.1 and 1.7, we conclude that

$$\omega(\tau(A)) = \omega\left((I - T)^{-1}S(A)\right) \leq \frac{1}{h - 1}\omega(S(A)). \tag{3.6}$$

Together with (3.6) and (v), one can see that $\tau : \mathcal{K} \to \mathcal{K}$ is weakly condensing. We denote by $C = \overline{\text{co}}(\mathcal{F}(\mathcal{K}, \mathcal{K}; T, S))$, then $C \subset \mathcal{K}$. Observe that $\tau(\mathcal{K}) \subset \mathcal{F}$. Thus, it is easy to know that $\tau(C) \subset C$. Let now $x_0 \in C$ and

$$\mathcal{A} = \{A : x_0 \in A \subset C, A \text{ is a closed convex set and } \tau(A) \subset A\}.$$

Then $\mathcal{A}$ is nonempty since $C \in \mathcal{A}$. Repeating the proof of Theorem 3.8 we know that $A_0 = \bigcap_{A \in \mathcal{A}} A$ is a nonempty, weakly compact convex subset of C and τ maps A_0 into A_0. We next show that $\tau : A_0 \to A_0$ is continuous and fulfils ($\mathcal{A}1$). Indeed, let (x_n) be a sequence in A_0 with $x_n \to x$ in A_0. Notice that $A_0 \subset C$ and S are continuous on C. Hence $Sx_n \to Sx$. Furthermore, we have by Lemma 1.7 that $(I - T)^{-1}$ is continuous. Since S fulfils the condition ($\mathcal{A}1$) on C and $(I - T)^{-1}$ is continuous, it follows easily that $\tau = (I - T)^{-1}S$ is continuous and satisfies the condition ($\mathcal{A}1$) on A_0. Now invoking Theorem 3.5, we achieve the proof. □

Corollary 3.5. *[30] Under the conditions of Theorem 3.10, if only the condition (iv) of Theorem 3.10 is replaced by T that maps $\mathcal{K}$ onto X, then there exists a point $x^* \in \mathcal{K}$ with $Sx^* + Tx^* = x^*$.*

Proof. Since T maps $\mathcal{K}$ onto X, which means $T(\mathcal{K}) = X$, we infer that if $z \in S(\mathcal{K})$ so $T(\mathcal{K}) + z \supset \mathcal{K}$; and we get the condition (iv) of Theorem 3.10. □

In general, the condition (iv) may be hard to be verified. The next result might be regarded as an improvement of Theorem 3.10.

Theorem 3.11. *[30] Let $\mathcal{K} \subset X$ be a nonempty closed, convex and bounded subset. Suppose that $T : X \to X$ and $S : \mathcal{K} \to X$ such that* *(i)* T *is expansive with constant* $h > 1$;

(ii) $(I - T)^{-1}$ *satisfies the hypothesis* $(\mathcal{A}2)$;

(iii) S *is continuous and fulfils the condition* $(\mathcal{A}1)$ *on* $\mathcal{K}$;

(iv) $S(\mathcal{K}) \subset (I - T)(X)$ *and* $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$ *(or* $S(\mathcal{K}) \subset (I - T)(\mathcal{K})$);

(v) S *is a strictly* $((h - 1), \omega)$*-weakly set contractive map (or a* (k, ω)*-weakly set contractive map with* $k < h - 1$).

Then there exists a point $x^* \in \mathcal{K}$ *with* $Sx^* + Tx^* = x^*$.

Proof. For each $y \in \mathcal{K}$, by the first part of (iv) there exists $x \in X$ such that

$$x - Tx = Sy.$$

By Lemma 1.7 and the second part of (iv) we have $x = (I - T)^{-1}Sy \in \mathcal{K}$. As is shown in Theorem 3.10 one has that $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is weakly condensing, continuous and fulfils the condition $(\mathcal{A}1)$. Consequently, there is a point $x^* \in \mathcal{K}$ with $x^* = (I - T)^{-1}Sx^*$. This completes the proof. □

With respect to the contractive map, thanks to Lemma 1.3 and Theorem 3.9 we obtain two types of such results. The first one is a complement to Theorem 3.11.

Theorem 3.12. *[30] Let* $\mathcal{K} \subset X$ *be a nonempty closed, convex and bounded subset. Suppose that* $T : X \to X$ *and* $S : \mathcal{K} \to X$ *such that.*

(i) T *is a ww-compact contraction with constant* $\alpha < 1$;

(ii) S *is ws-compact on* $\mathcal{K}$;

(iii) $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$ *(or* $S(\mathcal{K}) \subset (I - T)(\mathcal{K})$);

(iv) S *is a strictly* $((1 - \alpha), \omega)$*-weakly set contractive map (or a* (β, ω)*-weakly set contractive map with* $\beta < 1 - \alpha$).

Then there exists $x^* \in \mathcal{K}$ *with* $Sx^* + Tx^* = x^*$.

Proof. For each fixed $y \in \mathcal{K}$, the map of $X \to X$ defined by

$$x \to Tx + Sy,$$

is contractive. Hence there exists a unique $x \in X$ such that $x = Tx + Sy$. By (iii) and Lemma 1.3, we have $x = (I - T)^{-1}Sy \in \mathcal{K}$ for each $y \in \mathcal{K}$. Now, let $A \subset \mathcal{K}$ be bounded. By Lemma 3.1 together with the equality :

$$(I - T)^{-1}S = T(I - T)^{-1}S + S,$$

we obtain that

$$\omega\left(\left((I - T)^{-1}S\right)(A)\right) \leq \alpha\omega\left((I - T)^{-1}S(A)\right) + \omega(S(A)). \tag{3.7}$$

Thus, for all $A \subset$ with $\omega(A) > 0$, we have

$$\omega\left((I - T)^{-1}S(A)\right) \leq \frac{1}{1 - \alpha}\omega(S(A)) < \omega(A). \tag{3.8}$$

This shows that $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is a weakly condensing map.

Since $(I - T)^{-1}S : \mathcal{K} \to \mathcal{K}$ is continuous and fulfils the condition $(\mathcal{A}1)$, Theorem 3.5 works. □

Theorem 3.13. *[30] Let X be a Banach space and ψ be a measure of weak noncompactness on X. Let $\mathcal{K} \subset X$ be a nonempty closed, convex and bounded subset. Suppose that $T : X \to X$ and $S : \mathcal{K} \to X$ such that*

(i) T is a contraction;

(ii) S is ws-compact on $\mathcal{K}$;

(iii) $[x = Tx + Sy, y \in \mathcal{K}] \Longrightarrow x \in \mathcal{K}$;

(iv) $\psi(T(A) + S(A)) < \psi(A)$ for all $A \subset \mathcal{K}$ with $\psi(A) \neq 0$.

Then there exists a point $x^* \in \mathcal{K}$ with $Sx^* + Tx^* = x^*$.

Proof. For each fixed $y \in \mathcal{K}$, the map of $X \to X$ defined by

$$x \to Tx + Sy,$$

is contractive. Thus, there exists a unique $x \in X$ such that $x = Tx + Sy$. By (iii) and Lemma 1.3, we have $x = (I - T)^{-1}Sy := Ny \in \mathcal{K}$ for each $y \in \mathcal{K}$. Hence, we obtain $N(\mathcal{K}) \subset \mathcal{K}$. Let $x_0 \in \mathcal{K}$ and

$$\mathcal{A} = \{A : x_0 \in A \subset \mathcal{K}, A \text{ is a closed convex set and } \mathrm{N}(A) \subset A\}.$$

Clearly, $\mathcal{A} \neq \emptyset$ since $\mathcal{K} \in \mathcal{A}$. Moreover, for each $A \in \mathcal{A}$ we have that

$$(I - T)^{-1}S(A) \subset T(I - T)^{-1}S(A) + S(A) \subset T(A) + S(A).$$

This fact together with (iv) yields that $\psi(N(A)) < \psi(A)$ for all $A \in \mathcal{A}$ with $\psi(A) > 0$. Putting $A_0 = \bigcap_{A \in \mathcal{A}} A$ and repeating the proof of Theorem 3.7 one obtains that A_0 is a nonempty weakly compact convex subset of $\mathcal{K}$ and $N(A_0) \subset A_0$. It follows from Lemma 1.3 that $(I - T)^{-1}$ is continuous. Thus, it is easy to show that N is continuous and satisfies $(\mathcal{A}1)$ on A_0. Then Theorem 3.5 gives the desired result. □

The following results are consequences of Theorem 3.13. The proofs are more easier to be shown due to Lemma 1.3, and Theorem 3.8.

Corollary 3.6. *[30] Let $\mathcal{K}$ be a nonempty bounded convex convex subset of a Banach space X. Suppose that $S : \mathcal{K} \to X$ and $T : X \to X$ such that:*

(i) S is ws-compact;

(ii) there exists $\gamma \in [0,1)$ such that $\psi(T(A) + S(A)) \leq \gamma\psi(A)$ for all $A \subseteq \mathcal{K}$. Here ψ is an arbitrary measure of weak noncompactness on X;

(iii) T is a contraction;

(iv) $(x = Tx + Sy, y \in \mathcal{K})$ implies $x \in \mathcal{K}$.

Then there is a $x \in \mathcal{K}$ such that $Sx + Tx = x$.

Corollary 3.7. *[30] Let $\mathcal{K}, T, S$ and (iii) be the same as Theorem 3.13. In addition, assume that the following conditions are satisfied.*

(i) T is a contraction with constant $\alpha \in [0,1)$ and satisfies $(\mathcal{A}2)$;

(ii) $S(\mathcal{K})$ *is relative weakly compact,* S *is continuous and fulfils* $(\mathcal{A}1)$.

Then the conclusion of Theorem 3.13 holds.

Corollary 3.8. *[30] Let* $\mathcal{K}$ *be the same as Theorem 3.13. Assume that* $S: \mathcal{K} \to \mathcal{K}$ *is continuous and verifies the condition* $(\mathcal{A}1)$. *If* $S(\mathcal{K})$ *is relatively weakly compact, then there exists* $x^* \in \mathcal{K}$ *such that* $Sx^* = x^*$.

3.3 Application

Consider the following variant of Hammerstein's integral equation

$$x(t) = g(t, x(t)) + \lambda \int_0^1 \kappa(t, s) f(s, x(s)) \mathrm{d}s, \tag{3.9}$$

in $L^1(0,1)$, the space of Lebesgue integrable functions on $(0,1)$ with values in $\mathbb{R}$. Here $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $\kappa(\cdot,\cdot)$ are measurable functions and λ is a non-negative real parameter.

Let us mention that in general it is very difficult to express the measure ω with help of a convenient formula in a concrete nonreflexive Banach space X as like in the case of the Lebesgue space $L^1(0,1)$ [19]. But on the other hand in this space the following convenient criterion of weak compactness is known [19]:

Theorem 3.14. *A bounded set* B *in* $L^1(0,1)$ *is weakly sequentially compact if and only if*

$$\lim_{\substack{\mathrm{meas}(E) \to 0 \\ E \subset (0,1)}} \int_E x(s) ds = 0,$$

uniformly with respect to $x \in B$.

Let us notice that the above theorem may be rewritten in the following equivalent form:

Theorem 3.15. *[12] A bounded set* B *in* $L^1(0,1)$ *is weakly sequentially compact if and only if*

$$\lim_{\varepsilon \to 0} \left\{ \sup \left[\int_a^b |x(t)| dt : 0 \leqslant a \leqslant b \leqslant 1,\, b - a \leqslant \varepsilon \right] \right\} = 0,$$

uniformly with respect to $x \in B$.

The following definition and lemma give a characterization of $\omega(B)$ for any bounded subset B of L^1.

Definition 3.2. Let B be a bounded subset of $L^1\,(0{,}1)$. We call the following real number

$$\pi_1(B) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in K} \left\{ \sup \left\{ \int_a^b |x(t)| dt : 0 \leqslant a \leqslant b \leqslant 1, b - a \leqslant \varepsilon \right\} \right\} \right\},$$

the measure of nonequiabsolute continuity of B.

This formula was obtained by Appell and De Pascale [5].

Lemma 3.4. *[23] Let B is a bounded subset of $L^1\,(0{,}1)$, then $\omega(B) = \pi_1(B)$.*

Recall that a function $f : (0{,}1) \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if

$$\begin{cases} t \longrightarrow f(t,x) \text{ is measurable on } (0{,}1) \text{ for all } x \in \mathbb{R}, \\ x \longrightarrow f(t,x) \text{ is continuous on } \mathbb{R} \text{ for almost all } t \in (0{,}1). \end{cases}$$

To every function x being measurable on $(0{,}1)$, we may assign the function $(\mathbf{N}_f x)\,(t) = f(t, x(t)), t \in (0{,}1)$. The operator $\mathbf{N}_f$ defined in such a way is called Nemytskii operator or the superposition operator generated by the function f. The superposition operator enjoys several nice properties. We recall the following results which states a basic fact for the theory of these operators on L^1 spaces.

Lemma 3.5. *[30] Let $f : (0{,}1) \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the superposition operator $\mathbf{N}_f$ maps $L^1(0{,}1)$ into $L^1(0{,}1)$ if and only if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+(0{,}1)$ such that*

$$|f(t,x)| \leq a(t) + b|x|,$$

where $L^1_+(0{,}1)$ denotes the positive cone of the space $L^1(0{,}1)$.

Remark 3.3. Under the conditions of Lemma 3.5 the operator $\mathbf{N}_f$ is obviously continuous and maps bounded sets of $L^1(0{,}1)$ into bounded sets of $L^1(0{,}1)$.

Lemma 3.6. *[30] Suppose $f : (0{,}1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\mathbf{N}_f$* maps $L^1(0{,}1)$ *into itself, then $\mathbf{N}_f$ satisfies $(\mathcal{A}2)$.*

The problem (3.9) will be discussed under the following assumptions:

($\mathcal{H}_1$) there is a $h > 1$ such that for all $x, y \in L^1(0,1)$ and for all $t \in (0,1)$ we have

$$|g(t, x(t)) - g(t, y(t)| \geq h|x(t) - y(t)|, \tag{3.10}$$

($\mathcal{H}_2$) for each $x \in L^1(0,1), g(\cdot, x(\cdot))$ is integrable on $(0,1)$ and for each $t \in (0,1)$, $g(t,\cdot) : L^1(0,1) \to L^1(0,1)$ is onto;

($\mathcal{H}_3$) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist a constant $b > 0$ and a function $a(\cdot) \in L^1_+(0,1)$ such that $|f(t, x)| \leq a(t) + b|x|$ for all $t \in (0,1)$ and $x \in \mathbb{R}$;

($\mathcal{H}_4$) The function $\kappa : (0,1) \times (0,1) \to \mathbb{R}$ is strongly measurable and $\int\limits_0^1 \kappa(\cdot, s)x(s)\mathrm{d}s \in L^1(0,1)$ whenever $x \in L^1(0,1)$ and there exists a function $\rho : (0,1) \to \mathbb{R}$ belonging to $L^\infty(0,1)$ such that $|\kappa(t, s)| \leq \rho(t)$ for all $(t, s) \in (0,1) \times (0,1)$;

($\mathcal{H}_5$) $h - 1 - \lambda\|\rho\|b > 0$.

The following theorem provides an existence result for equation (3.9).

Theorem 3.16. *[30] Assume that the conditions ($\mathcal{H}_1$)-($\mathcal{H}_5$) are satisfied. Then the problem (3.9) has at least one solution in $L^1(0,1)$.*

First, notice that the problem (3.9) may be written abstractly in the form:

$$x = Tx + Sx, \quad x \in L^1(0,1),$$

where T is defined on $L^1(0,1)$ by $(Tx)(t) = g(t, x(t)) - g(t, 0)$ for $x \in L^1(0,1)$ $t \in (0,1)$. The map S is defined for $x \in L^1(0,1)$ by

$$Sx = g(\cdot, 0) + K\mathbf{N}_f x, \tag{3.11}$$

where $\mathbf{N}_f$ is the superposition operator associated to f and K denotes the linear integral operator defined by

$$K : L^1(0,1) \to L^1(0,1) : u(t) \mapsto Ku(t) := \lambda \int_0^1 \kappa(t, s)u(s)\mathrm{d}s.$$

Note that, for any $x \in L^1(0,1)$, the function $Sx + Tx$ belongs to $L^1(0,1)$ which is a consequence of the assumptions ($\mathcal{H}_2$), ($\mathcal{H}_3$) and ($\mathcal{H}_4$).

Our strategy is to apply Theorem 3.11 to find a fixed point for the operator $S+T$ in L^1. The proof will be broken up into several steps.

Claim 1: We first show that S is continuous. To see this, first notice that the assumption $(\mathcal{H}_3)$ and Lemma 3.5 guarantee that $\mathbf{N}_f$ maps continuously $L^1(0,1)$ into itself. To complete the proof it remains only to show that K is continuous, to this end; let $u \in L^1(0{,}1)$ from the hypothesis $(\mathcal{H}_4)$, we get

$$\begin{aligned}
\|Ku\|_{L^1} &= \int_0^1 |Ku(t)|\mathrm{d}t \\
&= \int_0^1 \left|\lambda \int_0^1 \kappa(t,s)u(s)\mathrm{d}s\right| \mathrm{d}t \\
&\leq |\lambda| \int_0^1 \int_0^1 |\kappa(t,s)||u(s)|\mathrm{d}s \ \mathrm{d}t \\
&\leq \lambda \int_0^1 \rho(t) \int_0^1 |u(s)|\mathrm{d}s \ \mathrm{d}t \\
&\leq \lambda \|\rho\|_{L^\infty} \operatorname{meas}((0{,}1)) \|u\|_{L^1} \\
&= \lambda \|\rho\|_{L^\infty} \|u\|_{L^1},
\end{aligned}$$

which give us that K is bounded linear operator; hence S is continuous.

Claim 2: We then illustrate that S verifies $(\mathcal{A}1)$. To see this, let $(\theta_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of $L^1(0{,}1)$. Using Lemma 3.6 the sequence $(\mathbf{N}_f(\theta_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence, say $(\mathbf{N}_f(\theta_{n_k}))_{k\in\mathbb{N}}$. Let θ be the weak limit of $(\mathbf{N}_f(\theta_{n_k}))$. Accordingly, keeping in mind the boundedness of the mapping $\kappa(t,.)$ we get

$$\int_0^1 \kappa(t,s) f\left(s, \theta_{n_k}(s)\right) \mathrm{d}s \to \int_0^1 \kappa(t,s) f(s,\theta(s))\mathrm{d}s.$$

The use of the dominated convergence Theorem S 1.13 allows us to conclude that the sequence $(S\theta_{n_k})$ converges in $L^1(0{,}1)$.

Claim 3: We prove that S maps bounded sets of $L^1(0{,}1)$ into weakly compact sets. To this end, let A be a bounded subset of $L^1(0{,}1)$ and let $M>0$ such that $\|x\|_{L^1} \leq M$ for all $x \in A$. For $x \in A$ we have

$$\begin{aligned}
|(Sx)(t)| &\leq |g(t,0)| + \lambda \int_0^1 |\kappa(t,s)||f(s,x(s)|\mathrm{d}s \\
&\leq |g(t,0)| + \lambda \int_0^1 |\rho(t)|(a(s) + b|x(s)|)\mathrm{d}s \\
&\leq |g(t,0)| + \lambda \rho(t)(\|a\|_{L^1} + bM).
\end{aligned}$$

Consequently,

$$\int_E |(Sx)(t)|\mathrm{d}t \leq \int_E |g(t,0)|\,\mathrm{d}t + \lambda(\|a\|_{L^1} + bM)\int_E |\rho(t)|\,\mathrm{d}t,$$

for all measurable subsets E of (0,1). Taking into account the fact that any set consisting of one element is weakly compact and using Theorem 3.14, we obtain

$$\lim_{\mathrm{meas}(E)\to 0}\int_E |\rho(t)|\mathrm{d}t = 0 \quad \text{and} \quad \lim_{\mathrm{meas}(E)\to 0}\int_E |g(t,0)|\mathrm{d}t = 0.$$

Since $S(A)$ is bounded, applying the sufficient condition of Theorem 3.14 we infer that the set $S(A)$ is sequentially weakly compact, and by Eberlein–Šmulian's Theorem conclude that $S(A)$ is weakly compact.

Claim 4: We will examine that there is a $r > 0$ such that for all $x \in L^1(0,1)$ we have $(x = Tx + Sy, y \in \mathfrak{B}_r)$ that implies $x \in \mathfrak{B}_r$. To perform this, put

$$r = \frac{\|g(.,0)\| + \lambda\|\rho\|\|a\|}{h - 1 - \lambda\|\rho\|b},$$

and let $x \in L^1(0,1)$ and $y \in \mathfrak{B}_r$ such that $x = Tx + Sy$. Then for all $t \in (0,1)$ we have

$$x(t) = g(t, x(t)) + \lambda\int_0^1 \kappa(t,s)f(s,y(s))\mathrm{d}s.$$

Thus,

$$x(t) - g(t,x(t)) + g(t,0) = g(t,0) + \lambda\int_0^1 \kappa(t,s)f(s,y(s))\mathrm{d}s.$$

The use of condition $(\mathcal{H}_1)$ give us

$$\begin{aligned} |x(t) - g(t,x(t)) + g(t,0)| &\geq |g(t,x(t)) - g(t,0)| - |x(t)| \\ &\geq (h-1)|x(t)|. \end{aligned} \tag{3.12}$$

Therefore,

$$(h-1)|x(t)| \leq |x(t) - g(t,x(t)) + g(t,0)| \leq |g(t,0)| + \lambda\int_0^1 |\kappa(t,s)\|f(s,y(s))|\mathrm{d}s.$$

In view of our assumptions

$$|x(t)| \leq \frac{1}{h-1}\left(|g(t,0)| + \lambda|\rho(t)|\int_0^1 (|a(s)| + b|y(s)|)\mathrm{d}s\right).$$

Hence

$$|x(t)| \leq \frac{1}{h-1}(|g(t,0)| + \lambda|\rho(t)|(\|a\| + b\|y\|).$$

By integration on (0,1) we get

$$\|x\| \leq \frac{1}{h-1}(\|g(.,0)\| + \lambda\|\rho\|\|a\| + \lambda b\|\rho\|\|y\|).$$

As a consequence,

$$\|x\| \leq \frac{1}{h-1}(\|g(.,0)\| + \lambda\|\rho\|\|a\| + \lambda b\|\rho\| r) = r.$$

Thus, $x \in \mathfrak{B}_r$.

Claim 5: We shall illuminate that

$$\omega((I-T)(M)) \geq (h-1)\omega(M), \tag{3.13}$$

for any bounded subset M of $L^1(0,1)$. This obviously implies that $(I-T)^{-1}$ verifies $(\mathcal{A}2)$. To do so, note first that for all $x \in L^1(0,1)$ and for all $t \in (0,1)$ by (3.12) we have

$$(h-1)|x(t)| \leq |x(t) - Tx(t)|. \tag{3.14}$$

Hence,

$$(h-1)\int_D |x(t)|\mathrm{d}t \leq \int_D |x(t) - Tx(t)|\mathrm{d}t, \tag{3.15}$$

for any subset D of $L^1(0,1)$. This leads to (3.13).

Claim 6: We shall prove that $S(\mathfrak{B}_r) \subset (I-T)(L^1(0,1))$. To see this, let $y \in L^1(0,1)$ be fixed. We define $U : L^1(0,1) \to L^1(0,1)$ by

$$(U_y x)(t) = (Tx)(t) + y(t) = g(t, x(t)) - g(t,0) + y(t)$$

Then U_y is expansive with constant h. From Assumption $(\mathcal{H}_2)$ it follows that U_y is onto. By Lemma 1.6 we know there exists $x^* \in L^1(0,1)$ such that $U_y x^* = x^*$, that is $(I-T)x^* = y$. Hence $S(\mathfrak{B}_r) \subset L^1(0,1) \subset (I-T)(L^1(0,1))$.

Thus, the hypotheses of Theorem 3.11 are all fulfilled. This gives a fixed point for $S+T$ and hence an integrable solution to equation (3.9).

Example: Consider that following Hammerstein integral equation

$$x(t) = \frac{1}{t^2+1} + 2x(t) + \int_0^1 e^{-(t+s)}t\left[\frac{1}{s^2+1} + \frac{s}{2s+1}x(s)\sin(x(s))\right]\mathrm{d}s, \tag{3.16}$$

where $t \in (0,1)$. It is easily seen that equation (3.16) is a particular case of equation (3.9) where

$$\kappa(t,s) = e^{-(t+s)}t,$$

and

$$f(t,x) = \frac{1}{t^2+1} + \frac{t}{2t+1}x\sin x,$$

and

$$g(t,x(t)) = \frac{1}{t^2+1} + 2x(t).$$

Clearly, we have

$$|\kappa(t,s)| = |e^{-(t+s)}t| \leq t = \rho(t),$$

where $\rho \in L^\infty(0,1)$ and since f is continuous in $(0,1)\times\mathbb{R}$, so it is a Carathéodory function, and

$$\begin{aligned}|f(t,x)| &= |\frac{1}{t^2+1} + \frac{t}{2t+1}x\sin x| \\ &\leq |\frac{1}{t^2+1}| + |\frac{t}{2t+1}x\sin x| \\ &\leq \underbrace{\frac{1}{t^2+1}}_{a(t)} + \underbrace{\frac{1}{3}}_{b}|x|.\end{aligned}$$

Obviously, g satisfies the conditions $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ with constant $h = 2$, and

$$h - 1 - \|\rho\|b = 2 - 1 - \frac{1}{3} = \frac{2}{3} > 0.$$

Now, applying Theorem 3.16 we obtain that the equation 3.16 has at least one solution in the space $L^1(0,1)$.

Chapter 4

Fixed-Point Theorems for Block Operator Matrix

In this chapter, we are interesting to show some fixed point theorems for a 2×2 block operator matrix with nonlinear entries (in short BOM) acting on a product of two Banach spaces. We will prove some theorems by combining the results studied in Chapter 2, Chapter 3 and some ideas in the book [23].

4.1 Schauder's and Krasnoselskii's Fixed Point Theorems for BOM

Let $\mathcal{M}$ be a nonempty, bounded, closed, and convex subset of a Banach space X. We consider the 2×2 block operator matrix

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.1}$$

in the space $\mathcal{M} \times X$ that is, the operators

$$A, C : \mathcal{M} \longrightarrow X,$$
$$B, D : X \longrightarrow X.$$

In this section, we impose some conditions on the entries to discuss the existence of fixed points for the block operator matrix (4.1). This discussion is based on the invertibility or not of the diagonal terms of $\mathcal{I}-\mathcal{L}$. The study will be broken into three cases as follow:

The both of the diagonal entries of $\mathcal{I}-\mathcal{L}$ is invertible

We assume that the entries of (4.1) fulfils :

($\mathcal{H}_1$) The operator A fulfils the conditions of Lemma 2.1 or Lemma 2.3 (resp. the operator D fulfils the conditions of Lemma 2.1 or Lemma 2.3) with the constant γ_A (resp. with the constant γ_D);

($\mathcal{H}_2$) The operator B is k_B-Lipschitz and C is nonlinear contraction with the function ϕ_C;

($\mathcal{H}_3$) $(I-A)^{-1}B(I-D)^{-1}C(\mathcal{M})$ is a subset of $\mathcal{M}$.

Theorem 4.1. *Under the assumptions* $(\mathcal{H}_1)-(\mathcal{H}_3)$, *the block operator matrix*(4.1) *has, at least, a fixed point in* $\mathcal{M}\times X$ *provided that* $\gamma_A\cdot\gamma_D\cdot k_B<1$.

Remark 4.1. We are going to show only the case when the both of A and D satisfy all the conditions of Lemma 2.1, the other cases are similar.

Proof. Since D fulfils the conditions of Lemma 2.1 so, $(I-D)^{-1}$ exists on X and γ_D-Lipschitz. Using the condition ($\mathcal{H}_2$) we get

$$\begin{aligned}\|B(I-D)^{-1}Cx-B(I-D)^{-1}Cy\| &\leq k_B(\|(I-D)^{-1}Cx-(I-D)^{-1}Cy\|)\\ &\leq k_B\cdot\gamma_D\|Cx-Cy\|\\ &\leq k_B\cdot\gamma_D\cdot\phi_C(\|x-y\|)\\ &\leq k_B\cdot\gamma_D\|x-y\|.\end{aligned}$$

So, by the Lemma 2.2

$$S:=B(I-D)^{-1}C:\mathcal{M}\longrightarrow X$$

is $k_B\cdot\gamma_D$-set contractive.

But A is also verifies all the conditions of Lemma 2.1 so, $(I-A)$ is Lipschitz invertible with the constant $\gamma_A<\dfrac{1}{k_B\cdot\gamma_D}$, and maps X onto X. And because $S:\mathcal{M}\to X$, it follows that for every $y\in\mathcal{M}$, there exists $x\in X$ such that

$$\begin{aligned} x - Ax = Sy \quad &\Longleftrightarrow \quad (I - A)x = Sy \\ &\Longrightarrow x = (I - A)^{-1} Sy, \end{aligned}$$

now from $(\mathcal{H}_3)$, $x \in \mathcal{M}$. So all the conditions of Theorem 2.6 are satisfy which means that there exists $x_0 \in \mathcal{M}$ such that

$$\begin{aligned} Ax_0 + Sx_0 = x_0 &\Leftrightarrow (I - A)^{-1} Sx_0 = x_0 \\ &\Leftrightarrow (I - A)^{-1} B(I - D)^{-1} Cx_0 = x_0. \end{aligned}$$

Let $y_0 := (I - D)^{-1} Cx_0$, hence;

$$\begin{cases} Ax_0 + By_0 = x_0 \\ Cx_0 + Dy_0 = y_0 \end{cases} \Rightarrow \mathcal{L} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

□

Notation: We denote by $\Upsilon_{\mathcal{M}}^{(A,B,C,D)}$ the following set:

$$\Upsilon_{\mathcal{M}}^{(A,B,C,D)} = \left\{ x \in \mathcal{M} \text{ such that } x = Ax + B(I - D)^{-1} Cy; \text{ for some } y \in \mathcal{M} \right\}.$$

Theorem 4.2. *Let $\mathcal{M}$ be a nonempty,bounded, convex and closed subset of a Banach space X. Assume that $A, C : \mathcal{M} \longrightarrow X$, and $B, D : X \longrightarrow X$ are four operators satisfying the following conditions:*

$(\mathcal{H}_4)$ *$(I - A)$ is one-to-one;*

$(\mathcal{H}_5)$ *A is α-condensing with sequentially closed graph in $\Upsilon_{\mathcal{M}}^{(A,B,C,D)}$;*

$(\mathcal{H}_6)$ *$(I - D)$ is continuously invertible on $C(\mathcal{M})$;*

$(\mathcal{H}_7)$ *B and C are continuous mappings with $C(\mathcal{M})$ is relatively compact;*

$(\mathcal{H}_8)$ *$B(I - D)^{-1} C(\mathcal{M}) \subset (I - A)(\mathcal{M})$.*

Then the BOM (4.1) has at least one solution in $\mathcal{M} \times X$.

Proof. Since $(I - D)$ is continuously invertible, the operator

$$S =: B(I - D)^{-1} C : \mathcal{M} \to X,$$

is continuous. Let now Ω be a nonempty subset of $\mathcal{M}$ with $\alpha(\Omega) > 0$. By the subadditivity of the Kuratowski's measure of noncompactness, the first part of $(\mathcal{H}_5)$ and the last part of $(\mathcal{H}_7)$ we get

$$\begin{aligned}\alpha\left(A\Omega + B(I-D)^{-1}C\Omega\right) &\leq \alpha(A\Omega) + \alpha\left(B(I-D)^{-1}C\Omega\right)\\ &< \alpha(\Omega).\end{aligned}$$

Which means that the second condition (ii) of Theorem 2.7 is hold. Obviously the other conditions of Theorem 2.7 are satisfying, then BOM (4.1) has at least one solution in $\mathcal{M} \times X$. □

Just one of the diagonal entries of $\mathcal{I} - \mathcal{L}$ is invertible

We shall treat only the case of invertibility of $I - D$, the other case is similar just simply exchanging the roles of D and A and C and B. Assume that:

$(\mathcal{H}_9)$ The operator D fulfils the conditions of Lemma 2.1 with the constant γ_D;

$(\mathcal{H}_{10})$ B is nonlinear contraction with the functions ϕ_B;

$(\mathcal{H}_{11})$ C is contraction with the constant $k_C < \gamma_D^{-1}$;

$(\mathcal{H}_{12})$ A is a continuous compact operator;

$(\mathcal{H}_{13})$ $(A + B(I-D)^{-1}C)(\mathcal{M})$ is a subset of $\mathcal{M}$.

Theorem 4.3. *Under the assumptions $(\mathcal{H}_9) - (\mathcal{H}_{13})$, the BOM (4.1) has, at least, a fixed point in $\mathcal{M} \times X$.*

Proof. We only have to prove that the map

$$\Gamma := A + B(I-D)^{-1}C : \mathcal{M} \longrightarrow \mathcal{M},$$

is condensing with respect to α. (The continuity of Γ is obvious.)
Using the ideas used to prove Theorem 4.1, one easily derive that $S := B(I-D)^{-1}C$ is a $k_C \cdot \gamma_D$-set contractive.
Let now Ω be a nonempty subset of $\mathcal{M}$ with $\alpha(\Omega) > 0$. By the subadditivity of the Kuratowski's measure of noncompactness, we get

$$\begin{aligned}\alpha(\Gamma\Omega) &\leq \alpha\left(A\Omega + B(I-D)^{-1}C\Omega\right)\\ &\leq \alpha(A\Omega) + \alpha\left(B(I-D)^{-1}C\Omega\right)\\ &< \alpha(\Omega).\end{aligned}$$

This latter inequality means that Γ is a condensing map with respect to α, and Sadoveskii's fixed point theorem works. □

None of the diagonal entries of $(\mathcal{I} - \mathcal{L})$ is invertible

In this case, we discuss the existence of fixed points for the following perturbed block operator matrix by laying down some conditions on the entries.

$$\widetilde{\mathcal{L}} = \begin{pmatrix} A_1 & B \\ C & D_1 \end{pmatrix} + \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \tag{4.2}$$

Let $\mathcal{M}'$ be a nonempty, bounded, closed, and convex subset of a Banach space Y.

Assume that the operators A_1 and P_1 map X into X, B from $\mathcal{M}'$ into X, C from $\mathcal{M}$ into Y, and D_1 and P_2 from Y into itself. Suppose that the operator (4.2) respects the following assumptions:

($\mathcal{H}_{14}$) A_1 fulfils the conditions of Lemma 2.1 (resp. the operator D_1 fulfils the conditions of Lemma 2.1) with the constant γ_{A_1} (resp. with the constant γ_{D_1}),

($\mathcal{H}_{15}$) the operators P_1, P_2, B and C are k_{P_1}-Lipschitz, k_{P_2}-Lipschitz, k_B-Lipschitz, k_C-Lipschitz respectively,

($\mathcal{H}_{16}$) For every $x_1 \in X$, $x_2 \in \mathcal{M}$ and $y_1 \in Y$, $y_2 \in \mathcal{M}'$, $(I-A_1)^{-1}P_1x_1+(I-A_1)^{-1}By_2 \in \mathcal{M}$ and $(I-D_1)^{-1}Cx_2 + (I-D_1)^{-1}P_2y_1 \in \mathcal{M}'$.

Theorem 4.4. *Under the above assumptions* $(\mathcal{H}_{14}) - (\mathcal{H}_{16})$, *the block operator matrix* (4.2) *has, at least, a fixed point in* $\mathcal{M} \times \mathcal{M}'$ *provided that*

$$\max\left\{k_{P_1}\cdot\gamma_{A_1}, k_{P_2}\cdot\gamma_{D1}, \frac{\gamma_A\cdot k_B}{1-\gamma_A\cdot k_{P_1}}, \frac{\gamma_D\cdot k_C}{1-\gamma_D\cdot k_{P_2}}\right\} < 1.$$

To prove this theorem we need the following theorem which is define a MNC in Cartesian product $X = X_1 \times X_2 \times \ldots \times X_n$ based on its MNCs $\mu_1, \mu_2, \ldots, \mu_n$ respectively.

Theorem 4.5. *[9] Assume the function* $F :]0, +\infty)^n \to]0, +\infty)$ *is convex and* $F(x_1, x_2, \cdots, x_n) = 0$ *if and only if* $x_i = 0$ *for* $i = 1, 2, \ldots, n$*. Then*

$$\mu(B) = F\left(\mu_1\left(B^1\right), \mu_2\left(B^2\right), \ldots, \mu_n\left(B^n\right)\right) \quad \forall B \in \mathcal{B}_X,$$

defines the measure of noncompactness in $X = X_1 \times X_2 \times \ldots \times X_n$*.*
Here B^i *denotes the natural projection of* B *into* X_i*.*

Corollary 4.1. *The function*

$$F(B) := \max\{\mu_1\left(B^1\right), \mu_2\left(B^2\right), \ldots, \mu_n\left(B^n\right)\} \quad \forall B \in \mathcal{B}_X,$$

defines a MNC in Cartesian product $X = X_1 \times X_2 \times \ldots \times X_n$*.*

Now we are ready to prove Theorem 4.4.

Proof. Let us consider the following matrix equation

$$\begin{pmatrix} A_1 + P_1 & B \\ C & D_1 + P_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.3}$$

This latter is equivalent to

$$\begin{pmatrix} P_1 & B \\ C & P_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I - A_1 & 0 \\ 0 & I - D_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

By using the assumption $(\mathcal{H}_{14})$ we deduce that the right block matrix above is invertible and

$$\begin{pmatrix} (I - A_1)^{-1} & 0 \\ 0 & (I - D_1)^{-1} \end{pmatrix} \begin{pmatrix} P_1 & B \\ C & P_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

or, equivalently

$$\begin{pmatrix} (I - A_1)^{-1} P_1 & (I - A_1)^{-1} B \\ (I - D_1)^{-1} C & (I - D_1)^{-1} P_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, this latter may be transformed into

$$\mathcal{T}\begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{S}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\mathcal{T} = \begin{pmatrix} (I - A_1)^{-1} P_1 & 0 \\ 0 & (I - D_1)^{-1} P_2 \end{pmatrix}$$

and

$$\mathcal{S} = \begin{pmatrix} 0 & (I - A_1)^{-1} B \\ (I - D_1)^{-1} C & 0 \end{pmatrix}.$$

In one hand, we have

$$\max \{k_{P_1} \cdot \gamma_{A_1}, k_{P_2} \cdot \gamma_{D1}, \} < 1$$

it give us that the both operators $(I - A_1)^{-1} P_1$ and $(I - D_1)^{-1} P_2$ are contractions, so; $(\mathcal{I} - \mathcal{T})^{-1}$ exists in $X \times Y$ and Lipschitz with the constant $m_* = \max \left\{ \frac{1}{1 - \gamma_A \cdot k_{P_1}}, \frac{1}{1 - \gamma_D \cdot k_{P_2}} \right\}$.
In the other hand, $\mathcal{S}$ is m_*^{-1}-set contractive with respect to the measure of non compactness $\alpha(\cdot) = \max\{\alpha_X(.), \alpha_Y(.)\}$.
Using the condition $(\mathcal{H}_{16})$ we see that all the conditions of Theorem 2.6 are satisfy, hence, the block operator matrix (4.2) has at least one fixed point. □

4.2 Schauder's and Krasnoselskii's Fixed Point Theorems for BOM under Weak Topology

In this section, we will use the results studied in Chapter 3 in order to develop a general matrix fixed point theory under weak topology.

Theorem 4.6. *Let $\mathcal{M}$ be a nonempty, bounded, convex and closed subset of a Banach space X. Assume that $A, C : \mathcal{M} \longrightarrow X$, and $B, D : X \longrightarrow X$ are four operators satisfying the following conditions:*

(i) $(I - D)$ is weak sequential continuous invertible;

(ii) B and C are sequentially weakly continuous, in addition the operator $B(I - D)^{-1}C$ is k-contractive, $k > 0$;

(iii) A is ϕ-nonlinear contraction with $\phi(r) < (1 - k)r$, sequentially weakly continuous in $\Upsilon_{\mathcal{M}}^{(A,B,C,D)}$ and satisfies the condition $\mathcal{A}2$;

(iv) $B(I-D)^{-1}C(\mathcal{M}) \subset (I-A)(\mathcal{M})$.

Then BOM (4.1) has at lest one fixed point.

Proof. Using (i) and the first part of (ii) we deduce that

$$B(I-D)^{-1}C : \mathcal{M} \to X$$

is weakly sequentially continuous. The fact that A is ϕ-nonlinear contraction by Lemma 1.5, $(I-A)$ is a surjective operator and invertible, so the first and part of condition (i) and condition (ii) in Theorem 3.8 hold. We next claim that $(I-A)^{-1}B(I-D)^{-1}C$ is ω-weakly condensing. Indeed, we have the equality

$$\begin{aligned}(I-A)^{-1}B(I-D)^{-1}C &= (I-A+A)(I-A)^{-1}B(I-D)^{-1}C \\ &= B(I-D)^{-1}C + A(I-A)^{-1}B(I-D)^{-1}C,\end{aligned} \tag{4.4}$$

let Ω a bounded subset of $\mathcal{M}$, keeping in mind the first and the second part of (iii), and using the condition (ii) and Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}\omega((I-A)^{-1}B(I-D)^{-1}C(\Omega)) &\leq \omega(B(I-D)^{-1}C(\Omega)) + \omega(A(I-A)^{-1}B(I-D)^{-1}C(\Omega)) \\ &\leq k\omega(\Omega) + \phi(\omega((I-A)^{-1}B(I-D)^{-1}C(\Omega))) \\ &< k\omega(\Omega) + (1-k)\omega((I-A)^{-1}B(I-D)^{-1}C(\Omega)).\end{aligned}$$

This shown that, $(I-A)^{-1}B(I-D)^{-1}C$ is ω-weakly condensing. Now the use of Theorem 3.8 achieves the proof. □

Corollary 4.2. *Let $\mathcal{M}$ be a nonempty, bounded, convex and closed subset of a Banach space X. Assume that $A, C : \mathcal{M} \longrightarrow X$, and $B, D : X \longrightarrow X$ are four operators satisfying the following conditions:*

(i) $(I-D)$ *is weakly sequentially continuous invertible;*

(ii) B *and* C *are sequentially weakly continuous;*

(iii) A *is ϕ-nonlinear contraction and sequentially weakly continuous in* $\Upsilon_{\mathcal{M}}^{(A,B,C,D)}$;

(iv) the set $\Upsilon_{\mathcal{M}}^{(A,B,C,D)}$ *is relatively weakly compact;*

Then BOM (4.1) has at lest one fixed point.

Proof. Let $z \in B(I-D)^{-1}C(\mathcal{M})$, and we define the map $A_z : X \to X$ by $A_z x = Ax + z$. We claim that A_z has a unique fixed point in X. Keeping in the mind that A is ϕ-nonlinear contraction, and we have

$$\begin{aligned} \|A_z x - A_z y\| &= \|Ax + z - Ay + z\| \\ &\leq \phi(\|Ax - Ay\|). \end{aligned}$$

Which means that A_z is ϕ-nonlinear contraction and by Boyd and Wong Theorem 1.4, it has a unique fixed point.
By the second part of (iii) and (iv), together with Lemma 3.3 we conclude that $B(I-D)^{-1}C : \mathcal{M} \to X$ is weakly compact operator and $B(I-D)^{-1}C(\mathcal{M}) \subset (I-A)(\mathcal{M})$. Now, we claim that $(I-A)^{-1}B(I-D)^{-1}C$ is weakly compact. If this is not the case, then $d = \omega((I-A)^{-1}B(I-D)^{-1}C(\Omega)) > 0$, $\forall\, \Omega \in \mathcal{B}_{\mathcal{M}}$. The use of (4.4) yields

$$\begin{aligned} \omega((I-A)^{-1}B(I-D)^{-1}C(\Omega)) &\leq \underbrace{\omega(B(I-D)^{-1}C(\Omega))}_{=0} + \omega(A(I-A)^{-1}B(I-D)^{-1}C(\Omega)) \\ &\leq \phi(\omega((I-A)^{-1}B(I-D)^{-1}C(\Omega))) \\ &< \omega((I-A)^{-1}B(I-D)^{-1}C(\Omega)). \end{aligned}$$

which is a contradiction and consequently $\omega((I-A)^{-1}B(I-D)^{-1}C(\Omega))$ is relatively weakly compact whenever Ω is a bounded subset in $\mathcal{B}_{\mathcal{M}}$, that means $(I-A)^{-1}B(I-D)^{-1}C$ is weakly compact operator in the other words is 0-contractive, hence, Theorem 4.6 works.

□

Theorem 4.7. *[22] Let $\mathcal{M}$ be a nonempty, bounded, convex and closed subset of a Banach space X. Assume that $A, C : \mathcal{M} \longrightarrow X$, and $B, D : X \longrightarrow X$ are four weakly sequentially continuous operators satisfying the following conditions:*

(i) C is weakly compact operator;

(i) D is a ϕ-nonlinear contraction and $(I-D)^{-1}C(\mathcal{M})$ is bounded;

(iii) A is k-contraction, and;

(iv) $Ax + B(I-D)^{-1}Cx \in \mathcal{M}$ for all $x \in \mathcal{M}$.

Then BOM (4.1) has at lest one fixed point in $\mathcal{M} \times X$.

Proof. Since D is ϕ-nonlinear contraction, by Lemma 1.5 $(I-D)^{-1}$ exists on X. Now, we claim that $(I-D)^{-1}C(\mathcal{M})$ is relatively weakly compact. The use of

$$(I-D)^{-1}C = C + D(I-D)^{-1}C, \tag{4.5}$$

and also the weak compactness of $\overline{C(\mathcal{M})}^{w}$ yields

$$\omega\left((I-D)^{-1}C(\mathcal{M})\right) \leq \omega\left(D(I-D)^{-1}C(\mathcal{M})\right).$$

Let $\varepsilon > \omega\left((I-D)^{-1}C(\mathcal{M})\right)$. By using the definition of ω, there are $0 \leq \varepsilon_0 < \varepsilon$ and a set $K \in \mathcal{W}_X$ of $\mathcal{M}$ such that

$$D(I-D)^{-1}C(\mathcal{M}) \subseteq D(K) + \mathfrak{B}_{\phi(\varepsilon_0)}.$$

Taking into account that D is weakly sequential continuous and using the Eberlein-Šmulian's theorem, we infer that $D(K)$ is a relatively weakly compact subset of X and

$$\omega\left(D(I-D)^{-1}C(\mathcal{M})\right) < \phi(\varepsilon_0) < \phi(\varepsilon)$$

Letting $\varepsilon \to \omega\left((I-D)^{-1}C(\mathcal{M})\right)$, we obtain

$$\begin{aligned}\omega\left((I-D)^{-1}C(\mathcal{M})\right) \leq \omega\left(D(I-D)^{-1}C(\mathcal{M})\right) &\leq \phi(\omega\left((I-D)^{-1}C(\mathcal{M})\right))\\ &< \omega\left((I-D)^{-1}C(\mathcal{M})\right),\end{aligned}$$

which is a contradiction and consequently $(I-D)^{-1}C$ is weakly compact.
Next, let us show that the mapping $F : \mathcal{M} \longrightarrow X$ defined by the formula

$$F(x) = Ax + B(I-D)^{-1}Cx,$$

is weakly sequentially continuous. To do so, let $(\xi_n)_n$ be a sequence in $\mathcal{M}$ which converges weakly to ξ. Since $(I-D)^{-1}C(\mathcal{M})$ is relatively weakly compact, there exists a subsequence (ξ_{n_k}) of (ξ_n) such that $(I-D)^{-1}C(\xi_{n_k}) \rightharpoonup \gamma$. Taking into account the weak sequential continuity of the maps C and D and using the equality (4.5), to obtain $\gamma = (I-D)^{-1}C(\xi)$. Thus,

$$(I-D)^{-1}C(\xi_{n_k}) \rightharpoonup (I-D)^{-1}C(\xi).$$

Now, we show that

$$(I-D)^{-1}C\left(\xi_n\right) \rightharpoonup (I-D)^{-1}C(\xi).$$

Suppose the contrary, then there exists a weak neighbourhood V^w of $(I-D)^{-1}C(\xi)$ and a subsequence $\left(\xi_{n_j}\right)$ of (ξ_n) such that $(I-D)^{-1}C\left(\xi_{n_j}\right) \notin V^w$ for all $j \geq 1$, since $\left(\xi_{n_j}\right)$ converges weakly to ξ, and arguing as before, we find a subsequence $\left(\xi_{n_{j_k}}\right)$ of $\left(\xi_{n_j}\right)$ such that $(I-D)^{-1}C\left(\xi_{n_{j_k}}\right) \rightharpoonup (I-D)^{-1}C(\xi)$. Which is absurd, since $(I-D)^{-1}C\left(\xi_{n_{j_k}}\right) \notin V^w$. As a result, $(I-D)^{-1}C$ is weakly sequentially continuous.

We only have to show that the operator F is a weakly condensing operator with respect to ω. Indeed, let Ω be a subset of $\mathcal{M}$ with $\omega(\Omega)>0$. From the above discussion, it is easy to see that $F(\Omega)$ is a bounded subset of $\mathcal{M}$. Besides, since A and B are weakly sequentially continuous, it follows that

$$\begin{aligned}
\omega(F(\Omega)) &\leq \omega\left[A(\Omega)+B(I-D)^{-1}C(\Omega)\right] \\
&\leq \omega(A(\Omega))+\omega\left(B(I-D)^{-1}C(\Omega)\right) \\
&\leq k(\omega(\Omega)).
\end{aligned}$$

This inequality means that is F is weakly condensing with respect to ω. Hence, F has, at least, one fixed point x in $\mathcal{M}$ in view of Theorem 3.7. □

Theorem 4.8. *Let $\mathcal{M}$ be a nonempty, bounded, convex and closed subset of a Banach space X. Assume that A, $C:\mathcal{M}\longrightarrow X$, and B, $D:X\longrightarrow X$ are four operators satisfying the following conditions:*

(i) D is expansive mapping with the constant $h>1$, and $C(\mathcal{M})\subset(I-D)(\mathcal{M})$;

(ii) C is sequentially weakly-strongly continuous and B is weakly sequentially continuous;

(iii) B and C are Lipschitzian with Lipschitz constants k_B and k_C respectively, such that $k_B.k_C<h-1$;

(iv) A is weakly compact operator, and sequentially weakly continuous;

(v) $Ax+B(I-D)^{-1}Cx\in\mathcal{M}$, for all $x\in\mathcal{M}$.

Then BOM (4.1) has at lest one fixed point in $\mathcal{M}\times\mathcal{M}$.

Proof. Since D is expansive, the inverse operator $(I-D)^{-1}$ exists on $(I-D)(X)$ and, for all $x, y \in (I-D)(X)$, and $(I-D)^{-1}$ is $\frac{1}{h-1}$-Lipschitz. Moreover the use of (ii) give us that $B(I-D)^{-1}C$ is sequentially weakly continuous, and by (iii) this latter is contractive.Thus, $B(I-D)^{-1}C(\mathcal{M})$ is bounded. Now, let Ω a bounded subset of $\mathcal{M}$, using the subadditivity of ω and Lemma 3.1 we obtain

$$\begin{aligned}\omega((A\Omega + B(I-D)^{-1}C(\Omega)) &\leq \omega(B(I-D)^{-1}C(\Omega)) + \omega(A(\Omega))\\ &\leq \frac{k_B \cdot k_C}{h-1}\omega(\Omega).\end{aligned}$$

Now, Theorem 3.7 completes the proof. □

4.3 Application

Let X be a Banach space. Consider the following system of nonlinear integral equations occurring in some biological problems, and also in ones dealing with physics:

$$\begin{cases} x(t) = f(t, x(t)) + \left[\left(\int_0^{\sigma_1(t)} \kappa(t,s) f_1(s, y(\eta(s)))ds\right) \cdot u\right], \\ y(t) = \left[\left(q(t) + \int_0^{\sigma_2(t)} p(t,s,x(s),x(\lambda s))ds\right) \cdot v\right] + g(t, y(t)). \end{cases} \tag{4.6}$$

where $u \in X\backslash\{0\}$ and $v \in X\backslash\{0\}$. We will seek the solutions of the system (4.6) in the space $\mathcal{C}(J, X)$ of all continuous functions on $J = [0, T], 0 < T < \infty$ endowed with the norm $\|\cdot\|_\infty$. Looking that, we can rewrite th equation (4.6) in the following form

$$\begin{cases} x(t) = Ax(t) + By(t), \\ y(t) = Cx(t) + Dy(t); \end{cases}$$

where

$$\begin{cases} (Ax)(t) = f(t, x(t)), t \in J; \\ (Bx)(t) = \left(\int\limits_0^{\sigma_1(t)} k(t,s) f_1(s, x(\eta(s)))ds\right) \cdot u; \ t \in J \text{ and } u \in X\backslash\{0\}; \\ (Cx)(t) = \left(q(t) + \int\limits_0^{\sigma_2(t)} p(t,s,x(s))ds\right) \cdot v \\ \text{where } t \in J, 0 < \lambda < 1, \ v \in X\backslash\{0\}, \text{ and} \\ (Dx)(t) = g(t, x(t)), t \in J \end{cases} \tag{4.7}$$

Let us assume that the functions involved in Eq.(4.6) satisfy the following assumptions:

($\mathcal{H}_1$) $\kappa : J \times J \to \mathbb{R}$ is nonnegative and continuous function.

($\mathcal{H}_2$) $\sigma_1, \sigma_2, \eta : J \longrightarrow J$ are continuous,

($\mathcal{H}_3$) $q : J \longrightarrow \mathbb{R}$ is continuous,

($\mathcal{H}_4$) The function $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ is weakly sequentially continuous such that, for an arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longrightarrow p(t, s, x, y)$ is continuous.

($\mathcal{H}_5$) The mapping $f : J \times X \longrightarrow X$ is such that:

(a) f is weakly sequentially continuous, and

(b) f is a contraction operator with a constant k'.

($\mathcal{H}_6$) The function $f_1 : J \times X \longrightarrow \mathbb{R}$ is such that:

(a) f_1 is weakly sequentially continuous with respect to the second variable, and

(b) $\|f_1(\cdot, x(\cdot))\| \leq \lambda r$, if $\|x\|_\infty \leq r$, for $r > 0$.

($\mathcal{H}_7$) The function $g : J \times X \longrightarrow X$ is such that:

(a) g is weakly sequentially continuous with respect to the second variable,

(b) g is a Φ-nonlinear contraction with respect to the second variable, and

(c) $\Phi(r) < (1 - \lambda)r$, for all $r > 0$.

Theorem 4.9. *[23] Suppose that the assumptions $(\mathcal{H}_1) - (\mathcal{H}_7)$ are satisfied. Moreover, assume that there exists a real number $r_0 > 0$ such that*

$$\begin{cases} |p(t,s,x(s),x(\lambda s))| \leq r_0, \text{ for } x \in \mathcal{C}(J,X) \text{ such that } \|x\|_\infty \leq r_0, \text{ and} \\ \|f(t,x(t))\| \leq k'\|x(t)\|, \quad \text{for } t \in J \text{ and } x \in \mathcal{C}(J,X) \text{ such that } \|x\|_\infty \leq r_0 \,. \\ \|g(\cdot,x(.))\| \leq \lambda\|x\|_\infty, \text{ for } x \in \mathcal{C}(J,X) \text{ such that } \|x\|_\infty \leq r_0 \,. \\ \delta \leq \frac{(1-k')r_0}{KT\lambda\|u\|_\infty}, \text{ with } u \in X\backslash\{0\}, \\ \text{where } K = \sup\limits_{t,s\in J} \kappa(t,s), \lambda\delta = (\|q\|_\infty + Tr_0)\,\|v\|_\infty + r_0, \text{ and } v \in X\backslash\{0\}. \end{cases} \tag{4.8}$$

Then, the nonlinear system (4.6) has at least, one solution in $\mathcal{C}(J,X) \times \mathcal{C}(J,X)$.

Proof. Let $\mathcal{M}$ be the closed ball $\mathfrak{B}_{r_0}$ on $\mathcal{C}(J,X)$ with $r_0>0$. In order to apply Theorem 4.7, we have to verify the following steps.

Claim 1: $(I-D)^{-1}C(\mathcal{M})$ is bounded. Indeed, since D is a Φ-nonlinear contraction, then the inverse operator $(I-D)^{-1}$ is well-defined on $\mathcal{C}(J,X)$. Let $(x,y)\in\mathcal{M}\times\mathcal{C}(J,X)$ be such that $y=(I-D)^{-1}Cx$. Then, for all $t\in J$, we have

$$y(t)=\left(q(t)+\int_0^{\sigma_2(t)}p(t,s,x(s),x(\lambda s))ds\right)\cdot v+g(t,y(t)).$$

since $y\in\mathcal{C}(J,X)$ then, there is $t^*\in J$ such that

$$\begin{aligned}\|y\|_\infty&=\|y(t^*)\|\\&\leq\left|q(t^*)+\int_0^{\sigma_2(t^*)}p(t^*,s,x(s),x(\lambda s))\,ds\right|\|v\|\\&+\|g(t^*,y(t^*))-g(t^*,x(t^*))\|+\|g(t^*,x(t^*)\|\\&\leq(\|q\|_\infty+Tr_0)\|v\|+(1-\lambda)(\|y(t^*)\|+\|x(t^*)\|)+\lambda\|x\|_\infty\\&<(\|q\|_\infty+Tr_0)\|v\|+(1-\lambda)\|y(t^*)\|+\|x\|_\infty\\&\leq(\|q\|_\infty+Tr_0)\|v\|+r_0+(1-\lambda)\|y\|_\infty.\end{aligned}$$

Consequently,

$$\|y\|_\infty<\delta,$$

where

$$\delta=\frac{1}{\lambda}\left[(\|q\|_\infty+Tr_0)\|v\|+r_0\right].$$

Hence, $(I-D)^{-1}C(\mathcal{M})$ is bounded with a bound δ which end the first claim. It should be noted that the operators defined in (4.8) are well-defined. Indeed, the maps Ax and Dy are continuous on J in view of assumptions $(\mathcal{H}_5)\,(b)$ and $(\mathcal{H}_7)\,(b)$, for all $(x,y)\in\mathcal{M}\times\mathcal{C}(J,X)$. Now, we claim that the two maps Cx and By are continuous on J for all $(x,y)\in\mathcal{M}\times(I-D)^{-1}C(\mathcal{M})$. To see this, let $\{t_n\}$ be any sequence in J converging to a point t in J. Then

$$\begin{aligned}\|(By)(t_n)-(By)(t)\|\leq&\left[\int_0^{\sigma_1(t_n)}|\kappa(t_n,s)-\kappa(t,s)|\,|f_1(s,y(\eta(s)))|\,ds\right]\|u\|\\&+\left|\int_{\sigma_1(t_n)}^{\sigma_1(t)}\kappa(t,s)f_1(s,y(\eta(s)))ds\right|\|u\|.\end{aligned}$$

Moreover, taking into account that $(I-D)^{-1}C(\mathcal{M})$ is bounded with a bound δ, and using the assumption $(\mathcal{H}_6)\,(b)$, we get

$$\begin{aligned}\|(By)(t_n)-(By)(t)\| &\leq \left[\int_0^T |\kappa(t_n,s)-\kappa(t,s)|\,\lambda\delta ds\right]\|u\| + \left[\left|\int_{\sigma_1(t_n)}^{\sigma_1(t)} K\lambda\delta ds\right|\right]\|u\| \\ &\leq \left[\int_0^T |\kappa(t_n,s)-\kappa(t,s)|\,ds + K\,|\sigma_1(t_n)-\sigma_1(t)|\right]\lambda\delta\|u\|.\end{aligned}$$

The continuity of κ and σ_1 on $[0,T]$ implies that the function By is continuous. And we have,

$$\begin{aligned}\|(Cx)(t_n)-(Cx)(t)\| &\leq \left|\left[q(t_n)+\int_0^{\sigma_2(t_n)} p(t_n,s,x(s),x(\lambda s))ds - q(t) - \int_0^{\sigma_2(t)} p(t,s,x(s),x(\lambda s))ds\right]\right|\|v\| \\ &\leq \left[\int_0^{\sigma_2(t_n)} |p(t_n,s,x(s),x(\lambda s))-p(t,s,x(s),x(\lambda s))|\,ds\right]\|v\| \\ &+ \left|\int_{\sigma_2(t)}^{\sigma_2(t_n)} |p(t,s,x(s),x(\lambda s))|\,ds\right|\|v\| + |q(t_n)-q(t)|\|v\| \\ &\leq \left[\int_0^T |p(t_n,s,x(s),x(\lambda s))-p(t,s,x(s),x(\lambda s))|\,ds\right]\|v\| \\ &+ (r_0\,|\sigma_2(t_n)-\sigma_2(t)| + |q(t_n)-q(t)|)\|v\|.\end{aligned}$$

Since $t_n \to t$, so, $(t_n,s,x(s),x(\lambda s)) \to (t,s,x(s),x(\lambda s))$, for all $s \in J$. Taking into account $(\mathcal{H}_4)$ the hypothesis, we obtain

$$p(t_n,s,x(s),x(\lambda s)) \to p(t,s,x(s),x(\lambda s)) \text{ in } \mathbb{R}.$$

Moreover, the use of the first inequality in (4.8) leads to

$$|p(t_n,s,x(s),x(\lambda s) - p(t,s,x(s),x(\lambda s)| \leq 2r_0,$$

for all $t,s \in J, \lambda \in (0,1)$. Consider

$$\begin{cases} \varphi : J & \to \mathbb{R} \\ \quad\quad s & \to \varphi(s) = 2r_0. \end{cases}$$

Clearly $\varphi \in L^1(J)$. Therefore, from the dominated convergence Theorem S 1.13 and assumption $(\mathcal{H}_2)-(\mathcal{H}_3)$, we obtain

$$(Cx)(t_n) \to (Cx)(t) \text{ in } X.$$

It follows that $Cx \in \mathcal{C}(J,X)$.

Claim 2: In the proof of Theorem 4.7 we need to show that B is weakly sequentially continuous on $(I-D)^{-1}C(\mathcal{M})$ and A and C are weakly sequentially continuous on $\mathcal{M}$. We begin to show the property for the operator B. Let $\{x_n\}_{n=0}^{\infty}$ be a weakly convergent sequence of $(I-D)^{-1}C(\mathcal{M})$ to a point x. Since $(I-D)^{-1}C(\mathcal{M})$ is bounded, we can apply the Dobrakov's theorem 1.7 in order to get

$$x_n(t) \rightharpoonup x(t) \quad \text{in } X.$$

The use of assumption $(\mathcal{H}_6)$ and the Dobrakov's theorem 1.7 allows us we obtain

$$f_1\left(t, x_n(t)\right) \rightharpoonup f_1(t, x(t)) \quad \text{in } \mathbb{R}.$$

Which implies that

$$\kappa(t,s) f_1\left(t, x_n(t)\right) \rightharpoonup \kappa(t,s) f_1(t, x(t)) \quad \text{in } \mathbb{R}.$$

Moreover, the use of the condition (b) $\mathcal{H}_6$, the boundedness of κ the, and dominated convergence Theorem W 1.14 leads to

$$\lim_{n\to\infty} \int_0^{\sigma_1(t)} \kappa(t,s) f_1\left(s, x_n(\eta(s))\right) ds = \int_0^{\sigma_1(t)} \kappa(t,s) f_1(s, x(\eta(s))) ds.$$

Hence,

$$(Bx_n)(t) \rightharpoonup (Bx)(t)$$

Since $(Bx_n)_n$ is bounded by $TK\lambda\delta\|u\|$, then by using Dobrakov's Theorem 1.7, we get that $Bx_n \rightharpoonup Bx$ and so B is weakly sequentially continuous on $(I-D)^{-1}C(\mathcal{M})$. Therefore, since g is weakly sequentially continuous with respect to the second variable and by the third inequality in (4.8) $g\left(\cdot, x_n\right)$ is bounded, then the operator D defined in (4.7) is also weakly sequentially continuous. Moreover, taking into account that $\mathcal{M}$ is bounded and using the Dobrakov's theorem 1.7 we show that A is a weakly sequentially continuous operator on $\mathcal{M}$.

Now, we show that C is weakly sequentially continuous on $\mathcal{M}$. To see this, let $\{x_n\}_{n=0}^{\infty}$ be any sequence in $\mathcal{M}$ weakly converging to a point $x \in \mathcal{M}$. Then by using the Dobrakov's Theorem 1.7, we get for all $t \in J$, $x_n(t) \rightharpoonup x(t)$. Then, by assumption $(\mathcal{H}_4)$ and the dominated convergence Theorem W 1.14, we obtain

$$\left(Cx_n\right)(t) \rightharpoonup (Cx)(t) \quad \text{in } X.$$

Thus, $Cx_n \rightharpoonup Cx$. As a result, C is weakly sequentially continuous on $\mathcal{M}$.

Claim 3: Next, let us show that C is weakly compact and that A is condensing on $\mathcal{M}$. We should prove that $C(\mathcal{M})$ is relatively weakly compact. By definition, we have

$$\text{for all } t \in J, \quad C(\mathcal{M})(t) = \{(Cx)(t);\ \|x\|_\infty \leq r_0\}.$$

Then, $C(\mathcal{M})(t)$ is sequentially relatively weakly compact in X. To see this, let $\{x_n\}_{n=0}^{\infty}$ be any sequence in $\mathcal{M}$, we have $(Cx_n)(t) = r_n(t) \cdot v$, where

$$r_n(t) = q(t) + \int_0^{\sigma_2(t)} p(t, s, x_n(s), x_n(\lambda s))\, ds,$$

since $|r_n(t)| \leq \|q\|_\infty + Tr_0$ in view of the first inequality in (4.8), it follows that that $\{r_n\}$ is a uniformly bounded sequence in $\mathcal{C}(J, \mathbb{R})$. Next, we show that $\{r_n(t)\}$ is an equicontinuous set. Let $t_1, t_2 \in J$. Then, we have

$$\begin{aligned}
|r_n(t_1) - r_n(t_2)| \leq{}& |q(t_1) - q(t_2)| + \left| \int_{\sigma_2(t_1)}^{\sigma_2(t_2)} |p(t_2, s, x(s), x(\lambda s))|\, ds \right| \\
&+ \left| \int_0^{\sigma_2(t_1)} |p(t_1, s, x(s), x(\lambda s)) - p(t_2, s, x(s), x(\lambda s))|\, ds \right| \\
\leq{}& \left| \int_0^{T} |p(t_1, s, x(s), x(\lambda s)) - p(t_2, s, x(s), x(\lambda s))|\, ds \right| \\
&+ |q(t_1) - q(t_2)| + r_0 |\sigma_2(t_1) - \sigma_2(t_2)|,
\end{aligned}$$

since p, q, and σ_2 are uniformly continuous functions, we conclude that $\{r_n\}$ is an equicontinuous set, thus by Arzelà-Ascoli's Theorem S 1.8, $\{r_n\}$ is a compact set. As a result, $C(\mathcal{M})(t)$ is sequentially relatively weakly compact. Next, we will show that $C(\mathcal{M})$ is a weakly equi-continuous set. If we take $\varepsilon > 0, x \in \mathcal{M}$, $x^* \in X^*$ and $t, t' \in J$ such that $t \leq t', t' - t \leq \varepsilon$, and using the first inequality in (4.8) we obtain

$$\begin{aligned}
\| x^*((Cx)(t) - (Cx)(t')) \| \leq{}& \left[\int_0^{\sigma_2(t)} |p(t, s, x(s), x(\lambda s)) - p(t', s, x(s), x(\lambda s))|\, ds \right] \|x^*(v)\| \\
&+ |q(t) - q(t')|\, \|x^*(v)\| + \left[\int_{\sigma_2(t)}^{\sigma_2(t')} |p(t', s, x(s), x(\lambda s))|\, ds \right] \|x^*(v)\| \\
\leq{}& (w(q, \varepsilon) + Tw(p, \varepsilon) + r_0 w(\sigma_2, \varepsilon)) \|x^*(v)\|,
\end{aligned}$$

where

$$\begin{cases}
w(q, \varepsilon) = \sup \{|q(t) - q(t')| : t, t' \in J; |t - t'| \leq \varepsilon\}, \\
w(p, \varepsilon) = \sup\limits_{\substack{t,t',s \in J,\\ x,y \in \mathcal{M}}} \{|p(t, s, x, y) - p(t', s, x, y)| : |t - t'| \leq \varepsilon\}, \text{ and} \\
w(\sigma_2, \varepsilon) = \sup \{|\sigma_2(t) - \sigma_2(t')| : t, t' \in J; |t - t'| \leq \varepsilon\}.
\end{cases}$$

By taking into account the assumption $(\mathcal{H}_4)$, and in view of the uniform continuity of the functions q and σ on the set J, it follows that $w(q,\varepsilon) \to 0, w(p,\varepsilon) \to 0$ and $w(\sigma_2,\varepsilon) \to 0$ as $\varepsilon \to 0$. By applying Arzelà-Ascoli's Theorem W 1.9 we conclude that $C(\mathcal{M})$ is sequentially weakly relatively compact in X, Again, an application of Eberlein-Šmulian's theorem implies that $C(\mathcal{M})$ is relatively weakly compact. As a result, C is weakly compact. Now, the use of the assumption $(\mathcal{H}_5)$ and Lemma 3.1 allows us to deduce that the operator A is weakly condensing.

Claim 4: To finish, it is sufficient to show that

$$Ax + B(I-D)^{-1}Cx \in \mathcal{M} \text{ for all } \ x \in \mathcal{M}.$$

Let $y \in \mathcal{C}(J,X)$ be arbitrary, with

$$y = Ax + B(I-D)^{-1}Cx,$$

for some $x \in \mathcal{M}$. Then, for all $t \in J$, we have

$$\|y(t)\| \leq \|(Ax)(t)\| + \left\|B(I-D)^{-1}Cx(t)\right\|.$$

We should notice that, for all $x \in (I-D)^{-1}C(\mathcal{M})$, there exists a unique $z \in C(J,X)$ such that $z = x$, with $\|z\| \leq \delta$. Therefore

$$\begin{aligned}
\|y(t)\| &\leq \|f(t,x(t))\| + \left\|\left(\int_0^{\sigma_1(t)} \kappa(t,s) f_1(s,z(\eta(s)))ds\right)\cdot u\right\| \\
&\leq k'\|x(t)\| + \left(\int_0^T |\kappa(t,s)\lambda\|z(\eta(s))\|ds\right)\|u\|_\infty \\
&\leq k'r_0 + \lambda KT\delta\|u\|_\infty.
\end{aligned}$$

where

$$K = \sup_{t,s\in J} |\kappa(t,s)|,$$

since $y \in \mathcal{C}(J,X)$, there is $t^* \in J$ such that $\|y\|_\infty = \|y(t^*)\|$ and so, $\|y\|_\infty \leq r_0$ in view of the last inequality in (4.8). Hence, the hypothesis (iii) of Theorem 4.7 is satisfied, which achieves the proof. □

Conclusion and perspective

The fixed point theory plays an important role to proof the existence of solutions of different types of equations.

The work presented in this Master thesis focused on the study several extensions of Schauder and Krasnoselskii fixed point theorems in Banach spaces endowed with its strong and weak topologies in the context that the involved operators are not (weakly) compact, invoking the technique of measures of (weak) noncompactness.

Besides that, under which conditions we also study some theorems which unsure the existence of fixed points for a 2×2 block operator matrix. In addition, this memoir gives three applications of fixed point theorems in previous cases.

Consequently, in the direct continuity of our work, in the near future, we aim at investigating the fixed point theorems of Krasnoselskii in Fréchet spaces under its weak topology by using the family of measures of (weak) noncompactness - which is used the first time by Olszowy [26]- and show some applications.

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ملخص

الهدف من هذه المذكرة هو دراسة بعض تعميمات نظريات النقطة الصامدة لشودر و كراسنسلسكي في فضاء بناخي المزود بنظيم أو بطوبولوجيته الضعيفة، باستخدام تقنيتي قياس عدم التراص و القياس الضعيف لعدم التراص، كما تطرقنا الى دراسة النقط الصامدة لكتلة مصفوفة 2×2 لتابعين، و قد استخدمنا بعض هذه النظريات لاثبات وجود حلول لبعض المعادلات التكاملية الغير خطية.

الكلمات المفتاحية. نظرية النقطة الصامدة، قياس عدم التراص، القياس الضعيف لعدم التراص، التوابع (k, μ)-مجموعة التقلصية، التوابع التكثيفية، التوابع التمددية، مستمر بشكل ضعيف متتالي، المعادلة التكاملية الغير خطية، كتلة مصفوفة لتابعين.

Abstract

The propose of this memoir is to study some generalisation versions of fixed point theorem of Schauder and Krasnoselskii on Banach spaces and product of two Banach spaces furnished with its norm and weak topology by using the most useful technique of Measure of (Weak) Noncompactness. Besides that we applicate this results to find a solution for nonlinear integral equations.

Key words. Fixed point theorem, Measure of (Weak) Noncompactness, (k, μ)-set contraction mapping, Condensing mapping, Expansive mapping, Sequentially weakly continuous, Nonlinear integral equation, Block Operator Matrix.

Résumé

Le but de ce mémoire est d'étudier quelque généralisation des théorèmes de point fixe de Schauder et Krasnoselskii dans des espaces de Banach et produit de deux espaces de Banach munis de leurs topologies forts et faibles, par l'usage des techniques de la mesure de non-compacité et mesure de non-compacité faible. Ensuite on applique ces théorèmes pour prouver des résultats d'existence de solutions pour certaines équations intégrales non linéaires.

Mots clés. Theorem de point fixe, Mesure de non-compacité (faible), Application (k, μ)-ensemble contractive, Application condensante, Application expansive, sequentiellement faiblement continue, équation intégral non linéaire, Bloc des opérateurs matrices.