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Théorèmes du Point Fixe de Kransnoselskii sous la Topologie Faible et Applications

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I would like to dedicate this work

To my parents

To the people who paved our way of science and knowledge

All our distinguished teachers

To every persons who supported me in my studies.

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Abstract

The aim of this memoir is first to study new version of Krasnoselskii fixed point theorems under weak topology of Banach spaces then to provide as applications some existence results for some nonlinear integral equation.

This work is divided into fours chapters. The first chapter provides some definitions and auxiliary results that will be used later on.

In Chapter 2, we give an exposition of Krasnoselskii fixed point theorems for contraction mappings involving the weak topology. Our main result is applied to solve the following nonlinear integral equation

$$x(t) = f(x) + \int_0^T g(s, x(s))ds, \quad t \in [0,T].$$

In chapter 3, we provide some expansive Krasnoselskii-type fixed point theorems under weak topology and apply our result to prove the existence of solution of above equation.

In the last chapter we present an existence theories for the two operator equations $AxBx = x$ and $AxBx + Cx = x,\ x \in M$, in Banach algebra endowed with weak topology. Where M is bounded, closed and convex subset of Banach algebra. A, B and C three operators defined on M. Then we apply this result to the following integral equations.

$$x(t) = a(t) + T(x)(t) \left[\left(q(t) + \int\limits_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right) .u \right] ,0 < \lambda < 1.$$

Key words: Fixed point of Krasnoselskii, Contraction mapping, Expansive mapping Weakly continuous, Sequentially weakly continuous, Integral equation, Banach algebra.

Résumé

Le but de ce mémoire est d'étudier des nouvelles versions du théorème de point fixe de Krasnoselskii dans l'espace de Banach sous la topologie faible puis on va appliquer ces théorèmes sur les équations intégrales non linéaires.

Le mémoire est divisé en quatre chapitres. Dans le premier chapitre on va donner certains notations, définitions et des théorèmes dont on aura besoin dans les chapitres suivants.

Dans le chapitre 2, nous présentons quelleques théorèmes du point fixe de Krasnoselskii pour les applications contractions sous la topologie faible. Notre résultat principal est appliqué pour prouver l'existence des solutions de l'équation intégrale non linéaire suivante,

$$x(t) = f(x) + \int_0^T g(s, x(s))ds, \quad t \in [0,T].$$

Dans le chapitre 3, nous éutidions des théorèmes du point fixe de type Krasnoselskii pour les applications expansives sous la topologie faible et appliquer notre résultat pour prouver l'existence de la solution de l'équation ci-dessus.

Dans le dernier chapitre, nous établissons des théories d'existence pour les équations suivantes $AxBx = x$ et $AxBx + Cx = x$, $x \in M$ dans l'Algèbre de Banach muni d'une topologie faible. Où M est un sous ensemble convexe, fermé et borné, de l'algèbre de Banach. A, B et C trois opérateurs définis sur M. Puis nous appliquons ce résultat à l' équation intégrale suivante,

$$x(t) = a(t) + T(x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds\right).u\right] ,0 < \lambda < 1.$$

Mots clés: point fixe de Krasnoselskii, Application contraction, Application expansive Faiblement continu, Séquentiellement faiblement continu, Équation intégrale, Algèbre de Banach.

ملخص

الهدف من هذه المذكرة هو أولا دراسة نسخ جديدة من نظريةالنقطة الثابتة لكراسنسلسكي لفضاء بناخي مزود بطوبولوجيته الضعيفة ، ومن ثم تطبيق هده النتائج لاثبات وجودحلول لبعض المعادلات التكاملية غير الخطية.

تنقسم المذكرة إلى أربع فصول .يقدم الفصل الأول بعض التعريفات والنتائج المهمة التي سنستخدمها طوال هذا العمل .

في الفصل الثاني، نقدم عرضا لنظريات جديدة حول النقطة الثابتة لكراسنسلسكي تتمتع بتوابع قابلة للتقلص .ثم نطبق النتيجة الرئيسية لدينا لحل المعادلة التكاملية غير الخطية التالية:

$$x(t) = f(x) + \int_0^T g(s, x(s))ds, \quad t \in [0,T].$$

في الفصل الثالث ، نقدم بعض امتدادات النقطةالثابتة لكراسنسلسكي تتمتع بتوابع قابلة للتوسع تحت طوبولوجيا ضعيفة وتطبيق النتائج لإثبات وجود حل للمعادلة أعلاه.

في الفصل الأخير نقدم نظرية الوجود لكراسنسلسكي للمعادلات التالية:

$$AxBx = x, x \in M \quad \text{و} \quad AxBx + Cx = x, x \in M.$$

في جبر بناخي مزود بطوبولوجيتيه الضعيفة. حيث M مجموعة فرعية محدودة ،محدبة و مغلقة من جبر بناخي. B, A و C ثلاثة توابع معرفة على M . ثم نطبق هذه النتيجة لأيجاد حلول للمعادلةالتكامل التالية:

$$x(t) = a(t) + (T(x)(t) \left[\left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right) .u \right] ,0 < \lambda < 1.$$

الكلمات المفتاحية: نتائج نقاط ثابتة لكراسنسلسكي ، توابع قابلة للتقلص ، توابع قابلة للتوسع، مستمر بشكل ضعيف ،مستمر بشكل ضعيف متتالي ، المعادلة التكاملية ، جبر بناخي.

Contents

Acronyms

$d(x,y)$	The distance between x and y.
(X,d)	Metric space.
$(X,\|.\|)$	Norm vector space.
X^*	Dual topology of X.
X^{**}	Bidual topology of X.
$\sigma(X,X^*)$	The weak topology.
$\sigma(X^*,X)$	The weak star topology.
B_R	The closed ball of center x and radius R.
$[M]$	The linear span of M.
$\overline{M}^{\tau}$	Closures of M with respect to the topology τ.
L^P	The vector space of classes of functions whose exponent power P is integrable in the sense of Lebesgue, where $p > 0$.
$\mathcal{L}(X)$	The set of continuous linear functions from X into X.
$\overline{co}(M)$	The closed convex hull of M .
$C(J,X)$	The set of continuous functions from J into X.
$C(X)$	The set of continuous functions from X into X .

Introduction

Fixed point theory is a branch of topology that studies the conditions under which a function from a set to itself has a fixed point. Advancements in fixed point theory enrich many scientific fields such as biology, chemistry, computer science, economics and game theory.

The fixed point theory is at the heart of the nonlinear analysis then that it provides the necessary tools to have theorems of existence in many different non-linear problems. It uses its tools of analysis and topology. Depending on the nature of the assumptions involved, we can divide fixed point theory into two main branches fixed point and metric theory. Or, fixed point and topological theory.

With respect to the metric approach, the most important metric fixed point result is the Banach fixed point theorem (also known the contraction mapping theorem or the contraction mapping principle). It was first stated by Stefan Banach in 1922. This theorem guarantees the existence and uniqueness of fixed points of certain self maps of a metric space, and provides a constructive method to find those fixed points .

Concerning the topological branch, results are obtained using topological properties of the set X. The main result is Schauder fixed point theorem which stated by Schauder in 1930. This theorem is a generalization of Brower's fixed point theorem.

Although historically the two branches of the fixed point theory had separate development. In 1958, Krasnoselskii established that the sum of two operators $A + B$ has a fixed point in a nonempty closed convex subset M of a Banach space X, whenever A and B satisfy :

(i) $A(x) + B(y) \in M$ for all $x,y \in M$,

(ii) A continuous and compact ,

(iii) B is contraction on X.

This is a captivating result and it has a number of interesting applications. The proof of this result combines the Banach contraction principle and Schauder fixed point theorem and thus it is a blend of the two branches.

Recently, as a tentative approach to overcoming such difficulties, many interesting works have appeared with different ways and directions of weakening conditions (i). In 1998, Burton [9] noticed that the Krasnoselskii fixed point theorem remains valid if the condition (i) is replaced by the following less restrictive one, $\forall y \in M$, $x = Bx + Ay$ imply that $x \in M$. In [4], Barroso proposed the following improvement for condition (i).

$$\text{If} \quad \lambda \in (0,1), \quad x = \lambda Bx + Ay, \quad y \in M \Longrightarrow x \in M.$$

His result applies to a problem from stability theory and covers cases where Theorem of Krasnoselskii does not work.

The aim of this memoir is first to studies some new versions of fixed point theorems of Krasnoselskii types involving the weak topology of Banach spaces, and then to provide as applications some existence results for some nonlinear integral equations in Banach spaces endowed with their weak topologies.

The memoir is divided into four chapters. The first one provides some notations that will be used, we starting from some definitions and auxiliary results. The weak topology and it's proprieties are presented in section 1. Elementary fixed point theory is discussed in Section 2. In section 3, we present the fixed point theory involving the weak topology of a Banach space.

In the second chapter we present a version of Krasnoselskii's fixed point theorem for contraction maps. Starting from a version of Krasnoselskii's theorem for weakly continuous maps [4]. In section 2, Barroso [3] established new versions of the Krasnoselskii's fixed point theorem were obtained for sequentially weakly continuous mappings (i.e. operators which map weakly convergent sequences into weakly convergent sequences). In Section 3 we prove the existence of a solution to the nonlinear integral equations in the form,

$$x(t) = f(x) + \int_0^T g(s, x(s))ds, \quad t \in [0, T]. \tag{1}$$

Where x takes values in a reflexive Banach space. By imposing some conditions on f and g, we are able to prove the existence of a solution to equation (1).

In chapter three, we provide some expansive Krasnoselskii-type fixed point theorem for weakly continuous and sequentially weakly continuous mapping and it's application to nonlinear integral equations of the form (1).

In chapter four we presented some fixed Point theory in Banach algebras satisfying certain sequential conditions under the weak topology. We starting by given some definitions and elementary proprieties which using in this chapter. In section 2 and 3 respectively, we offer some fixed point theorems in Banach algebras to get the solution for the following operator equation:

$$AxBx = x, \quad x \in M, \tag{2}$$

and

$$AxBx + Cx = x, \quad x \in M. \tag{3}$$

Where M is a closed, bounded and convex subset of a Banach algebras X. A, B and C are three operators defined on M verified some conditions. Our main conditions are formulated in terms of a weak sequential continuity related to the three nonlinear operators A, B, and C involved in the previous equations. In section 4, we gives new version of Krasnoselskii fixed point theorem in Banach algebras for $\mathcal{D}$-Lipschitzian mappings (see Definition 4.3) under weak topology. In [8] D. W. Boyd and J. S. W. Wong extend the essential theorem for nonlinear contractions mapping, which used to proved the main result in this chapter. In the last section we studies solution of the following nonlinear integral equation:

$$x(t) = a(t) + T(x)(t) \left[\left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right) .u \right] ,0 < \lambda < 1, \tag{4}$$

using the result of section 2 and 3, Where $(X, \| \ . \ \|)$ be a Banach algebra satisfying condition $(\mathcal{P})$ (see Definition 4.2) and $u \neq 0$ is a fixed vector of X and the functions a, q, σ, p, T are given and verified some conditions.

Chapter 1

Preliminaries

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the memoir. We recall some classical results from functional analysis ([13],[14],[15],[19]).

1.1 Basics of weak topology

The topology induced by a norm on a vector space is a very strong topology in the sense that it has many open sets. This brings advantages to the functions whose domain is such a space because for them is easy to be continuous but it brings disadvantages to compactness because the richness of open sets makes it difficult for a set to be compact.

The weaker topology is the more compact sets we have, this fact motivates us to search for a topology defined on a normed space X which is the weakest topology among all topologies which we can define on X, so we can have the biggest class of compact sets. The topology which gives us the desired result is the weak topology defined on X. Another useful topology is the weak* topology of X^*. These two topologies help us to characterize properties of topological spaces with infinite dimensions and provide simple means to check their nature.

Definition 1.1. Let X be a Banach space and, X^* its dual. The weak topology denoted $\sigma(X,X^*)$, is the weakest topology in X such that each map $f : X \to \mathbb{R}, f \in X^*$, is continuous.

Proposition 1.1. *[15] The weak topology $\sigma(X,X^*)$ is Hausdorff.*

Notation: If a sequence (x_n) in X converges to x in the weak topology $\sigma(X,X^*)$ we shall write $x_n \rightharpoonup x$.

Proposition 1.2. *[15] Let (x_n) be a sequence in X. Then,*

i) $[x_n \rightharpoonup x$ *weakly in* $\sigma(X,X^*)] \Leftrightarrow [\langle f, x_n\rangle \to \langle f, x\rangle \quad , \forall f \in X^*]$,

ii) If $x_n \to x$ strongly, then $x_n \rightharpoonup x$ weakly in $\sigma(X,X^*)$,

iii) If $x_n \rightharpoonup x$ weakly in $\sigma(X,X^*)$, then (x_n) is bounded and $\|x\| \leq \liminf \|x_n\|$,

iv) If $x_n \rightharpoonup x$ weakly in $\sigma(X,X^*)$ and if $f_n \to f$ strongly in X^* (i.e., $\|f_n - f\|_{X^*} \to 0$), then $\langle f_n, x_n\rangle \to \langle f,x\rangle$.

Remark 1.1. When X is finite-dimensional, the weak topology $\sigma(X,X^*)$ and the usual topology are the same.

Theorem 1.1. *[15] Let X and Y be two Banach spaces and let f be a linear operator from X into Y. Assume that f is continuous in the strong topologies. Then f is continuous from X weak $\sigma(X,X^*)$ into Y weak $\sigma(Y,Y^*)$ and conversely.*

Definition 1.2. The weak* topology of the dual space of a normed space X is the smallest topology for X^* such that, for each x in X, the linear functional $x^* \mapsto \langle x^*, x\rangle$ on X^* is continuous.

Remark 1.2. All properties which are stated for weak topology of X can be adapted and hold for the weak* topology of X^*.

Definition 1.3. Let X be a Banach space and let $J : X \to X^{**}$ be the canonical injection from X into X^{**} defined as follows: given $x \in X$ the map $f \longmapsto \langle f, x\rangle$ is a continuous linear functional on X^*, thus it is an element of X^{**}, which we denote by J_x. We have

$$\langle J_x, f\rangle_{X^{**},X^*} = \langle f, x\rangle_{X^*,X}, \quad \forall x \in X, \forall f \in X^*.$$

The space X is said to be reflexive if J is surjective, i.e., $J(X) = X^{**}$.

Now, let us recall the following definitions:

Definition 1.4. Let A be a subset of topological space X.

1. The set A is said to be compact if any cover of A by open sets admits a finite subcover

   A is said to be relatively compact if its closure is a compact subset of X.

2. A is said to be precompact if every sequence in A contains a convergent subsequence.

3. A is said to be countably compact if any countable cover of A by open sets admits a finite subcover i.e. $\forall (x_n)_{n\in N} \subset A, \exists (x_\alpha)_{\alpha\in A} \subset (x_n)_{n\in N}$ such that $x_\alpha \to x \in A$.

   A is called relatively countably compact if its closure is countably compact.

4. A is called limit-point compact if every infinite subset of A has at least one accumulation point that belongs to A. A is called relatively limit-point compact if every infinite subset of A has, at least, one accumulation point.

5. A is said to be sequentially compact if any sequence in A has a subsequence converging to some element of A i.e. $\forall (x_n)_{n\in N} \subset A, \exists (x_n)_k \subset (x_n)_{n\in N}$ such that $(x_n)_k \to x \in A$. A is called relatively sequentially compact if every sequence in A has convergent subsequence.

Remark 1.3. We recall that a set is weakly compact, if it is compact in the topology $\sigma(X,X^*)$.

Definition 1.5. We said that a subset $M \subset X$ is convex if $\forall x, y \in M, \forall t \in [0,1]$ $tx + (1-t)y \in M$.

Lemma 1.1. *[15] Let M be a convex subset of X. Then M is closed in the weak topology $\sigma(X,X^*)$ if and only if it is closed in the strong topology.*

Theorem 1.2. *(Kakutani, [15]) A Banach space X is reflexive if and only if $\overline{B}_X$ is weakly compact. Where*

$$\overline{B}_X = \{x \in X \quad \| x \| \leq 1\}.$$

Lemma 1.2. *Let X be a reflexive Banach space. Let $K \subset X$ be a bounded, closed, and convex subset of X. Then K is compact in the topology $\sigma(X,X^*)$.*

Proof. If K is closed and convex, by Lemma 1.1 it is weakly closed. Since it is bounded, it is included in $B(0,R)$ for some $R > 0$ and this set is weakly compact by Kakutani theorem. Closed subsets of compact sets are compact, so K is weakly compact. □

Lemma 1.3. *[15] Assume that X is a reflexive Banach space and let (x_n) be a bounded sequence in X. Then there exists a subsequence that converges in the weak topology $\sigma(X,X^*)$.*

Lemma 1.4. *(Bihari, [25]) Let $x : [a,b] \to \mathbb{R}^+$ be a continuous function that satisfies the inequality:*

$$x(t) \leq M + \int_0^t \psi(s)\omega(x(s))ds, t \in [a,b],$$

where $M \leq 0$, $\psi : [a,b] \to \mathbb{R}^+$ is continuous and $\omega : \mathbb{R}^+ \to R_*^+$ is continuous and monotone-increasing. Then the estimation

$$x(t) \leq \phi^{-1}\left(\phi(M) + \int_0^t \psi(s)\right) ds, t \in [a,b],$$

holds, where $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(u) := \int_a^t \frac{ds}{\omega(s)}, u \in R.$$

1.2 Some definitions and fundamentals theorems

In this section we will present some definitions and theorems which will be used during the proofs that will be presented during the memoir.

Definition 1.6.

- A map $f : X \to X$ is said to be weakly continuous if for every $\varphi \in X^*$, we have $\varphi \circ f : X \to \mathbb{R}$ is continuous.
- A map $f : X \to X$ is said to be sequentially weakly continuous, if for every sequence $x_n \subset X$ and $x \in X$ such that $x_n \rightharpoonup x$ we have that $fx_n \rightharpoonup fx$.

Definition 1.7. The mapping $f : X \to X$ is called demicontinuous if it maps strongly convergent sequences into weakly convergent sequences i.e., if $\{x_n\} \subset X$ and $x \in X$ such that $x_n \to x$ then $fx_n \rightharpoonup fx$.

Definition 1.8. The mapping $f : X \to X$ is called to be strongly continuous on X, if for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have $fx_n \to fx$.

Definition 1.9. let f be a function from a set X into Y .

(i) The function f is said to be one -to- one or injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for $x_1 , x_2 \in X$,

(ii) The function f is said to be onto or surjective if for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$,

(iii) The function f is said to be bijective if it is both one -to- one and onto .

Theorem 1.3. *(Dominated convergence theorem, Lebesgue, [15]) Let Ω a nonempty set and (f_n) be a sequence of functions in L^1 that satisfy*

(i) $f_n(x) \to f(x)$ a.e.on Ω,

(ii) there is a function $g \in L^1$ such that for all $n, |f_n(x)| \leq g(x)$ a.e.on Ω.

Then $f \in L^1$ and $\| f_n - f \|_1 \to 0$. where $\| f \|_1 = \int_{\Omega} |f(x)|dx$

Definition 1.10. let $f : X \to Y$, where (X, Σ,μ) is a measure space and Y is a topological vector space. We say that f is pettis integrable if

$$\varphi \circ f \in L^1(X,\Sigma,\mu) \text{ for all } \varphi \in Y^*.$$

And there exist a vector $e \in Y$ so that

$$\forall \varphi \in Y^* : \langle \varphi, e\rangle = \int_X \langle \varphi, f(x)\rangle d\mu(x).$$

Theorem 1.4. *[17] Let (M, Σ, μ) be a finite perfect measure space and let f be a bounded function from M into X. satisfying the following two conditions:*

(a) *There exists a sequence of Pettis integrable functions*

$f_n : M \to X$, $n \in N$, such that $\lim_n x^* f_n = x^* f$ in measure, for each $x^* \in X^*$,

(b) *There exists a Pettis integrable function $g : M \to X$ such that $|x^* f_n| \leq |x^* g|$ μ.a.e for each $x^* \in X^*$ and $n \in N$ (the exceptional set depends on x^*) .*

Then f is Pettis integrable and $\lim_n \int_E f_n d\mu = \int_E f d\mu$ weakly for all $E \in \Sigma$.

Definition 1.11. Let X be a Banach space. An operator $f : X \to X$ is said to be weakly compact if $f(B)$ is relatively weakly compact for every bounded subset $B \subset X$.

Theorem 1.5. *(Arzela-Ascoli's Theorem)[14, Theorem A.2.1.] A subset F in $C([a,b],X)$ is relatively compact if and only if.*

(i) *F is equicontinuous on $[a,b]$, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ for all $\varphi \in F$ whenever $|x - y| < \delta$, and $x, y \in [a,b]$,*

(ii) *There exists a dense subset D in $[a,b]$ such that, for each $t \in D$, $F(t) = \{f(t); f \in F\}$ is relatively compact in X.*

Definition 1.12. A family $F = \{f_i, i \in J\}$, where J is some index set, is said to be weakly equicontinuous if, given $\varepsilon > 0$ and $\varphi \in X^*$, there exists $\delta > 0$ such that, for $s, t \in [0,T]$, if $|t - s| < \delta$, then

$$|\varphi(f_i(t) - f_i(s))| < \varepsilon, \quad \forall i \in J,$$

i.e. $\varphi(F)$ is equicontinuous for all $\varphi \in X^*$.

Clearly, we have

Proposition 1.3. *[21] If $F = \{f_i, i \in J\}$ is equicontinuous then, F is weakly equicontinuous.*

Definition 1.13. A sequence (x_n) is weakly Cauchy if for every $x^* \in X^*$, the sequence $(x^*(x_n))$ is Cauchy in the scalar filed.

Definition 1.14. Let X be a Banach space, we recall that it is sequentially weakly complete if any weakly Cauchy sequence in X is weakly convergent .

Theorem 1.6. *(Arzéla-Ascoli)[14, Theorem A.3.1.] Let X be a sequentially weakly complete Banach space. A family F in the space $C([a,b];X)$, endowed with the uniform weak convergence topology, is sequentially relatively compact if and only if:*

(i) F is weakly equicontinuous on $[a, b]$,

(ii) there exists a dense subset D in $[a, b]$ such that, for each $t \in D$, the section $J(t) = \{f(t); / f \in F\}$ is sequentially weakly relatively compact in X.

We present a direct consequence of the Hahn-Banach theorem.

Theorem 1.7. *[21] Let X be a normed space with $0 \neq x_0 \in X$. Then there exists $\phi \in X^*$ such that $\| \phi \|= 1$ and $\phi(x_0) =\| x_0 \|$.*

The following theorem play the key role in this memoir.

Theorem 1.8. *(Eberlein-Šmulian's Theorem, [18]) let A a subset of Banach space X the following assertions are equivalent*

(i) The set A is (relatively) weakly compact,

(ii) The set A is (relatively) weakly sequentially compact,

(iii) The set A is (relatively) weakly countably compact.

Before presenting the proof we present some important definitions and theories, we will use [18].

Definition 1.15. A sequence $(e_k)_{k=1}^{\infty}$ in a Banach space X is called a basic sequence if it is a basis for $[e_k]$, the closed linear span of $(e_k)_{k=1}^{\infty}$.

Lemma 1.5. *If A is a bounded set in X such that $\overline{A}^{weak*} \subset X$ then A is relatively weakly compact.*

Theorem 1.9. *Let M be a bounded subset of a Banach space X such that $0 \notin M^{\|\|}$. Then the following are equivalent*

(i) M fails to contain a basic sequence,

(ii) $\overline{M}^{weak}$ *is weakly compact and fails to contain* 0*.*

Definition 1.16. A point $x \in X$ is a cluster point of a sequence $(x_n)_{n\in N}$ if, for every neighborhood V of x, there are infinitely many natural numbers n such that $x_n \in V$. If the space is sequential (you can do topology in it using only sequence), this is equivalent to the assertion that x is a limit of some subsequence of the sequence $(x_n)_{n\in N}$.

Lemma 1.6. *If $(x_n)_{n=1}^{\infty}$ is a basic sequence in a Banach space X and $x \in X$ is a weak cluster point of $(x_n)_{n=1}^{\infty}$. Then, $x = 0$.*

Lemma 1.7. *Let A be a relatively weakly countably compact subset of a Banach space X. Suppose that $x \in X$ is the only weak cluster point of the sequence $(x_n)_{n=1}^{\infty} \subset A$. Then $(x_n)_{n=1}^{\infty}$ converges weakly to x.*

Now, we state the proof of Theorem 1.8.

Proof. Since every sequence can be considered as a special case of a net (i) and (ii) both imply (iii). We have to show that (iii) implies both (ii) and (i). First we will prove the relativized versions and then show that the result for non relativized versions can follow easily. Note that each of the statements implies that A is bounded.
(iii) $\Longrightarrow$(ii). Let $(x_n)_{n=1}^{\infty}$ be any sequence in A. Then, by hypothesis, there is a weak cluster point x of $(x_n)_{n=1}^{\infty}$. If x is in the norm closure of the set $\{x_n\}_{n=1}^{\infty}$, then there is a subsequence which converges and we are done. If not, applying Theorem 1.9, we construct a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ so that $(y_n - x)_{n=1}^{\infty}$ is a basic sequence. Since $(y_n - x)_{n=1}^{\infty}$ is in A which is relatively weakly compact then it has a weak cluster point y. So, if B is an arbitrary index set, there exists a net $(x_\beta)_{\beta\in B} \subset (y_n)_{n=1}^{\infty}$ with $x_\beta \to y$. We have

$$(x_\beta - x)_{\beta\in B} \subset (y_n - x)_{n=1}^{\infty} \quad \text{and} \quad (x_\beta - x) \to y - x,$$

therefore $y - x$ is a weak cluster point of the basic sequence $(y_n - x)_{n=1}^{\infty}$ By Lemma 1.6 $y-x = 0$, therefore $y = x$. Thus x is the only weak cluster point of $(y_n)_{n=1}^{\infty}$. By Lemma 1.7 $(y_n)_{n=1}^{\infty}$ converges to x. So we proved that every sequence $(x_n)_{n=1}^{\infty} \subset A$ has a convergent subsequence $(y_n)_{n=1}^{\infty}$ and therefore A is relatively weakly sequentially compact.
(iii) $\Longrightarrow$ (i). Suppose the opposite, that A is not relatively weakly compact by Lemma

1.5 then, the weak * closure W of A is not contained in X. Thus there exists $x^{**} \in W \backslash X$. Pick $x^* \in X^*$ so that $x^{**}(x^*) > 1$. Then consider the set $A_0 = \{x \in A : x^*(x) > 1\}$. The set is not relatively weakly compact since x^{**} is in its weak* closure. Theorem 1.9 gives us a basic sequence $(x_n)_{n=1}^{\infty}$ contained in A_0. Since $(x_n)_{n=1}^{\infty} \subset A_0 \subset A$ and A relatively countably compact then $(x_n)_{n=1}^{\infty}$ must have a weak cluster point x which by Lemma 1.6 should be $x = 0$. This is a contradiction since, by construction, $x^*(x) \geq 1$. □

Some consequence of the Eberlein Šmulian Theorem is the following result.

Theorem 1.10. *(Krein-Šmulian's Theorem, [19]) The closed, convex hull of weakly compact subset of a Banach space is weakly compact.*

1.3 Elementary fixed point theorems

In this section, we offer the main fixed point theorems, which can be found in many books of analysis, topology and functional analysis (see [8], [13]). At the beginning we recall several basic definitions and concepts used further on.

Definition 1.17. let f be a mapping of a set X into itself. Then a point $x \in X$ is said to be fixed point of f if $f(x) = x$.

Definition 1.18. A set X is called a locally convex linear topological spaces

- if X is a linear space and at the same time a topological space such that the two mappings $X \times X \to X : (x, y) \to x + y$ and $K \times X \to X : (\alpha, x) \to \alpha x$ are both continuous,
- And if any of its open sets $\mathcal{O} \ni 0$ contains a convex, balanced and absorbing open set (in other words, the element 0 has a base of convex, balanced and absorbing neighbored).

Recall that, a set $M \subset X$ is balanced if : $x \in M$ and $|\alpha| \leq 1$ imply $\alpha x \in M$. M is absorbing if for any $x \in X$, there exists $\alpha > 0$ such that $\alpha^{-1}x \in M$.

Definition 1.19. let (X, d) be a metric space and M be a subset of X. The mapping $f : M \to X$ is called contractive, if there exists a constant $\alpha < 1$ such that

$$d(fx, fy) \leq \alpha\, d(x, y) \quad \forall x, y \in M.$$

Lemma 1.8. *Let X be a linear vector space and $M \subset X$ a nonempty subset.*
If $f : M \to X$ a contraction. Then, the mapping $I - f : M \to (I - f)(M)$ is an homeomorphism.

Proof. Since

$$\|(I - f)\, x - (I - f)\, y\| \leq (1 + \alpha)\, \|x - y\|\,.$$

The mapping $I - f$ is continuous. In addition

$$\|(I - f)\, x - (I - f)\, y\| \geq (1 - \alpha)\, \|x - y\| \quad (0 < \alpha < 1).$$

Which proof that $I - f$ is one-to-one, thus the inverse of $I - f : M \to (I - f)(M)$ exists and continuous. □

Definition 1.20. let (X, d) be a metric space and M be a subset of X. The mapping $f : M \to X$ is called expansive, if there exists a constant $h > 1$ such that

$$d(fx, fy) \geq h\, d(x, y) \quad \forall x, y \in M.$$

Lemma 1.9. *Let $(X, \|\cdot\|)$ be a linear normed space, $M \subset X$. Suppose that the mapping $f : M \to X$ is expansive with constant $h > 1$. Then the inverse of F exists. Where $F := I - f : M \to (I - f)(M)$ and*

$$\|F^{-1}x - F^{-1}y\| \leq \frac{1}{h - 1}\|x - y\| \quad x, y \in F(M). \tag{1.1}$$

Proof. For each $x, y \in M$ we have

$$\| \, Fx - Fy \, \| = \| \, (fx - fy) - (x - y) \, \| \geq (h - 1) \, \| \, x - y \, \| \,. \tag{1.2}$$

Which shows that F is one-to-one, hence the inverse of $F : M \to F(M)$ exists. Now taking $x, y \in F(M)$, then $F^{-1}x$, $F^{-1}y \in M$ thus using $F^{-1}x$, $F^{-1}y$ substitute for x, y in (1.2) respectively, we obtain

$$\| \, F^{-1}x - F^{-1}y \, \| \leq \frac{1}{h - 1} \, \| \, x - y \, \| \quad \forall x, y \in F(M).$$

Definition 1.21. Let X be a Banach space. A mapping $f : X \to X$ is said to be nonexpansive if

$$\| fx - fy \| \leq \| x - y \| \text{ for all } x, y \in X.$$

In the first, we show theorem known as the Banach contraction principle was formulated by Banach in 1922. Let describe this theorem.

Theorem 1.11. *(Banach, [13]) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction with constant α. Then f has a unique fixed point $x \in X$.*

Theorem 1.12. *(Schauder, [13]) Let M be a closed convex set in a Banach space X and assume that $f : M \to M$ is continuous mapping such that $f(M)$ is relatively compact subset of M. Then, f has a fixed point.*

Theorem 1.13. *(Schauder-Tychonoff, [12]) Let M be a convex, and compact subset of a locally convex topological space X. If f is a continuous map on M into M, then f has, at least a fixed point.*

Krasnoselskii combined the two main fixed point theorems, Banach contraction mapping principle and Schauder fixed point theorem into the following result.

Theorem 1.14. *(Krasnoselskii, [23]) Assume that M is closed bounded convex subset of a Banach space X. Furthermore that A and B are mapping from M into X such that:*

(i) $A(x) + B(y) \in M$ *for all* $x, y \in C$,

(ii) *A continuous and compact ,*

(iii) B *is a* $\lambda-$*contraction on* X *with* $\lambda < 1$.

Then $\exists\, x \in M$ *such that* $A(x) + B(x) = x$.

An important fixed point theorem that has been commonly used in the theory of nonlinear differential and integral equations is the following result proved by D. W. Boyd and J. S. W. Wong [8]. This theorem extends the contractions to nonlinear contractions, and also generalizes the Banach fixed point principle .

Theorem 1.15. *(Boyd and Wong, [13]) Let X be a Banach space and $f : X \to X$ be a nonlinear contraction. Then, f has a unique fixed point x, and the sequence $(f^n x)_n$ of successive iterations of f converges to x for each $x \in X$.*

1.4 Fixed point theory relative to the weak topology

Now, we present fixed point theorems involving the weak topology of Banach space. The results presented in this section are importance and the most works study in Chapters 2, 3 and 4 are based on these theorems.

Theorem 1.16. *(Schauder-Tychonoff Theorem for the weak topology, [21]) Let X be a Banach space and M a weakly compact convex subset of X. Then, any weakly continuous map $f : M \to M$ has at least one fixed point.*

Theorem 1.17. *(Modified Schauder-Tychonoff Theorem, [21]) Let X be a Banach space and M a closed convex subset of X. Then, any weakly continuous and weakly compact map $f : M \to M$ has at least one fixed point.*

Let show the version of Schauder fixed point principal which was obtained by Arino Gautier and Penot in 1984 [1, Theorem 1].

Lemma 1.10. *let M be a weakly compact convex subset of a Banach space X. Then each sequentially weakly continuous map $f : M \to M$ has a fixed point in M.*

Proof. It suffice to proved that f is weakly continuous, so that the Schauder-Tychonoff fixed point Theorem 1.16 applies. Now for each weakly closed subset B of X, $f^{-1}(B)$ is sequentially closed in M, hence sequentially weakly compact (because M is weakly compact); and by the Eberlein Šmulian Theorem 1.8, $f^{-1}(B)$ is weakly compact. So $f^{-1}(B)$ is weakly closed, hence f is weakly continuous. □

Theorem 1.18. *[13, Theorem 2.2.1] Let X be a Banach space, M be a nonempty closed convex subset of X and $f : M \to M$ be a sequentially weakly continuous map. If $f(M)$ is relatively weakly compact, then f has a fixed point in M.*

Proof. Let $C = \overline{co}(f(M))$ be the closed convex hull of $f(M)$. Since $f(M)$ is relatively weakly compact, then C is a weakly compact convex subset of X. Moreover, $f(C) \subset f(M) \subset \overline{co}(f(M)) = C$ i.e., f maps C into itself. Since f is weakly sequentially continuous, and by using Lemma 1.10, it follows that f has, at least, one fixed point in C. □

Definition 1.22. let (X, τ) a Hausdorff topological vector space, M a nonempty subset of X. $f : M \to M$ be a mapping. A sequence x_n in M is called a τ-approximate fixed point sequence for f if $x_n - f(x_n) \xrightarrow{\tau} 0$, as $n \to \infty$.

Theorem 1.19. *[5] Let M be a weakly compact convex subset of a Banach space X. Then every demicontinuous mapping $f : M \to M$ has a weak-approximate fixed point sequence.*

Chapter 2

Krasnoselskii Fixed Point Theorems for Contraction Maps

In this chapter, several other attempts have been made in the literature in order to prove the analogousness of Krasnoselskii fixed point theorem under the weak topology.

2.1 Krasnoselskii's fixed point theorem for weakly continuous maps

In 2003, Barroso [4] established a version of Krasnoselskii fixed point theorem using the weak topology of a Banach space. His result only requires the weak continuity and weak compactness of A. While B is a linear operator satisfying $\|B^p\| \leq 1$.

Theorem 2.1. *let M be a closed convex subset of a Banach space $(X,\|.\|)$. Assume that $A : M \to X$, and $B \in \mathcal{L}(X)$ satisfies*

(i) $\|B^p\| < 1$ *for some* $p \geq 1$,

(ii) *A is weakly continuous and $A(M)$ is weakly precompact,*

(iii) $x = Bx + Ay,\ y \in M \Longrightarrow x \in M$.

Then there is $x \in M$ such that $Ax + Bx = x$.

Proof. By Banach's contraction principal for each $y \in M$ there exists a unique $x = x(y) \in X$ so that $x = Bx + Ay$, which we have $x \in M$ by assumption (iii).
Now, define

$$(I - B)^{-1} = (I - B^p)^{-1} \sum_{k=0}^{p-1} B^k. \tag{2.1}$$

Then $(I-B)^{-1}$ is well defined, by (2.1) we have $(I-B)^{-1} \in \mathcal{L}(X)$ and $(I-B)^{-1}(I-B) = I$. By Theorem 1.1, $(I-B)^{-1}$ is a weakly continuous operator on X. Hence, because (ii), the map from M into itself $y \longmapsto (I-B)^{-1}A(y) = x$ is weakly continuous in M. Since $A(M)$ is weakly precompact, so $(I-B)^{-1}A(M)$ is. Once M is closed convex, we may apply Theorem 1.18 and conclude that $(I-B)^{-1}A$ has a fixed point in M. So, this completes the proof. ☐

If assumption (i) is interchanged by $\|B^p\| = 1$, the same technique is used in the proof of Theorem 2.1 namely Banach contraction principle, cannot be applied. Thus, in order to study such cases some additional hypothesis may be required. Inspired by proof of Theorem 2.1 we introduce the following condition:

$$[\lambda \in (0,1) \text{ and } x = \lambda Bx + Ay, y \in M] \Longrightarrow x \in M. \tag{2.2}$$

Now, we can state the following result .

Theorem 2.2. *Let X be a Banach space and M a weakly compact convex, subset of X. Assume (2.2) with $B \in \mathcal{L}(X)$, $\|B^p\| \leq 1$, for some $p \geq 1$, and $A : M \to X$ being weakly continuous. Then fixed points for $A + B$ and A are achieved in M.*

Proof. For $0 < \lambda < 1$ define $B_\lambda = \lambda B$. Then, $B_\lambda \in \mathcal{L}(X)$, $\|B_\lambda^p\| < 1$. Now, arguing as in the proof of Theorem 2.1 by using condition (2.2), we find a family $x_\lambda \subset M$ such that $Ax_\lambda + \lambda Bx_\lambda = x_\lambda$ for all $\lambda \in (0,1)$. Taking now a sequence $0 < \lambda_n < 1$ so that $\lambda_n \to 1$ and considering the respective sequence $x_n \subset M$ satisfying

$$Ax_n + \lambda_n Bx_n = x_n \quad \text{for all} \quad n \in N. \tag{2.3}$$

As M is weakly compact, we can find a subsequence x_{n_i} such that $x_{n_i} \rightharpoonup x$ in X with $x \in M$. A is weakly continuous so $Ax_{n_i} \rightharpoonup Ax$ in X. Besides, $Bx_{n_i} \rightharpoonup Bx$ in X since

$B \in \mathcal{L}(X)$. So, passing the limit in (2.3) we conclude that x is a fixed point for $A + B$. Similarly, for a sequence $0 < \lambda_n < 1$ converging to 0 on obtains a fixed point for A. □

Definition 2.1. Let X be a Banach space. An operator A with domain $D(A) \in X$ is called dissipative if

$$\forall \lambda > 0, \forall x \in D(A), \quad \| x \| \leq \| (I - \lambda A)x \| \ .$$

In a Hilbert space, A is called dissipative if

$$Re(Ax, x) \leq 0, \quad \forall x \in D(A).$$

Next, we shall provide a sufficient condition for getting (2.2) fulfilled.

Proposition 2.1. *Let X be a Banach space and suppose that $A : X \to X$ is a mapping such that $AB_R \subseteq B_R$, for some $R > 0$. If $B \in \mathcal{L}(X)$ is a dissipative operator, then condition (2.2) holds.*

Proof. Since B is dissipative operator on X we have,

$$\|x\| \leq \|(I - \lambda B)x\| \quad \forall x \in X, \quad \forall \lambda > 0. \tag{2.4}$$

Now, let $\lambda \in (0,1)$ and suppose that $x = \lambda Bx + Ay$, with $y \in B_R$. From (2.4) we have $\|x\| \leq \|(I - \lambda B)x\| = \|Ay\| \leq R$, Since $Ay \in B_R$. So $x \in B_R$ and the proposition is proved. □

Remark 2.1. Proposition 2.1 also implies condition (iii) of Theorem 2.1.

By Lemma 1.2, Proposition 2.1 combined with Theorem 2.2 immediately yields the following result.

Theorem 2.3. *Suppose X is reflexive and assume that $B \in \mathcal{L}(X)$ with $\|B^P\| \leq 1$, $p \geq 1$ is a dissipative Operator on X. If $A : X \to X$ is weakly continuous mapping such that $AB_R \subseteq B_R$ for some $R > 0$, then there is $x \in B_R$ such that $Ax + Bx = x$.*

2.2 Krasnoselskii's fixed point theorem for weakly sequentially continuous maps

In 2005, new versions of the Krasnoselskii's fixed point theorem were obtained for sequentially weakly continuous mappings [3]. Consequently we have the following result which will be used in this form in Section 3.

Theorem 2.4. *Let M be a closed, convex subset of a Banach space X. Assume that $A, B : M \to X$ satisfies:*

(i) A is sequentially weakly continuous,

(ii) B is λ-contraction,

(iii) If $x = Bx + Ay$ for some $y \in M$, then $x \in M$,

(iv) If x_n is a sequence in $\mathbb{F}$ where

$$\mathbb{F} := \{x \in X : x = B(x) + A(y) \quad \text{for some} \quad y \in M\}.$$

Such that $x_n \rightharpoonup x$, for some $x \in M$, then $Bx_n \rightharpoonup Bx$,

(v) The set $\mathbb{F}$ is relatively weakly compact.

Then $A + B$ has a fixed point in M.

Proof. Fix a point $x \in M$ and let Tx be the unique point in X such that $Tx = BTx+Ax$. By (iii), we have $Tx \in M$. So that the mapping $T : M \longrightarrow M$ given by $x \longmapsto Tx$ is well-defined. Notice that $Tx = (I - B)^{-1}Ax$, for all $x \in M$. In addition, we observe that $T(M) \subset \mathbb{F}$. We claim now that T is sequentially weakly continuous in M. Indeed, let x_n be a sequence in M such that $x_n \rightharpoonup x$ in M. Since $Tx_n \subset \mathbb{F}$, the assumption (v) guarantees, up to a subsequence, that $Tx_n \rightharpoonup y$, for some $y \in M$. By (iv), we have $BTx_n \rightharpoonup By$. Also, from (i) it follows that $Ax_n \rightharpoonup Ax$ and hence the equality $Tx_n = BTx_n + Ax_n$ give us $y = By + Ax$. By uniqueness, we conclude that $y = Tx$. This proves the claim. Take now the subset $C = \overline{co}(\mathbb{F}) \subset M$. Krein-Šmulian Theorem 1.10 implies that C is a weakly compact set. Furthermore, it is easy to see that $T(C) \subset C$. Applying lemma 1.10, we find a fixed point $x \in C$ for T. Consequently, this proves Theorem . $\square$

Let us now state some consequences of Theorem 2.4. The first one is the following result for reflexive Banach spaces, where closed, convex and bounded sets are weakly compacts.

Corollary 2.1. *Assume that the conditions (i)-(iv)of Theorem 2.4 are fulfilled for A and B. If M is a closed, convex and bounded subset of a reflexive Banach space, then $A+B$ has a fixed point in M.*

Corollary 2.2. *Let M be a convex and weakly compact subset of a Banach space X and let $A, B : M \to X$ be sequentially weakly continuous operators such that*

(i) B is nonexpansive,

(ii) If $\lambda \in (0,1)$ and $x = \lambda Bx + Ay$ with $y \in M$, then $x \in M$.

Then $A+B$ has a fixed point in M.

Finally, we give the following asymptotic version of the Krasnoselskii fixed point theorem.

Theorem 2.5. *Let M be a nonempty bounded, closed and convex subset of a reflexive Banach space X. Suppose that $A : M \to X$ and $B : X \to X$ are continuous maps satisfying:*

(i) A is weakly sequentially continuous and $A(M)$ is relatively weakly compact,

(ii) B is a strict contraction,

(iii) $x = Bx + Ay, y \in M$ implies $x \in M$.

Then, $x_n - (A+B)x_n \rightharpoonup 0$ for some sequence (x_n) in M.

Proof. First notice that M is a bounded closed convex subset of a reflexive Banach space, hence M is weakly compact. On the other hand, since B is a strict contraction, then by

Lemma 1.8 the mapping $(I-B)$ is a homeomorphism from X into $(I-B)X$. Next, let y be fixed in M. The map which assigns to each $x \in X$ the value $Bx+Ay$ defines a strict contraction from X into X. So, by the Banach fixed point principle, the equation $x = Bx + Ay$ has a unique solution $x \in X$. By hypothesis (iii) we have $x \in M$. Hence, $x = (I-B)^{-1}Ay \in M$ which implies that the mapping $(I-B)^{-1}A$ is well defined and $(I-B)^{-1}AM \subseteq M$. By (i) the mapping $(I-B)^{-1}A$ is demicontinuous. Thanks to Theorem 1.19 there is a sequence (y_n) in M such that

$$y_n - (I-B)^{-1}Ay_n \rightharpoonup 0. \tag{2.5}$$

Since M is weakly compact, then by extracting a subsequence if necessary, we may assume that $y_n \rightharpoonup y$. So, (2.5) implies $(I-B)^{-1}Ay_n \rightharpoonup y$. Since A is sequentially weakly continuous, then $Ay_n \rightharpoonup Ay$ and $A(I-B)^{-1}Ay_n \rightharpoonup Ay$. Now, let (x_n) be the sequence of M defined by $x_n = (I-B)^{-1}Ay_n$. Then, we have

$$\begin{aligned} x_n - (Ax_n + Bx_n) &= (I-B)x_n - Ax_n \\ &= Ay_n - Ax_n \\ &= Ay_n - A(I-B)^{-1}Ay_{n.} \end{aligned}$$

We obtain that $x_n - (Ax_n + Bx_n) \rightharpoonup 0$. This completes the proof. □

2.3 Application to a non linear integral equation

The purpose of this section is to study the existence of solutions of some nonlinear functional integral equations in the space of continuous functions under some conditions. To do this, we will use Krasnoselskii fixed point theorem. In this section we deal with the following integral equation

$$x(t) = f(x) + \int_0^t g(s, x(s))ds, \quad x \in C(J, X). \tag{2.6}$$

Where X is a reflexive space and $I = [0,T]$. Assume that the functions involved in (2.6) satisfy the following conditions

(H_1) $f : X \to X$ is sequentially weakly continuous and onto,

(H_2) $\|f(x) - f(y)\| \leq \lambda \|x - y\|$, $(\lambda \leq 1)$ for all $x,y \in X$,

(H_3) $\| x \| \leq \| x - (f(x) - f(0)) \|$,

(H_4) for any $t \in J$, the map $g_t = g(t, \cdot) : X \to X$ is sequentially weakly continuous,

(H_5) for each $x \in C(J, X)$, $g(\cdot, x(\cdot))$ is Pettis integrable on $[0, T]$,

(H_6) there exist $\alpha \in L^1[0, T]$ and a nondecreasing continuous function ϕ from $[0, \infty)$ to $(0, \infty)$ such that $\|g(t, x)\| \leq \alpha(t)\phi(\|x\|)$ for a.e. $t \in [0, T]$ and all $x \in X$. Further, assume that

$$\int_0^T \alpha(s)ds < \int_{\|f(0)\|}^{\infty} \frac{dr}{\phi(r)}.$$

$$\beta(t) = \int_{\|f(0)\|}^{t} \frac{dr}{\phi(r)} \quad \text{and} \quad b(t) = \beta^{-1}\Big(\int_0^t \alpha(s)ds\Big).$$

Our existence result for (2.6) is as follows.

Theorem 2.6. *Under assumptions (H_1)-(H_6), equation (2.6) has at least one solution $x \in C(I, X)$.*

Proof. We have

$$\int_{\|f(0)\|}^{b(t)} \frac{dr}{\phi(r)} = \int_0^t \alpha(r)dr. \tag{2.7}$$

It follows from (2.7) and the final part of (H_6) that $b(T) < \infty$. Now we define the set

$$M = \Big\{x \in C(J, X) : \|x(t)\| \leq b(t) \text{ for all } t \in I\Big\}.$$

Our strategy is to apply Theorem 2.4 in order to find a fixed point for the operator $A+B$ in M, where $A, B : M \to C(I, X)$ are defined by

$$Ax(t) = f(0) + \int_0^t g(s, x(s))ds,$$
$$Bx(t) = f(x(t)) - f(0).$$

The proof will be given in several steps.

Step 1: M is bounded, closed and convex in $C(I, X)$.
The fact that M is bounded and closed comes directly from its definition. Let us show M is convex. Let x, y be any two points in M. Then, there holds

$$\| sx(t) + (1-s)y(t) \| \leq b(t),$$

for all $t \in I$, which implies that $sx + (1-s)y \in M$, for all $s \in [0,1]$. This shows that M is convex.

Step 2: $A(M) \subseteq M$, $A(M)$ is weakly equicontinuous and relatively weakly compact.

(i) Let $x \in M$ be an arbitrary point. We shall prove $Ax \in M$. Fix $t \in I$ and consider $Ax(t)$. Without loss of generality, we may assume that $Ax(t) \neq 0$. By Theorem 1.7 there exists $\psi_t \in X^*$ with$\| \psi_t \| = 1$ such that $\langle \psi_t, Ax(t) \rangle = \|Ax(t)\|$. Thus,

$$\begin{aligned} \|Ax(t)\| &= \langle \psi_t, f(0) \rangle + \int_0^t \langle \psi_t, g(s, x(s)) \rangle ds \\ &\leq \|f(0)\| + \int_0^t \alpha(s)\phi(\|x(s)\|) ds \\ &\leq \|f(0)\| + \int_0^t \alpha(s)\phi(b(s)) ds. \end{aligned} \tag{2.8}$$

According to Bihari's inequality we have :

$$\|Ax(t)\| \leq b(t).$$

Then, (2.8) implies that $A(M) \subseteq M$. Analogously one shows that,

$$\begin{aligned} \|Ax(t) - Ax(s)\| &\leq \int_s^t \alpha(\tau)\phi(\|x(\tau)\|) d\tau \\ &\leq \int_s^t \alpha(\tau)\phi(b(\tau)) d\tau \\ &\leq \int_s^t b'(\tau) d\tau \\ &\leq |b(t) - b(s)|. \end{aligned} \tag{2.9}$$

For all $t, s \in I$. Thus it follows from (2.9) that $A(M)$ is equicontinuous.

(ii) Let (Ax_n) be any sequence in $A(M)$. Notice that M is bounded. By reflexiveness, for each $t \in I$ the set $\{Ax_n(t) : n \in N\}$ is relatively weakly compact. As before, one shows that $\{Ax_n : n \in N\}$ is a equicontinuous subset of $C(I, X)$. It follows now from the Eberlein-Šmulian's Theorem 1.8 and Ascoli-Arzela Theorem 1.6 that $A(M)$ is relatively weakly compact, which proves the third assertion of Step 2.

Step 3: A is sequentially weakly continuous.

Let (x_n) be a sequence in M such that $x_n \rightharpoonup x$ in $C(I,X)$, for some $x \in M$. Then, $x_n(s) \rightharpoonup x(s)$ in X for all $s \in I$. By assumption (H_5) one has that

$$g(s, x_n(s)) \rightharpoonup g(s, x(s)) \text{ in } X \text{ for all } s \in I.$$

The Lebesgue dominated convergence Theorem yields that $Ax_n(t) \rightharpoonup Ax(t)$ in X for all $t \in I$. On the other hand, it follows from (2.9) that the set $\{Ax_n : n \in N\}$ is a weakly equicontinuous subset of $C(I,X)$. Hence, by the Ascoli-Arzela Theorem there exists a subsequence (x_{n_j}) of (x_n) such that $Ax_{n_j} \rightharpoonup y$ for some $y \in C(I,X)$. Consequently, we have that $y(t) = Ax(t)$ for all $t \in I$ and hence $Ax_{n_j} \rightharpoonup Ax$. Now, a standard argument shows that $Ax_n \rightharpoonup Ax$. This proves Step 3.

Step 4: B satisfies conditions (ii) and (iv) of Theorem 2.4.

By (H_2) clearly we see that B is a λ contraction in $C(I,X)$. Now, in order to verify condition (iv) to B, we first remark that by combining (2.9) with (H_2), it follows that $\mathbb{F}$ is equicontinuous in $C(I,X)$. So is $B(\mathbb{F})$. Let now $(x_n) \subset \mathbb{F}$ be sequence such that $x_n \rightharpoonup x$, for some $x \in M$. Then by assumption (H_1), we obtain $Bx_n(t) \rightharpoonup Bx(t)$. Since (Bx_n) is equicontinuous in $C(I,X)$ and $\| (Bx_n)(t) \| \leq \lambda \| x_n(t) \|$ holds for all $n \in N$, we may apply the Ascoli-Arzela Theorem and concludes that there exists a subsequence (x_{n_j}) of (x_n) such that $Bx_{n_j} \rightharpoonup y$, for some $y \in C(I,X)$. Hence, $Bx = y$ and by standard arguments we have $Bx_n \rightharpoonup Bx$ in $C(I,X)$. This completes Step 4.

Step 5: Condition (iii) of Theorem 2.4 holds.

Suppose that $x = Bx + Ay$ for some $y \in M$. We will show that $x \in M$. By condition (H_3) it follows that $\| x(t) \leq \| x(t) - Bx(t) \| = \| Ay(t) \|$. Once $y \in M$ implies $Ay \in M$, we conclude $x \in M$.

Step 6: Condition (v) of Theorem 2.4 holds.

Let $(x_n) \subset \mathbb{F}$ be an arbitrary sequence. Then,(x_n) is equicontinuous in $C(I,X)$. Also, one has that

$$\| x_n(t) \| \leq (1-\lambda)^{-1} b(t),$$

for all $t \in I$, that is, for each $t \in I$ the set $\{x_n(t)\}$ is relatively weakly compact in X. Thus, invoking again the Ascoli-Arzela Theorem we obtain a subsequence of (x_n) which converges weakly in $C(I, X)$. By the Eberlein-Šmulian Theorem, it follows that $\mathbb{F}$ is relatively weakly compact.

Theorem 2.4 now gives a fixed point for $A + B$ in M, and hence a solution to (2.6).□

Chapter 3

Expansive Krasnoselskii-type Fixed Point Theorems

The point of this chapter is to replace the contractiveness of B by the expansiveness and derive new fixed point theorems. We will extends some results presented in [23].

Lemma 3.1. *Let M be a closed subset of a complete metric space X. Assume that the mapping $B : M \to X$ is expansive and $B(M) \supset M$, then there exists a unique point $x \in M$ such that $Bx = x$.*

Proof. B is expansive then the inverse of $B : M \to B(M)$ exists, $B^{-1} : B(M) \to M$ is contractive and hence continue, so $B(M)$ is closed set. Recalling that $M \subset B(M)$ by Banach fixed point theorem there exists $x \in B(M)$ such that $B^{-1}x = x$. Thus $x \in M$ and $Bx = x$. This completes the proof. □

Lemma 3.2. *let $B : X \to X$ be a map such that B^n is an expansive for some $n \in \mathbb{N}$ Assume further that there exist a closed subset M of X such that M is contained in $B(M)$. Then there exists a unique fixed point of B.*

Proof. Since B^n is expansive map and $M \subseteq B^n(M)$. From Lemma 3.1 there exists unique fixed point of B^n. Let $x \in M$ be a fixed point of B^n. Using the fact that B^n is expansive map, then there exist $k > 1$ such that

$$d(B^n(x), B^n(y)) \geq kd(x, y) \quad \forall x, y \in M.$$

Hence

$$d(x, B(x)) = d(B^n(x), B^{n+1}(x)) \geq kd(x, B(x)) \Longrightarrow d(x,B(x)) = 0.$$

Then, B has a unique fixed point in M. □

Remark 3.1. If the mapping $B : X \to X$ is expansive and onto, then there exists a unique point $x \in X$ such that $Bx = x$.

3.1 Krasnoselskii's fixed point theorem for weakly continuous maps

In this section we study some fixed point results of Krasnoselskii type fixed point theorem for the sum of $A + B$, where A is a weakly continuous and B is expansive linear operator.

Theorem 3.1. *Let X be a Banach space, M be a weakly compact convex subset of X, $A : M \to X$ be an weakly continuous map and $B \in \mathcal{L}(X)$ be a linear continuous operator. Assume that A and B satisfy the following hypotheses:*

(i) B is an expansive mapping,

(ii) For each $z \in A(M)$ we have $M \subset B(M) + z$, where

$$B(M) + z = \{y + z : y \in A(M)\}.$$

(iii) for each $x, y \in coA(M)$ such that

$$x = B(x) + A(y) \Longrightarrow x \in coA(M).$$

Then the equation $x = B(x) + A(x)$ has a solution.

Proof. Let $y \in M$. Let $F_y : M \to X$ be a operator defined by

$$F_y(x) = B(x) + A(y), x \in M.$$

From Lemma 3.1 there exist unique $x(y) \in M$ such that $x(y) = B(x(y)) + A(y)$.
By (i) and Lemma 1.9 we obtain that

$$x(y) = (I - B)^{-1} A(y).$$

Let us define

$$\begin{cases} N : M & \rightarrow X \\ \quad N(y) & \rightarrow x(y) \end{cases}$$

which is weakly continuous. Since $(I - B)^{-1} A(y)$ and A are weakly continuous.
Let $\widetilde{M} = \overline{co} A(M)$ be weakly compact convex. Now we prove only that $N(\widetilde{M}) \subseteq \widetilde{M}$. Indeed, let $x \in N(\widetilde{M})$. Then there exists $y \in \widetilde{M}$ such that $x = N(y)$. Hence

$$x = (I - B)^{-1} A(y) \Longrightarrow x \in \overline{co} A(M).$$

Then, $\quad N(\widetilde{M}) \subseteq \widetilde{M}$ So, by Theorem 1.16, there exists $x \in X$ which is fixed point of N.□

Theorem 3.2. *Let X be a Banach space, M be a weakly compact convex subset of X. $A : M \rightarrow X$ be an weakly continuous map and $B \in \mathcal{L}(X)$ be a linear continuous operator. Assume that A and B satisfy the conditions of Theorem 3.1. If only replaced the condition (i) by that B^n is expansive. We get the same result*

3.2 Krasnoselskii's fixed point theorem for weakly sequentially continuous maps

We are now ready to state and prove the first main result .

Theorem 3.3. *let $M \subset X$ be a nonempty closed convex subset. Suppose that A and B map M into X such that*

(i) A is sequentially weakly continuous,

(ii) B is an expansive mapping,

(iii) $z \in A(M)$ implies $B(M) + z \supset M$, where $B(M) + z = \{y + z \quad / y \in B(M)\}$,

(iv) If x_n is a sequence in $\mathbb{F}$ where

$$\mathbb{F} = \mathbb{F}(M, M; B, A) := \{x \in X : x = B(x) + A(y) \quad \text{for some} \quad y \in M\}.$$

such that $x_n \rightharpoonup x$ and $Bx_n \rightharpoonup y$ then $y = Bx$,

(v) The set $\mathbb{F}(M, M; A, B)$ is relatively weakly compact.

Then there exists a point $x \in M$ with $Ax + Bx = x$.

Proof. From (ii) and (iii), for each $y \in M$, we see that the mapping $B + Ay : M \to X$ is expansive and $B(M) + Ay \supset M$ by Lemma 3.1 we have,

$$Bx + Ay = x, \tag{3.1}$$

which has a unique solution $x = \tau(Ay) \in M$, so that the mapping $\tau Ay : M \to M$ given by $y \to \tau Ay$ is well defined. In view of Lemma 1.9, we obtain that $\tau Ay = (I - B)^{-1}Ay$ for all $y \in M$. In addition, we observe that $\tau A(M) \subset \mathbb{F} \subset M$. We claim that τA is sequentially weakly continuous in M. To see this, let x_n be a sequence in M with $x_n \rightharpoonup x$ in M. Notice that $\tau A(x_n) \in \mathbb{F}$. Thus, up to a subsequence, we may assume by (v) that $\tau A(x_n) \rightharpoonup y$ for some $y \in M$. It follows from (i) that $Ax_n \rightharpoonup Ax$. From the equality

$$B(\tau Ax_n) + Ax_n = \tau Ax_n. \tag{3.2}$$

Passing the weak limit in (3.2) yields

$$B(\tau Ax_n) \rightharpoonup y - Ax.$$

The assumption (iv) now implies that $y - Ax = By$ (i.e., $y = \tau Ax$) since $x \in M$ this proves the assertion. Let the set $C = \overline{co}(\mathbb{F})$, where $\overline{co}(\mathbb{F})$ denotes the closed convex hull of $\mathbb{F}$. Then $C \subset M$ and is a weakly compact set by Krein -Šmulian Theorem 1.10. Furthermore, it is straightforward to see that τA maps C into C. In virtue of Lemma 1.10, there exists $x \in C$ such that $\tau Ax = x$. From (3.1) we deduce that

$$B(\tau Ax) + Ax = \tau Ax.$$

That is, $Bx + Ax = x$. The proof is complete. □

Remark 3.2. We note that B may not be continuous since it is only expansive. It is worthy of pointing out that the condition (iii) may be a litter restrictive and the next result might be regarded as an improvement of Theorem 3.3.

Corollary 3.1. *Under the conditions of Theorem 3.3, if only the condition (iii) of Theorem 3.3 is replaced by that B maps M onto X, then there exists a point $x \in M$ with $Ax + Bx = x$.*

Theorem 3.4. *Let $M \subset X$ be a nonempty closed convex subset. Suppose that $A : M \to X$ and $B : X \to X$ such that*

(i) A is sequentially weakly continuous,

(ii) B is an expansive mapping,

(iii) $A(M) \subset (I-B)(X)$ and $[x = Bx+Ay, y \in M] \Longrightarrow x \in M$(or $A(M) \subset (I-B)(M)$),

(iv) If x_n is a sequence in $\mathbb{F}(X, M; B, A)$ such that $x_n \rightharpoonup x$ and $Bx_n \rightharpoonup y$ then $y = Bx$,

(v) The set $\mathbb{F}(X, M; B, A)$ is relatively weakly compact.

Then there exists a point $x \in M$ with $Ax + Bx = x$.

Proof. For each $y \in M$, by (iii), there exists $x \in X$ such that $x - Bx = Ay$. By (ii), Lemma 1.9 and the second part of (iii), we have $x = (I - B)^{-1}Ay \in M$. As is shown in Theorem 3.3 one obtains that $(I - B)^{-1}A : M \to M$ is sequentially weakly continuous and there is a point $x \in M$ with $x = (I - B)^{-1}Ax$. This completes the proof. □

Let us now state some consequences of Theorem 3.4. First, the case when X is a reflexive Banach space is considered, so that a closed, convex and bounded set is weakly compact. Rechecking the proof of Theorem 3.3, we find that it is only required $\overline{co}(\mathbb{F})$ to be weakly compact.

Corollary 3.2. *Suppose that the conditions (i)-(iv) of Theorem 3.4 for A and B are fulfilled. If $\mathbb{F}(X, M; B, A)$ is a bounded subset of a reflexive Banach space X, then, $B + A$ has at least one fixed point in M.*

The second consequence of Theorem 3.4 is concerned the case when B is non contractive on $M \in X$, i.e.: $\|Bx - By\| \geq \|x - y\|$ for all $x, y \in M$.

Corollary 3.3. *Let $M \subset X$ be a nonempty convex and weakly compact subset. Suppose that $B : X \to X$ and $A : M \to X$ are sequentially weakly continuous such that*

(i) *B is non-contractive on X (or M),*

(ii) *There is a sequence $\lambda_n > 1$ with $\lambda_n \to 1$ such that $A(M) \subset (I - \lambda_n B)(X)$ and $[x = \lambda_n Bx + Ay, y \subset M] \Longrightarrow x \in M$(or $A(M) \subset (I - \lambda_n B)(M)$.*

Then $B + A$ has a fixed point in M.

Proof. For each $x, y \in X$ we have

$$\|\lambda_n Bx - \lambda_n By\| = \lambda_n \|Bx - By\| \geq \lambda_n \|x - y\| .$$

Notice that $\lambda_n B : X \to X$ is expansive with constant $\lambda_n > 1$. By Theorem 3.4, there exists $x_n \in M$ such that

$$Ax_n + \lambda_n Bx_n = x_n. \tag{3.3}$$

Up to a subsequence we may assume that $x_n \rightharpoonup x$ in M since M is convex and weakly compact. Passing the weak limit in (3.3) we proved that $Ax + Bx = x$. Which complete the proof. $\square$

Given by Lemma 1.9, Theorem 3.4 and Theorem 2.4, the following weak type Krasnoselskii fixed point theorem may be easily formulated, which clearly contains, but not limited to Theorem 3.4 and Theorem 2.4.

Theorem 3.5. *Let $M \subset X$ be a nonempty closed convex subset. Suppose that $B : X \to X$ and $A : M \to X$ such that*

(i) *A is sequentially weakly continuous,*

(ii) *$(I - B)$ is one-to-one,*

(iii) *$A(M) \subset (I-B)(X)$ and $[x = Bx+Ay, y \in M] \Longrightarrow x \in M$(or $A(M) \subset (I-B)(M)$),*

(iv) If x_n is a sequence in $\mathbb{F}(X, M; B, A)$ such that $x_n \rightharpoonup x$ and $Bx_n \rightharpoonup y$, then $y = Bx$,

(v) The set $\mathbb{F}(X, M; B, A)$ is relatively weakly compact.

Then, there exists a point $x \in M$ with $Ax + Bx = x$.

Remark 3.3. If $B : X \to X$ is a contraction mapping, then $(I - B)(X) = X$ and hence $A(M) \subset (I - B)(X)$. It can be easily seen by (ii) and (iii) that $\mathbb{F}(X, M; B, A) = (I - B)^{-1}A(M)$. The condition (iv) is weaker than the condition that B is sequentially weakly continuous.

Taking advantage of the linearity of the operator B, we derive the following result.

Theorem 3.6. *Let X be a reflexive Banach space, $B : X \to X$ a linear operator and $A : X \to X$ a sequentially weakly continuous map. Assume that the following conditions are satisfied*

(i) $(I - B)$ is continuously invertible,

(ii) There exists $R > 0$ such that $A(B_R) \subset B_{\beta R}$, where $\beta \leq\| (I - B)^{-1} \|^{-1}$,

(iii) $A(B_R) \subset (I - B)(X)$.

Then $B + A$ possesses a fixed point in B_R.

Proof. Let $F = I - B : X \to (I - B)(X)$. By (i), one can easily see from the fact that B is linear and $\beta \leq\| (I - B)^{-1} \|^{-1}$ that

$$\left\|F^{-1}x - F^{-1}y\right\| \leq \frac{1}{\beta} \|x - y\| \ \forall x,y \in F(X). \tag{3.4}$$

It follows from (3.4) that $F^{-1} : F(X) \to X$ is continuous. Recall that F^{-1} being linear implies that F^{-1} is weakly continuous. Consequently, one knows from (iii) that
$F^{-1}A : B_R \to X$ is sequentially weakly continuous. For any $x \in B_R$, one easily derive from (3.4) and (ii) that $\| F^{-1}Ax \|\leq R$. Hence, $F^{-1}A$ maps B_R into itself. Applying Lemma 1.10 we obtain that $F^{-1}A$ has a fixed point in B_R. Which implied that $\exists\, x \in B_R$ such that $Bx + Ax = x$. □

Next, we shall present some concrete mappings which fulfill the condition (i) of Theorem 3.6. Before stating the consequences, we introduce the following two lemmas.

Lemma 3.3. *Let $(X, \| \, . \, \|)$ be a linear normed space, $M \subset X$. Assume that the mapping $B : M \to X$ is contractive with constant $\alpha < 1$, then the inverse of $F := I - B : M \to (I - B)(M)$ exists and*

$$\left\| F^{-1}x - F^{-1}y \right\| \leq \frac{1}{1-\alpha} \|x - y\| \quad \forall x,y \in F(X). \tag{3.5}$$

Proof. For each $x, y \in M$, we have

$$\|Fx - Fy\| \geq (1-\alpha) \|x - y\| ,$$

which proves that F is one-to-one, thus the inverse of $F : M \to F(M)$ exists. Now we set

$$G := F^{-1} - I : F(M) \to X.$$

From the identity

$$I = F \circ F^{-1} = (I - B) \circ (I + G) = (I + G) - B \circ (I + G),$$

we obtain that: $G = B \circ (I + G)$.
Hence

$$\begin{aligned}
\|Gx - Gy\| &= \|B(x) + B(G(x)) - B(y) - B(G(y))\| \\
&\leq \|B(x) - B(y)\| + \|B(G(x)) - B(G(y))\| \\
&\leq \alpha(\|x - y\| + \|Gx - Gy\|) \\
&\leq \frac{\alpha}{1-\alpha} \|x - y\| .
\end{aligned}$$

And so

$$\begin{aligned}
\left\| F^{-1}x - F^{-1}y \right\| &\leq \|Gx - Gy\| + \|x - y\| \\
&\leq \frac{1}{1-\alpha} \|x - y\| .
\end{aligned}$$

□

Lemma 3.4. *Let X be a Banach space. Assume that $B : X \to X$ is linear and bounded and B^p is a contraction for some $p \in \mathbb{N}$. Then $(I - B)$ maps X onto X, the inverse of $F := I - B : X \to X$ exists and*

$$\|F^{-1}x - F^{-1}y\| \leq \gamma_p \|x - y\| \; x, y \in X. \tag{3.6}$$

Where

$$\gamma_p = \begin{cases} \frac{p}{1-\|B^p\|}, & \text{if } \|B\| = 1, \\ \frac{1}{1-\|B\|}, & \text{if } \|B\| < 1, \\ \frac{\|B\|^p - 1}{(1-\|B^p\|)(\|B\|-1)}, & \text{if } \|B\| > 1. \end{cases}$$

Proof. Let $y \in X$ be fixed and define the map $B_y : X \to X$ by

$$B_y x = Bx + y.$$

We first show that B_y^p is a contraction. To this end, let $x_1, x_2 \in X$. Notice that B is linear. One has

$$\|B_y x_1 - B_y x_2\| = \|Bx_1 - Bx_2\|.$$

Again

$$\|B_y^2 x_1 - B_y^2 x_2\| = \|B^2 x_1 - B^2 x_2\|.$$

By induction,

$$\|B_y^p x_1 - B_y^p x_2\| = \|B^p x_1 - B^p x_2\| \leq \|B^p\| \|x_1 - x_2\|.$$

So B_y^p is a contraction on X. Next, we claim that both $(I - B)$ and $(I - B^p)$ map X onto X. Indeed, by Banach contraction mapping principle, there is a unique $x \in X$ such that $B_y^p x = x$. It then follows that $B_y x$ is also a fixed point of B_y^p. In view of uniqueness we obtain that $B_y x = x$ and x is the unique fixed point of B_y. Hence, we have

$$(I - B)x = y \in X,$$

which implies that $(I - B)$ maps X onto X. It is clear that $(I - B^p)$ maps X onto X. The claim is proved. Next, for each x,$y \in X$ and $x \neq y$, one easily obtain that

$$\|(I - B^p)x - (I - B^p)y\| \geq (1 - \|B^p\|)\|x - y\| > 0.$$

Which shows that $(I - B^p)$ is one-to-one. Summing the above arguments, we derive that $(I - B^p)^{-1}$ exists on X. Therefore, we infer that $(I - B)^{-1}$ exists on X due to the fact that

$$(I - B)^{-1} = (I - B^p)^{-1} \sum_{k=0}^{p-1} B^k. \tag{3.7}$$

Since B^p is a contraction, we know from (3.5) that

$$\|(I - B^p)^{-1}\| \leq \frac{1}{1 - \|B^p\|}. \tag{3.8}$$

We conclude from Lemma 3.3, (3.7) and (3.8) that

$$\|(I - B)^{-1}\| \leq \begin{cases} \frac{p}{1-\|B^p\|}, & \text{if } \|B\| = 1, \\ \frac{1}{1-\|B\|}, & \text{if } \|B\| < 1, \\ \frac{\|B\|^p - 1}{(1-\|B^p\|)(\|B\|-1)}, & \text{if } \|B\| > 1. \end{cases} \tag{3.9}$$

This proves the Lemma. □

Together Lemmas 1.9, 3.3, 3.4 and Theorem 3.6 immediately yield the following results.

Corollary 3.4. *Let X, A be the same as Theorem 3.6. Assume that $B : X \to X$ is a linear expansion with constant $h > 1$ such that $A(B_R) \subset B_{(h-1)R}$ for some $R > 0$ and $A(B_R) \subset (I - B)(X)$. Then fixed point for $B + A$ is achieved in B_R.*

Corollary 3.5. *Let X, A be the same as Theorem 3.6. Assume that $B : X \to X$ is a linear contraction with constant $\alpha < 1$ such that $A(B_R) \subset B_{(1-\alpha)R}$ for some $R > 0$. Then the equation $Bx + Ax = x$ has at least one solution in B_R.*

Corollary 3.6. *Let X, A be the same as Theorem 3.6. Assume that $B : X \to X$ is linear and bounded and B^p is a contraction for some $p \in \mathbb{N}$ such that $A(B_R) \subset B_{\gamma_p^{-1}R}$ for some $R > 0$, where γ_p is given in Lemma 3.4. Then the equation $Bx + Ax = x$ has at least one solution in B_R.*

Remark 3.4. Given by Lemma 3.4, it is easily verified that, under the conditions in Theorem 2.1 all the assumptions of Theorem 3.5 are fulfilled. Furthermore, when $B \in$

$\mathcal{L}(X)$ and $\|B^p\| \leq 1$ for some $p \geq 1$, instead of requiring $[x = Bx + Ay, y \in M] \Longrightarrow x \in M$, we assume the following condition holds in Theorem 3.5.

$$[\lambda \in (0,1) \text{ and } x = \lambda Bx + Ay, y \in M] \Longrightarrow x \in M.$$

Then Theorem 3.5 also covers the main result Theorem 2.2. However, it does not necessarily require that B is linear in Theorem 3.5.

Finally, inspired by the work of Barroso [5], we give the following asymptotic version of the Krasnoselskii fixed point theorem .

Theorem 3.7. *Let M, X, A, B and the conditions (ii), (iii) and (v) for A and B be the same as Theorem 3.5. In addition, assume that the following hypotheses are fulfilled.*

(a) A is demicontinuous ,

(b) B is sequentially weakly continuous and $B(0) = 0$.

Then there exists a sequence $\{x_n\}$ in M so that $(x_n - (A + B)x_n)_n$ converges weakly to zero.

Proof. Keeping the conditions (a) and (b) in mind , we claim that $(I - B)^{-1}A$ is demicontinuous in M. For each $y \in M$, there exists $x \in X$ such that

$$Bx + Ay = x. \tag{3.10}$$

By (ii) and the second part of (iii) we have, $x = (I - B)^{-1}Ay \in M$. Let $\{x\}_n$ be a sequence in M, with $x_n \to x$ in M; $(I - B)^{-1}Ax_n \in \mathbb{F}$ thus up to a subsequence we may assume by (v) that $(I - B)^{-1}Ax_n \rightharpoonup y$, for some $y \in M$. By hypothesis (a) and (b) we fined $B((I - B)^{-1}Ax_n) \rightharpoonup By$ and $Ax_n \rightharpoonup Ax$. By (3.10) we have

$$B((I - B)^{-1}Ax_n) + Ax_n = (I - B)^{-1}Ax_n. \tag{3.11}$$

Passing the weak limit in (3.11) we have $B((I - B)^{-1}Ax_n) \rightharpoonup y - Ax$, by uniqueness $By = y - Ax$ (i.e : $y = (I - B)^{-1}Ax$), which proved the claim. Let the set $C = \overline{co}(\mathbb{F})$, where $C = \overline{co}(\mathbb{F})$ denotes the closed convex hull of $\mathbb{F}$. Then $C \subset M$ and is a weakly compact set by Krein-Šmulian Theorem 1.10. Due to the Theorem 1.19 there is a sequence $\{x_n\}$ in C such that $x_n - (I - B)^{-1}Ax_n \rightharpoonup 0$, i.e., $(I - B)^{-1}[x_n - (A + B)x_n] \rightharpoonup 0$. Invoking again the item (b), one can readily deduce that $x_n - (A + B)x_n \rightharpoonup 0$. This ends the proof.□

3.3 Application to a non linear integral equation

In this section, our aim is to present some existence results for the following nonlinear integral equation

$$x(t) = f(x) + \int_0^t g(s, x(s))ds, \quad x \in C(J, X). \tag{3.12}$$

Where X is a reflexive Banach space and $J = [0, T]$. The integral in (3.12) is understood to be the Pettis integral. To study (3.12), we assume for the remained of this section the following hypotheses are satisfied:

(H_1) $f : X \to X$ is sequentially weakly continuous and onto,

(H_2) $\|f(x) - f(y)\| \geq h\|x - y\|$, $(h \geq 2)$ for all $x, y \in X$; and f maps relatively weakly compact sets into bounded sets and is uniformly continuous on weakly compact sets,

(H_3) for any $t \in J$, the map $g_t = g(t, \cdot) : X \to X$ is sequentially weakly continuous,

(H_4) for each $x \in C(J, X)$, $g(\cdot, x(\cdot))$ is Pettis integrable on $[0, T]$,

(H_5) there exist $\alpha \in L^1[0, T]$ and a nondecreasing continuous function ϕ from $[0, \infty)$ to $(0, \infty)$ such that $\|g(t, x)\| \leq \alpha(t)\phi(\|x\|)$ for a.e. $t \in [0, T]$ and all $x \in X$. Further, assume that

$$\int_0^T \alpha(s)ds < \int_{\|f(0)\|}^{\infty} \frac{dr}{\phi(r)}.$$

$$\beta(t) = \int_{\|f(0)\|}^{t} \frac{dr}{\phi(r)} \quad \text{and} \quad b(t) = (h-1)^{-1}\beta^{-1}\Big(\int_0^t \alpha(s)ds\Big).$$

We now state and prove an existence principle for (3.12).

Theorem 3.8. *Suppose that the conditions (H_1)-(H_5) are fulfilled. Then (3.12) has at least one solution $x \in C(J, X)$.*

Proof. We have

$$\int_{\|f(0)\|}^{(h-1)b(t)} \frac{dr}{\phi(r)} = \int_0^t \alpha(s)ds. \tag{3.13}$$

It follows from (3.13) and the final part of (H_5) that $b(T) < \infty$. We define the set

$$M = \{x \in C(J, X) : \|x(t)\| \le (h-1)b(t) \text{ for all } t \in J\}.$$

Then M is a closed and bounded subset of $C(J, X)$. Let us show that M is convex. Let $x, y \in M$ and $\lambda \in [0,1]$

$$\| \lambda x(t) + (1-\lambda)y(t) \| \le (h-1)b(t),$$

for all $t \in J$, which implies that $\lambda x + (1-\lambda)y \in M$, this shows that M is convex. Let us now introduce the nonlinear operators A and B as follows:

$$(Ay)(t) = f(0) + \int_0^t g(s, y(s))ds,$$
$$(Bx)(t) = f(x(t)) - f(0).$$

The conditions (H_1) and (H_4) imply that A and B are well defined on $C(J,X)$, respectively.

Our idea is to use Theorem 3.4 to find the fixed point for the sum $A + B$ in M. The proof will be shown in several steps.

Step 1: Prove that A maps M into M, $A(M)$ is equicontinuous and relatively weakly compact.

For any $y \in M$, we shall show that $Ay \in M$. Let $t \in J$ be fixed. Without loss of generality, we may assume that $(Ay)(t) \neq 0$. In view of the Hahn-Banach Theorem 1.7 there exists $y_t^* \in X^*$ with $\|y_t^*\| = 1$ such that $\langle y_t^*, (Ay)(t)\rangle = \|(Ay)(t)\|$. Thus, one can deduce from (H_5) and (3.13) that

$$\begin{aligned} \|(Ay)(t)\| &= \langle y_t^*, f(0)\rangle + \int_0^t \langle y_t^*, g(s, y(s))\rangle ds \\ &\le \|f(0)\| + \int_0^t \alpha(s)\phi(\|y(s)\|)ds \\ &\le \|f(0)\| + \int_0^t \alpha(s)\phi((h-1)b(s))ds \\ &= (h-1)b(t). \end{aligned} \tag{3.14}$$

It shows from (3.14) that $A(M) \subset M$ and hence is bounded. This proves the first claim of Step 1. Next, let $t, s \in J$ with $s \neq t$. We may assume that $(Ay)(t) - (Ay)(s) \neq 0$. Then there exists $x_t^* \in X^*$ with $\|x_t^*\| = 1$ and $\langle x_t^*, (Ay)(t) - (Ay)(s)\rangle = \|(Ay)(t) - (Ay)(s)\|$.

Consequently,

$$\begin{aligned}\|(Ay)(t)-(Ay)(s)\| &\leq \int_s^t \alpha(\tau)\phi(\|y(\tau)\|)d\tau \\ &\leq \int_s^t \alpha(\tau)\phi((h-1)b(\tau))d\tau \\ &\leq (h-1)\Big|\int_s^t b'(\tau)d\tau\Big| = (h-1)|b(t)-b(s)|.\end{aligned} \tag{3.15}$$

It follows from (3.15) that $A(M)$ is equicontinuous. The reflexiveness of X implies that $A(M)(t)$ is relatively weakly compact for each $t \in J$, where $A(M)(t) = \{z(t) : z \in A(M)\}$. It follows now from the Ascoli-Arzela Theorem that $A(M)$ is relatively weakly compact in $C(J, X)$, This completes Step 1.

Step 2: Prove that $A : M \to M$ is sequentially weakly continuous. Let $\{x_n\}$ be a sequence in M with $x_n \rightharpoonup x$ in $C(J, X)$, for some $x \in M$. Then $x_n(t) \rightharpoonup x(t)$ in X for all $t \in J$. Fix $t \in (0, T]$. From the item (H_3) one sees that $g(t, x_n(t)) \rightharpoonup g(t, x(t))$ in X. Together with (H_5) and the Lebesgue dominated convergence Theorem for the Pettis integral yield for each $\varphi \in X^*$ that

$$\langle \varphi, (Ax_n)(t)\rangle \to \langle \varphi, (Ax)(t)\rangle,$$

i.e., $(Ax_n)(t) \rightharpoonup (Ax)(t)$ in X. We can do this for each $t \in J$ and notice that $A(M)$ is equicontinuous, hence, by the Ascoli-Arzela Theorem there exists a subsequence (x_{n_j}) of (x_n) such that $Ax_{n_j} \rightharpoonup y$ for some $y \in C(I, X)$. Consequently, we have that $y(t) = Ax(t)$ for all $t \in I$ and hence $Ax_{n_j} \rightharpoonup Ax$. Now, a standard argument shows that $Ax_n \rightharpoonup Ax$. The Step 2 is proved.

Step 3: Prove that the conditions (ii) and (iii) of Theorem 3.4 hold. Since X is reflexive and f is continuous on weakly compact sets, it shows that B transforms $C(J,X)$ into itself. This, in conjunction with the first part of (H_2), one easily gets that $B : C(J, X) \to C(J, X)$ is expansive with constant $h \geq 2$. For all $x, y \in C(J, X)$, one can see from the first part of (H_2) that

$$\|(I-B)x(t)-(I-B)y(t)\| \geq (h-1)\|x(t)-y(t)\| \geq \|x(t)-y(t)\|,$$

where I is identity map. Thus, one has

$$\|(I-B)x(t)\| \geq (h-1)\|x(t)\| \geq \|x(t)\|, \quad \forall x \in C(J,X). \tag{3.16}$$

Assume now that $x = Bx + Ay$ for some $y \in M$. We conclude from (3.14) and (3.16) that

$$\|x(t)\| \leq \|(I-B)x(t)\| = \|(Ay)(t)\| \leq (h-1)b(t),$$

which shows that $x \in M$. Therefore, the second part of (iii) in Theorem 3.4 is fulfilled. Next, for each $y \in C(J,X)$, we define $B_y : C(J,X) \to C(J,X)$ by

$$(B_y x)(t) = (Bx)(t) + y(t).$$

Then B_y is expansive with constant $h \geq 2$ and onto since f maps X onto X. By Lemma 3.1, we know there exists $x^* \in C(J,X)$ such that $B_y x^* = x^*$, that is $(I-B)x^* = y$. Hence $A(M) \subset (I-B)(X)$. This completes Step 3.

Step 4: Prove that the condition (v) of Theorem 3.4 is satisfied. For each $x \in \mathbb{F}(X, M; B, A)$, then by the definition of $\mathbb{F}$ and Lemma 1.9 there exists $y \in M$ such that

$$x = (I-B)^{-1}Ay. \tag{3.17}$$

Hence, for $t, s \in J$, we obtain from Lemma 1.9, (3.17) and (3.15) that

$$\begin{aligned}\|x(t) - x(s)\| &\leq \| (I-B)^{-1}Ay(t) - (I-B)^{-1}Ay(s) \| \\ &\leq |b(t) - b(s)|,\end{aligned}$$

which illustrates that $\mathbb{F}(X, M; B, A)$ is equicontinuous in $C(J,X)$. Let $\{x_n\}$ be a sequence in $\mathbb{F}$. Then $\{x_n\}$ is equicontinuous in $C(J,X)$ and there exists $\{y_n\}$ in M with $x_n = Bx_n + Ay_n$. Thus, one has from (3.14) and (3.16) that

$$\|x_n(t)\| \leq \frac{1}{h-1}\|(Ay_n)(t)\| \leq b(t), \forall t \in J.$$

It follows that, for each $t \in J$, the set $\{x_n(t)\}$ is relatively weakly compact in X. The above discussion tells us that $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact. The Eberlein-Šmulian Theorem 1.8 implies that $\mathbb{F}$ is relatively weakly compact. This achieves Step 4

Step 5: Prove that B fulfills the condition (iv) of Theorem 3.4. By the second part of (H_2) and the fact that $\mathbb{F}$ is relatively weakly compact we obtain that $B(\mathbb{F})$ is bounded. Again by the second part of (H_2) and the fact that $\mathbb{F}$ is equicontinuous, one can readily deduce that

$B(\mathbb{F})$ is also equicontinuous. Now, let $\{x_n\} \subset \mathbb{F}$ with $x_n \rightharpoonup x$ in $C(J, X)$ for some $x \in M$. It follows from (H_1) that $(Bx_n)(t) \rightharpoonup (Bx)(t)$. Since $\{Bx_n : n \in \mathbb{N}\}$ is equicontinuous in $C(J, X)$, we may apply the Ascoli-Arzela Theorem 1.6 and concludes that there exists a subsequence (x_{n_j}) of (x_n) such that $Bx_{n_j} \rightharpoonup y$, for some $y \in C(J, X)$. Hence, $Bx = y$ and by standard arguments we have $Bx_n \rightharpoonup Bx$ in $C(J, X)$. This completes Step 5.

Now, invoking Theorem 3.4 we obtain that there is $x^* \in M$ with $Bx^* + Ax^* = x^*$. i.e., x^* is a solution to (3.12). This accomplishes the proof. □

Chapter 4

Fixed Point Theory In Banach Algebras

In recent years, many authors have focused on the resolution of integral equations and differential equations in Banach algebras satisfying certain sequential conditions, and obtained a lot of valuable results (see [7], [2]). Some of this equation the following nonlinear operator equation,

$$AxBx = x, \quad x \in M, \tag{4.1}$$

and,

$$AxBx + Cx = x, \quad x \in M. \tag{4.2}$$

Where M is a closed, bounded and convex subset of a Banach algebra, A, B and C are three operators defined on M.

The study of the nonlinear integral equations in Banach algebras was initiated by Dhage [10]. These studies were mainly based on the convexity of the bounded domain, and properties of the operators A, B and C (weakly continuous, contractive, $\mathcal{D}$-Lipschitzian).

In this chapter we extend a new version for Some nonlinear problems involve the study of solutions of nonlinear operator equations of the form (4.1) and (4.2) under weak topology. We start with give some definitions and proprieties and preliminary results which are useful for our analysis.

Definition 4.1. An algebra X is a vector space endowed with an internal composition law denoted by (.), i.e.,

$$\begin{cases} (\cdot) : X \times X & \to X \\ \qquad (x, y) & \to x.y \end{cases}$$

which is associative and bilinear. A normed algebra is an algebra endowed with a norm satisfying the following property

$$\|xy\| \leq \|x\|\|y\| \text{ for all } x, y \in X.$$

A complete normed algebra is called a Banach algebra.

Definition 4.2. We will say that a Banach algebra X satisfies condition $(\mathcal{P})$ if

$$(\mathcal{P}) \begin{cases} \text{For any sequences } (x_n)_n \text{ and } (y_n)_n \text{ in } X \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\ \text{we have } x_n y_n \rightharpoonup xy. \end{cases}$$

Proposition 4.1. *If X satisfies condition $(\mathcal{P})$ then $C(M, X)$ is also a Banach algebra satisfying condition $(\mathcal{P})$, where M is a compact Hausdorff space.*

The proof is based on Dobrokov's theorem:

Theorem 4.1. *[11] Let M be a compact Hausdorff space and X be a Banach space. Let $(f_n)_n$ be a bounded sequence in $C(M, X)$ and $f \in C(M, X)$. Then $(f_n)_n$ is weakly convergent to f if and only if $(f_n(t))_n$ is weakly convergent to $f(t)$ for each $t \in M$.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in $C(M, X)$, such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$. So, for each $t \in M$, we have $x_n(t) \rightharpoonup x(t)$ and $y_n(t) \rightharpoonup y(t)$. Since X satisfies the condition $(\mathcal{P})$, then

$$x_n(t)y_n(t) \rightharpoonup x(t)y(t),$$

because $(x_n y_n)_n$ is a bounded sequence. Moreover, this implies with Theorem 4.1, that

$$x_n y_n \rightharpoonup xy,$$

Which shows that the space $C(M, X)$ verifies condition $(\mathcal{P})$. □

4.1 Fixed point theorems in Banach algebras satisfying the condition $(\mathcal{P})$

Now, we introduce a class of Banach algebras satisfying certain sequential conditions called here the condition $(\mathcal{P})$. The main goal of this section is to prove some new fixed point theorems in a nonempty closed convex subset of any Banach algebras or Banach algebras satisfying the condition $(\mathcal{P})$ under weak topology setting. Our main conditions are formulated in term of weak sequential continuity to the three nonlinear operators A, B and C involved in equation (4.1), (4.2).

Theorem 4.2. *X be a Banach algebras satisfies condition $(\mathcal{P})$. Let M be a closed, convex subset of X. Assume that $A, B : M \to X$ satisfies:*

(i) A is sequentially weakly continuous,

(ii) B is λ-contraction,

(iii) If $x = BxAy$ for some $y \in M$, then $x \in M$,

(iv) If x_n is a sequence in $\mathbb{F}$ where

$$\mathbb{F} := \{x \in X : x = BxAy \quad \text{for some} \quad y \in M\}.$$

Such that $x_n \rightharpoonup x$, for some x in M, then $Bx_n \rightharpoonup Bx$,

(v) The set $\mathbb{F}$ is relatively weakly compact.

Then, the operator equation (4.1) has a solution, whenever $\lambda K < 1$, where

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. Let $y \in M$ we consider the operator

$$\begin{cases} L_y : M & \to M \\ \qquad y & \to L_y(x) = BxAy. \end{cases}$$

Let x_1, $x_2 \in M$, by (ii) we have $\| L_y(x_1) - L_y(x_2) \| \leq \lambda K \| x_1 - x_2 \|$. By Banach fixed point Theorem there exists unique point Ty in M such that $Ty = BTyAy$. By assumption (iii), the mapping $T : M \longrightarrow M$ given by $y \longmapsto Ty$ is well-defined. We observe that $T(M) \subset \mathbb{F}$, we claim now that T is sequentially weakly continuous in M. Indeed, let y_n be a sequence in M such that $y_n \rightharpoonup y$ in M. Since $Ty_n \subset \mathbb{F}$, thus up to a subsequence we may assume by (v), that $Ty_n \rightharpoonup x$, for some $x \in M$. By (iv), we have $BTy_n \rightharpoonup Bx$. Also, from (i) it follows that $Ay_n \rightharpoonup Ay$. Due to property $\mathcal{P}$ of space X and the equality $Ty_n = BTy_nAy_n$ we obtain $x = BxAy$. By uniqueness, we conclude that $x = Ty$. This proves the claim. Take now the subset $C = \overline{co}(\mathbb{F}) \subset M$. Krein-Šmulian Theorem 1.10 implies that C is a weakly compact set. Furthermore, it is easy to see that $T(C) \subset C$. Applying Lemma 1.10, we find a fixed point $x \in C$ for T, hence $AxBx = x$ this proves Theorem . □

Corollary 4.1. *X be a Banach algebras satisfies condition $(\mathcal{P})$. Let M be a convex and weakly compact subset of X and let $A, B : M \to X$ be sequentially weakly continuous operators such that*

(i) B is nonexpansive,

(ii) If $\lambda \in (0,1)$ and $x = \lambda BxAy$ with $y \in M$, then $x \in M$.

Then the operator equation (4.1) has a solution.whenever $K < 1$, where

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. For $0 < \lambda < 1$ define $B_\lambda = \lambda B$. Let x_1, $x_2 \in X$,

$$\|B_\lambda x_1 - B_\lambda x_2\| \leq K \|x_1 - x_2\| .$$

So B_λ is contraction. By Theorem 4.2, we find a family $y_\lambda \subset M$ such that $\lambda By_\lambda Ay_\lambda = y_\lambda$ for all $\lambda \in (0,1)$. Taking now a sequence $0 < \lambda_n < 1$ so that $\lambda_n \to 1$, and considering the respective sequence $y_n \subset M$ satisfying

$$\lambda_n By_nAy_n = y_n \quad \text{for all} \quad n \in N. \tag{4.3}$$

As M is weakly compact, we can find a subsequence y_{n_i} such that $y_{n_i} \rightharpoonup y$ in X with $y \in M$. A and B are sequentially weakly continuous so, the property $\mathcal{P}$ of space X and the equation (4.3) give us $ByAy = y$. □

Theorem 4.3. *X be a Banach algebras satisfies condition $(\mathcal{P})$. Let M be a closed, convex subset of X. Assume that $A, B : M \to X$ satisfies the conditions of Theorem 4.2. If only replaced the condition (ii) by that $\left(\frac{I}{B}\right)^{-1}$ exists on $A(M)$.*

Then, the operator equation (4.1) has a solution whenever $\lambda K < 1$, where

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. Let $y \in M$ there is unique point $Ty \in X$ such that $Ty = BTyAy$. By (ii) and (iii) the mapping

$$\begin{cases} T : M & \to M \\ \quad y & \to Ty = \left(\frac{I}{B}\right)^{-1} Ay \end{cases}$$

is well defined. We claim that the operator $y \longmapsto \left(\frac{I}{B}\right)^{-1} y$ is continuous. To see this, Let (x_n) be a sequence converging to $x \in M$. We set $y_n = \left(\frac{I}{B}\right)^{-1} x_n$ and $y = \left(\frac{I}{B}\right)^{-1} x$ Then

$$\begin{cases} y_n & = By_n x_n \\ y & = Byx \end{cases}$$

Hence

$$\begin{aligned} \|y_n - y\| &= \|By_n x_n - Byx\| \\ &\leq \|By_n - By\| \, \|x_n\| + \|By\| \, \|x_n - x\| \\ &\leq \alpha \, \|x_n\| \, \|y_n - y\| + \|By\| \, \|x_n - x\| \\ &\leq \frac{\|By\|}{(1 - \alpha \parallel x_n \parallel)} \|x_n - x\| \end{aligned}$$

Therefore $\|y_n - y\| \longrightarrow 0$ as $n \longrightarrow +\infty$. This proves the claim. Since T is a composition of a continuous and a sequentially weakly continuous operator, then it is a sequentially weakly continuous operator on M. By Lemma 1.10 the operator $T : C \to C$ has a fixed point where C is defined in the proof of Theorem 4.2. This complete the proof. □

Theorem 4.4. *Let $M \subset X$ be a nonempty closed convex subset. Suppose that $A : M \to X$ and $B : X \to X$ such that*

(i) A is sequentially weakly continuous,

(ii) $\left(\frac{I}{B}\right)$ *is continuously invertible,*

(iii) $A(M) \subset \left(\frac{I}{B}\right)(X)$ *and* $[x = BxAy, y \in M] \Longrightarrow x \in M$*,*

(iv) The set $\mathbb{F}(X, M; B,A)$ *is relatively weakly compact.*

Then the operator equation (4.1) has a solution.

Proof. Let $y \in M$, by (iii) there exists $x \in X$ such that $\left(\frac{I}{B}\right) x = Ay$. By (ii) and the second part of (iii) we have $x = \left(\frac{I}{B}\right)^{-1} Ay \in M$. Let T the operator defined in the proof of Theorem 4.3. By (i) and (ii) T is a composition of a continuous and a sequentially weakly continuous operator, then it is a sequentially weakly continuous. By Lemma 1.10 the operator $T : C \to C$ has a fixed point where C is defined in the proof of Theorem 4.2. This complete the proof. □

Theorem 4.5. *Let* X *be a reflexive Banach algebra,* $A : X \to X$ *and* $B : X \to X$*. Assume that the following conditions are satisfied*

(i) A *is sequentially weakly continuous ,*

(ii) $\left(\frac{I}{B}\right)$ *is continuously invertible,*

(iii) There exists $R > 0$ *such that* $A(B_R) \subset B_{\beta R}$ *, where* $\beta \leq \left\| \left(\frac{I}{B}\right)^{-1} \right\|^{-1}$ *,*

(iv) $A(B_R) \subset \left(\frac{I}{B}\right)(X)$*.*

Then the operator equation (4.1) has a solution.

Proof. Let $y \in B_R$, by (ii) and (iv) there exists $x \in X$ such that $x = \left(\frac{I}{B}\right)^{-1} Ay$. We consider

$$\begin{cases} T : B_R & \to X \\ \quad\quad y & \to Ty = \left(\frac{I}{B}\right)^{-1} Ay. \end{cases}$$

By (i) and (ii) T is sequentially weakly continuous. For any $x \in B_R$ by (iii) we have $\|T(x)\| \leq R$. Hence T maps B_R into itself. By Lemma 1.2 and Lemma 1.10 we obtain that there exists $x \in B_R$ such that $Tx = x$. So $AxBx = x$. □

Theorem 4.6. *Let X be a reflexive Banach algebras satisfies condition $\mathcal{P}$. M be a bounded, closed convex subset of X. Assume that A, $B : M \longrightarrow X$, satisfies*

(i) A is weakly continuous ,

(ii) B is λ-contraction ,

(iii) If $x = BxAy, y \in M \Longrightarrow x \in M$.

Then, the operator equation (4.1) has a solution whenever $\lambda K < 1$, where

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. Let $y \in M$, let $L_y : M \to M$ defined by $L_y(x) = BxAy$, $x \in M$. Let x_1, $x_2 \in M$

$$\| L_y(x_1) - L_y(x_2) \| \leq \lambda K \| x_1 - x_2 \| .$$

By Banach contraction principal there exists unique $x(y) \in M$ such that $x(y) = Bx(y)Ay$. We consider $L : M \to M$ defined by $L(y) = x(y)$. Now we assume that L is weakly continuous. To see this let F in M be weakly closed, by (i) and (ii) we have $A^{-1}(F)$ and $B^{-1}(F)$ are weakly closed, so $L^{-1}(F)$ is. Hence L is weakly continuous. By Theorem 1.16 there exists $y \in M$ such that $L(y) = y$. So $ByAy = y$. □

4.2 Fixed point theorems involving three operators

Now, we are ready to state our first fixed point theorems in Banach algebras to provide the existence results of equation (4.2).

Theorem 4.7. *[6, Theorem 3.1] Let X be a Banach algebra and M be a nonempty closed convex subset of X. Let $A : M \to X$ and B, $C : X \to X$ three operators such that*

(i) $\left(\frac{I-C}{B}\right)^{-1}$ exists on $A(M)$,

(ii) $\left(\frac{I-C}{B}\right)^{-1} A$ is sequentially weakly continuous,

(iii) $\left(\frac{I-C}{B}\right)^{-1} A(M)$ *is relatively weakly compact,*

(iv) $x = BxAy + Cx \Longrightarrow x \in M$, *for all* $y \in M$.

Then the equation (4.2) *has at least one solution in* M.

Proof. From assumption (i), it follows that for each $y \in M$, there is a unique $x_y \in X$ such that $\left(\frac{I-C}{B}\right) x_y = Ay$ or, equivalently $Bx_yAy + Cx_y = x_y$. Since the hypothesis (iv) holds, then $x_y \in M$. Therefore, we can defined

$$\begin{cases} T : M & \to M \\ \quad y & \to Ty = \left(\frac{I-C}{B}\right)^{-1} Ay. \end{cases}.$$

By using the hypothesis (ii), (iii) and Theorem 1.18, we conclude that T has a fixed point $y \in M$. Hence, y verifies equation (4.2). □

Theorem 4.8. *Let* M *be a closed, convex, and bounded subset of a Banach algebra* X *and let* $A, B, C : M \to M$ *be three operators such that:*

(i) B *and* C *are Lipschitzian with Lipschitz constants* α *and* β *respectively,*

(ii) A *is weakly continuous,*

(iii) $A(M)$ *is a relatively weakly compact subset of* X,

(iv) $BxAy + Cx \in M \quad \forall x, y \in M$.

Then, the operator equation (4.2) *has a solution whenever* $\alpha K + \beta < 1$, *where*

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. Let $y \in M$ and defined the mapping

$$\begin{cases} L_y : M & \to M \\ \quad x & \to L_y(x) = BxAy + Cx. \end{cases}$$

Let x_1, $x_2 \in M$, the use of assumption (i) leads to

$$\begin{aligned}\| L_y(x_1) - L_y(x_2) \| &\leq \| Bx_1Ay - Bx_2Ay \| + \| Cx_1 - Cx_2 \| \\ &\leq \| Bx_1 - Bx_2 \| \| Ay \| + \| Cx_1 - Cx_2 \| \\ &\leq (\alpha K + \beta) \| x_1 - x_2 \| .\end{aligned}$$

By Banach fixed point Theorem there is a unique $x_y \in M$ such that $L_y(x_y) = x_y$, so x_y verifies $Bx_yAy + Cx_y = x_y$, so the operator $x \longmapsto \left(\frac{I-C}{B}\right)^{-1} x$ is well defined and,

$$x_y = \left(\frac{I-C}{B}\right)^{-1} Ay.$$

Let $L : M \to M$ be defined by $L(y) = x_y$. We claim that the operator $y \longmapsto \left(\frac{I-C}{B}\right)^{-1}(y)$ is continuous. To see this, Let (x_n) be a sequence converging to $x \in M$. We set $y_n = \left(\frac{I-C}{B}\right)^{-1} x_n$ and $y = \left(\frac{I-C}{B}\right)^{-1} x$
Then

$$\begin{cases} y_n &= By_nx_n + Cy_n \\ y &= Byx + Cy \end{cases}$$

Hence

$$\begin{aligned}\|y_n - y\| &= \|By_nx_n + Cy_n - Byx - Cy\| \\ &\leq \|By_n - By\| \|x_n\| + \|By\| \|x_n - x\| + \|Cy_n - Cy\| \\ &\leq \alpha \|x_n\| \|y_n - y\| + \|By\| \|x_n - x\| + \beta \|y_n - y\| \\ &\leq \frac{\|By\|}{(1 - \alpha \| x_n \| - \beta)} \|x_n - x\|\end{aligned}$$

Therefore $\|y_n - y\| \longrightarrow 0$ as $n \longrightarrow +\infty$. This proves the claim. Since L is a composition of a continuous and a weakly continuous operator, then it's a weakly continuous operator on M. By virtue of the hypothesis (iv) $L(M)$ is relatively weakly compact, by Theorem 1.17 there exists $x \in M$ which is fixed point of L. This complete the proof. □

Theorem 4.9. *X be a Banach algebras satisfies condition $(\mathcal{P})$. Let M be a closed, convex subset of X. Assume that $A, B, C : M \longrightarrow X$ satisfies:*

(i) B and C are contraction with constants α and β respectively ,

(ii) A *and* C *are sequentially weakly continuous,*

(iii) *If* $x = BxAy + Cx$ *for some* $y \in M$ *, then* $x \in M$*,*

(iv) *If* x_n *is a sequence in* $\mathbb{F}$ *where*

$$\mathbb{F} := \{x \in X : x = BxAy + Cx \quad \text{for some} \quad y \in M\}.$$

Such that $x_n \rightharpoonup x$*, for some* x *in* M*, then* $Bx_n \rightharpoonup Bx$*,*

(v) *The set* $\mathbb{F}$ *is relatively weakly compact.*

Then the operator equation (4.2) has a solution, whenever $\alpha K + \beta < 1$*, where*

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. Let $y \in M$ there exists unique point $x(y) \in M$ such that $x(y) = Bx(y)Ay + Cx(y)$. We consider the operator $L : M \to M$ defined by $L(y) = x(y)$. Let y_n be sequence in M such that $y_n \rightharpoonup y$, $y \in M$. We have $L(M) \in \mathbb{F}$, thus up to a subsequence we may assume by (v) that $Ly_n \rightharpoonup x$, $x \in M$. By (ii), (iv) and the property $(\mathcal{P})$ of space X we have,

$$Bx(y_n)Ay_n + Cx(y_n) \rightharpoonup Bx(y)Ay + Cx(y),$$

So, $x = Bx(y)Ay + Cx(y)$ in view of uniqueness we have $x = x(y)$. This proves that L is sequentially weakly continuous. By Lemma 1.10 the operator $L : C \to C$ has a fixed point where C defined in proof of Theorem 4.2. □

Theorem 4.10. *Let* M *be a closed, convex subset of* X*. Assume that* $A, B, C : M \longrightarrow X$ *three operator satisfies:*

(i) B *and* C *are contraction with constants* α *and* β *respectively ,*

(ii) A *is sequentially weakly continuous,*

(iii) *If* $x = BxAy + Cx$ *, for some* $v \in M$ *, then* $y \in M$*,*

(iv) *The set* $\mathbb{F}$ *is relatively weakly compact. Where*

$$\mathbb{F} := \{x \in X : x = BxAy + Cx \quad \text{for some} \quad y \in M\}.$$

Then the operator equation (4.2) has a solution whenever $\alpha K + \beta < 1$, where

$$K = \| A(M) \| = sup\{\| Ax \| : x \in M\}.$$

Proof. Let $y \in M$ by assumption (i), the mapping $L_y : M \to X$, defined by $L_y(x) = BxAy + Cx$ is contraction so by Banach fixed point Theorem there exists $x_y \in M$ such that $x_y = Bx_yAy + Cx_y$ then,

$$x_y = \left(\frac{I-C}{B}\right)^{-1} Ay.$$

We consider $L : M \to M$ by $L(y) = \left(\frac{I-C}{B}\right)^{-1} Ay$. As is shown in Theorem 4.8, one obtains that $\left(\frac{I-C}{B}\right)^{-1}$ is continuous. So, L is sequentially weakly continuous operator. We complete the proof white the same proof of Theorem 4.9. □

Theorem 4.11. *Let $M \subset X$ be a nonempty closed convex subset. Suppose that $A : M \to X$ and $B, C : X \to X$ such that*

(i) A is sequentially weakly continuous,

(ii) $\left(\frac{I-C}{B}\right)$ is continuously invertible,

(iii) $A(M) \subset \left(\frac{I-C}{B}\right)(X)$ and $[x = BxAy + Cx, y \subset M] \Longrightarrow x \in M$,

(iv) The set $\mathbb{F}$ is relatively weakly compact. Where

$$\mathbb{F} := \{x \in X : x = BxAy + Cx \quad \text{for some} \quad y \in M\}.$$

Then the operator equation (4.2) has a solution.

Proof. For each $y \in M$, by (iii), there exists $x \in X$ such that $\left(\frac{I-C}{B}\right)x = Ay$. By (ii) and the second part of (iii), $x = \left(\frac{I-C}{B}\right)^{-1} Ay \in M$. By conditions (i) and (ii) we have that the operator $y \longmapsto \left(\frac{I-C}{B}\right)^{-1} Ay$ is sequentially weakly continuous. We complete proof withe the same proof of Theorem 4.9. □

Theorem 4.12. *Let X be a reflexive Banach algebra, A, B, $C : X \to X$. Assume that the following conditions are satisfied*

(i) *A is sequentially weakly continuous ,*

(ii) $\left(\dfrac{I-C}{B}\right)$ *is continuously invertible,*

(iii) *There exists $R > 0$ such that $A(B_R) \subset B_{\beta R}$, where* $\beta \leq \left\| \left(\dfrac{I-C}{B}\right)^{-1} \right\|^{-1}$,

(iv) $A(B_R) \subset \left(\dfrac{I-C}{B}\right)(X)$.

Then the operator equation (4.2) has a solution .

4.3 Fixed point theorems for $\mathcal{D}$-Lipschitzian mappings

Definition 4.3. A mapping $F : X \to X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_F : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\| Fx - Fy \| \leq \phi_F(\| x - y \|), \tag{4.4}$$

for all $x, y \in X$ with $\phi_F(0) = 0$. Sometimes we call the function ϕ_F a $\mathcal{D}$-function of F on X. If $\phi_F(r) = \alpha r$ for some constant $\alpha > 0$, then F is called a Lipschitzian with a Lipschitz constant α and further if $\alpha < 1$, then F is called a contraction with the contraction constant α. Again if ϕ_F satisfies $\phi_F(r) < r, r > 0$, then F is called a nonlinear $\mathcal{D}$-contraction on X.

Example 4.1. $f(x) = \sqrt{|x|}, x \in \mathbb{R}$ and consider $\phi(r) = \sqrt{r}, r \geq 0$. Clearly, ϕ is continuous and nondecreasing. First notice that f is subadditive. To see this, let $x, y \in \mathbb{R}$. Then,

$$\begin{aligned}(f(x+y))^2 = |x+y| &\leq |x|+|y| \\ &\leq \left(\sqrt{|x|} + \sqrt{|y|}\right)^2 \\ &\leq (f(x) + f(y))^2.\end{aligned}$$

Thus, for all $x, y \in \mathbb{R}$ we have :

$$f(x+y) \leq f(x) + f(y).$$

Using the subadditivity of f we get

$$|f(x) - f(y)| \leq f(x-y) = \phi(|x-y|),$$

for all $x, y \in \mathbb{R}$. Thus, f is $\mathcal{D}$- Lipschitzian with $\mathcal{D}$-function ϕ.

Remark 4.1. Obviously, every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian. The converse may not be true.

In example 4.1 f is $\mathcal{D}$-Lipschitzian but is not Lipschitzian. Indeed, suppose that f is Lipschitzian with constant k. Then, for all $x \in \mathbb{R}$ we have $f(x) \leq k|x|$. Hence, for all $x \neq 0$ we have $k \geq \dfrac{1}{\sqrt{|x|}}$. Letting x go to zero we obtain a contradiction. Consequently, f is not Lipschitzian.

Lemma 4.1. *[2] Let M be a nonempty bounded closed subset of a Banach algebra X and let $B, C : X \to X$ be $\mathcal{D}$-Lipschitzian mappings with $\mathcal{D}$- functions ϕ_B and ϕ_C respectively. Assume that for each $r > 0$ we have $\|M\|\phi_B(r) + \phi_C(r) < r$. Then $\left(\dfrac{I-C}{B}\right)^{-1}$ exists and is continuous.*

Proof. Let $y \in M$ be fixed. The map τy which assigns to each $x \in X$ the value $B(x).y + C(x)$ defines a nonlinear contraction with a contraction function $\psi(r)$, such that $\psi(r) = \| M \| \phi_B(r) + \phi_C(r)$, $r > 0$. Indeed, for all $x_1, x_2 \in X$ we have:

$$\begin{aligned} \| \tau y(x_1) - \tau y(x_2) \| &\leq \| Bx_1 - Bx_2 \| \| y \| + \| Cx_1 - Cx_2 \| \\ &\leq \| M \| \phi_B(\| x_1 - x_2 \|) + \phi_C(\| x_1 - x_2 \|). \end{aligned}$$

Now the Boyd and Wong fixed point Theorem 1.15 guarantees that there exists a unique point $x^* \in X$ such that $\tau y(x^*) = x^*$, i.e. $y = \left(\dfrac{I-C}{B}\right) x^*$. Thus, the operator

$$N := \left(\frac{I-C}{B}\right)^{-1} : M \to X,$$

is well defined. Now we show that N is continuous. To see this, let (x_n) be a sequence in M converging to a point x. Since M is closed, then $x \in M$. First notice that for each $z \in M$ we have

$$Nz = CNz + (BNz)z. \tag{4.5}$$

Hence,

$$\begin{aligned}\| Nx_n - Nx \| &\leq \| CNx_n - CNx \| + \| (BNx_n)x_n - (BNx)x \| \\ &\leq \| CNx_n - CNx \| + \| BNx \| \| x_n - x \| + \| BNx_n - BNx \| \| x_n \| \\ &\leq \phi_C(\| Nx_n - Nx \|) + \phi_B(\| Nx_n - Nx \|) \| M \| + \| BNx \| \| x_n - x \| .\end{aligned}$$

Thus,

$$\limsup_n \| Nx_n - Nx \| \leq \phi_C(\limsup_n \| Nx_n - Nx \|) + \phi_A(\limsup_n \| Nx_n - Nx \|) \| M \| .$$

This shows that $\lim_n \| Nx_n - Nx \| = 0$ and consequently N is continuous on M. This completes the proof. □

Theorem 4.13. *Let M be a nonempty, closed, convex and bounded subset of a Banach algebra X, and let $A : M \to X$ and $B, C : X \to X$, be three operators such that:*

(i) B and C are $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-functions ϕ_B and ϕ_C respectively,

(ii) A is sequentially weakly continuous and $A(M)$ is relatively weakly compact,

(iii) $x = BxAy + Cx \Longrightarrow x \in M$, for all $y \in M$.

Then the operator equation (4.2) has a solution whenever $K\phi_B(r) + \phi_C < r$, for $r > 0$ and is strictly increasing, where $K = \| A(M) \|$.

Proof. In view of Lemma 4.1 the operator $\tau := \left(\frac{I-C}{B}\right)^{-1} A$ is a well defined map from M into X. Notice also by assumption (iii) we have $\tau(M) \subset M$. By (ii) and Lemma 4.1, τ is composition of continuous and sequentially weakly continuous operators so τ is sequentially weakly continuous. We claim now that $\left(\frac{I-C}{B}\right)^{-1} A(M)$ is relatively weakly compact. To see this, let x_n be a sequence in M, and let

$$y_n = \left(\frac{I-C}{B}\right)^{-1} A(x_n).$$

Since $A(M)$ is relatively weakly compact, we deduce that there is a subsequence Ax_n weakly converging to an element z. And $y_n = \left(\frac{I-C}{B}\right)^{-1} Ax_n \rightharpoonup \left(\frac{I-C}{B}\right)^{-1} z$ We infer that $\left(\frac{I-C}{B}\right)^{-1} A(M)$ is sequentially relatively weakly compact. An application of Eberlein-Šmulian's Theorem 1.8 implies that $\left(\frac{I-C}{B}\right)^{-1} A(M)$ is relatively weakly compact, which proves our claim. By Theorem 1.18, N has a fixed point in M . □

Remark 4.2. The next result is consequence of Theorem 4.13.

Theorem 4.14. *Let M be a closed, convex and bounded subset of a Banach algebra X and let $A : M \to X$, $B : X \to X$ be two operators such that :*

(i) B is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function ϕ ,

(ii) A is sequentially weakly continuous, and $A(M)$ relatively weakly compact,

(iv) $x = BxAy \Longrightarrow x \in M$ for all $y \in M$.

Then the operator equation (4.1) has a solution, whenever $K\phi(r) < r$, $r > 0$, where $K = \|A(M)\|$.

The following proposition will be used throughout in this section .

Proposition 4.2. *[6] Let X be a Banach algebra and M be a nonempty closed convex subset of X. Let $A : M \to X$ and $B, C : X \to X$ be three operators such that*

(i) B and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_B and ϕ_C respectively,

(ii) B is regular on X, i.e., B maps X into the set of all invertible elements of X,

(iii) A is a bounded function with bound K. Then $\left(\frac{I-C}{B}\right)^{-1}$ exists on $A(M)$ as soon as $K\phi_B(r) + \phi_C(r) < r$, for $r > 0$.

Theorem 4.15. *[2] Let X be a Banach algebra and M be a nonempty closed convex subset of X. Let $A : M \to X$ and $B, C : X \to X$ be three operators such that*

(i) B and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_B and ϕ_C respectively,

(ii) B *is regular on* X*,*

(iii) A *is strongly continuous,*

(iv) $A(M)$ *is bounded with bound* K*,*

(v) $\left(\frac{I-C}{B}\right)$ *is weakly compact on* $A(M)$*,*

(vi) $x = BxAy + Cx \Longrightarrow x \in M$*, for all* $y \in M$*.*

Then the equation (4.2) *has at least one solution in* M *as soon as* $K\phi_B(r) + \phi_C(r) < r$*, for all* $r > 0$*.*

Proof. From Proposition 4.2, it follows that $\left(\frac{I-C}{B}\right)^{-1}$ exists on $A(M)$. By virtue of assumption (vi), we obtain $\left(\frac{I-C}{B}\right)^{-1} A(M) \subset M$.
Moreover, the use of hypotheses (iv) and (v) leads that $\left(\frac{I-C}{B}\right)^{-1} A(M)$ is relatively weakly compact. Now, we show that $\left(\frac{I-C}{B}\right)^{-1} A$ is sequentially weakly continuous. To see this, let x_n be any sequence in M such that $x_n \rightharpoonup x$ in M. By virtue of assumption (iii), we have

$$Ax_n \to Ax.$$

Since $\left(\frac{I-C}{B}\right)^{-1} A$ is a continuous mapping on $A(M)$, we deduce that

$$\left(\frac{I-C}{B}\right)^{-1} Ax_n \to \left(\frac{I-C}{B}\right)^{-1} Ax.$$

This shows that $\left(\frac{I-C}{B}\right)^{-1} A$ is sequentially weakly continuous. By Theorem 1.18, yields that the operator equation. (4.2) has a solution in M. □

Theorem 4.16. *[2] Let* M *be a nonempty closed convex subset of a Banach algebra* X*. Let* $A : M \to X$ *and* $B, C : X \to X$ *be three operators such that*

(i) B *and* C *are* $\mathcal{D}$*-Lipschitzians with the* $\mathcal{D}$*-functions* ϕ_B *and* ϕ_C *respectively,*

(ii) A *is sequentially weakly continuous and* $A(M)$ *is relatively weakly compact,*

(iii) B *is regular on* X*,*

(iv) $\left(\frac{I-C}{B}\right)^{-1}$ *is sequentially weakly continuous on* $A(M)$*,*

(v) $x = BxAy + Cx \Longrightarrow x \in M$*, for all* $y \in M$*.*

Then the equation (4.2) *has at least one solution in* M *as soon as* $K\phi_B(r) + \phi_C(r) < r$ *for all* $r > 0$

Proof. In view of proposition 4.2 and assumption (v) the operator

$$\tau := \left(\frac{I-C}{B}\right)^{-1} A : M \to M$$

is well defined. Since $\left(\frac{I-C}{B}\right)^{-1}$ and A are sequentially weakly continuous. So, by composition τ is sequentially weakly continuous. Now we show that $\tau(M)$ is relatively weakly compact. To see this, let x_n be any sequence in M and let $y_n = \left(\frac{I-C}{B}\right)^{-1} Ax_n$. Since $A(M)$ is relatively weakly compact, there is a renamed subsequence Ax_n weakly converging to an element z. This fact, together with hypothesis (iv) gives that

$$v_n = \left(\frac{I-C}{B}\right)^{-1} Ax_n \rightharpoonup \left(\frac{I-C}{B}\right)^{-1} z.$$

We infer that $\left(\frac{I-C}{B}\right)^{-1} A(M)$ is sequentially relatively weakly compact. An application of the Eberlein-Šmulian Theorem 1.8, yields that $\left(\frac{I-C}{B}\right)^{-1} A(M)$ is relatively weakly compact, which prove our claim. The result is concluded immediately from Theorem 1.18. □

4.4 Application to functional integral equations

Let $(X, \| \,.\, \|)$ be a Banach algebra satisfying condition $(\mathcal{P})$. Let $J = [0, 1]$ the closed and bounded interval in $\mathbb{R}$, the set of all real numbers. Let $E = C(J, X)$ the Banach algebra of all continuous functions from $[0, 1]$ to X, endowed with the sup-norm $\|\|_\infty$, defined by $\| f \|_\infty = \sup\{\| f(t) \|; t \in [0,1]\}$, for each $f \in C(J,X)$. We consider the nonlinear functional integral equation (in short, FIE):

$$x(t) = a(t) + (T(x)(t) \left[\left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right) .u \right] ,0 < \lambda < 1, \tag{4.6}$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions a, q, σ, p, T are given, while x in $C(J,X)$ is an unknown function. We shall obtain the solution of (FIE (4.6)) under some suitable conditions on the functions involved in (4.6). Suppose that the functions a, q, σ, p and the operator T verify the following conditions:

(H_1) $a : J \to X$ is a continuous function with $\| a \|_\infty < 1$,

(H_2) $\sigma : J \to J$ is a continuous and nondecreasing function.

(H_3) $q : J \to \mathbb{R}$ is a continuous function,

(H_4) The operator $T : C(J,X) \to C(J,X)$ is such that

- (a) T is Lipschitzian with a Lipschitzian constant α,
- (b) T is regular on $C(J,X)$,
- (c) $\left(\frac{I}{T}\right)^{-1}$ is well defined on $C(J,X)$,
- (d) $\left(\frac{I}{T}\right)^{-1}$ is sequentially weakly continuous on $C(J,X)$.

(H_5) The function $p : J \times J \times X \times X \to \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longmapsto p(t,s,x,y)$ is continuous uniformly for $(s,x,y) \in J \times X \times X$,

(H_6) There exists $r_0 > 0$ such that:

- (a) $|p(t,s,x,y)| \leq r_0 - \| q \|_\infty$ for each $t,s \in J$; x, $y \in X$ such that $\| x \| \leq r_0$ and $\| y \| \leq r_0$,
- (b) $\| Tx \|_\infty \leq \left(1 - \frac{\| a \|_\infty}{r_0}\right) \frac{1}{\| u \|}$ for each $x \in C(J,X)$,
- (c) $\alpha r_0 \| u \| < 1$.

Theorem 4.17. *Under assumptions* $(H_1)-(H_6)$, *equation* (4.6) *has at least one solution* $x = x(t)$ *which belongs to the space* $C(J, X)$.

Proof. In view of proposition 4.1 we show that $C(J, X)$ verifies condition $(\mathcal{P})$.
Let us define the subset M of $C(J, X)$ by

$$M := \{y \in C(J, X), \| y \|_\infty \leq r_0\} = Br_0.$$

Obviously M is nonempty, convex and closed. Let us consider three operators A, B and C defined on $C(J, X)$ by

$$\begin{aligned}
(Ax)(t) &= (Tx)(t),\\
(Bx)(t) &= \left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right) .u, \ 0 < \lambda < 1\\
(Cx)(t) &= a(t)x(t).
\end{aligned}$$

We shall prove that the operators A, B and C satisfy all the conditions of Theorem 4.16.

(i) From assumption $(H_4)(a)$, A is Lipschitzian with a Lipschitz constant α. Next, we show that C is Lipschitzian on $C(J,X)$. To see this, fix arbitrarily $x, y \in C(J,X)$. Then, if we take an arbitrary $t \in J$, we get

$$\begin{aligned}
\| (Cx)(t) - (Cy)(t) \| &= \| a(t)x(t) - a(t)y(t) \|\\
&\leq \| a \|_\infty \| x(t) - y(t) \| .
\end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\| Cx - Cy \|_\infty \leq \| a \|_\infty \| x - y \|_\infty .$$

This proves that C is Lipschitzian with a Lipschitz constant $\| a \|_\infty$.

(ii) Now we show that B is sequentially weakly continuous on M and $B(M)$ is relatively weakly compact. Firstly, we verify that if $x \in M$, then $Bx \in C(J,X)$. Let t_n be any sequence in J converging to a point $t \in J$. Then

$$\begin{aligned}
\| (Bx)(t_n) - (Bx)(t) \| &= \left\| \left[\int_0^{\sigma(t_n)} p(t_n,s,x(s),x(\lambda s))ds - \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right] .u \right\| \\
&\leq \left[\int_0^{\sigma(t_n)} |p(t_n,s,x(s),x(\lambda s)) - p(t,s,x(s),x(\lambda s))| \, ds \right] \|u\| \\
&+ \left| \int_{\sigma(t)}^{\sigma(t_n)} |p(t,s,x(s),x(\lambda s))| \, ds \right| \|u\| \\
&\leq \left[\int_0^{1} |p(t_n,s,x(s),x(\lambda s))ds - p(t,s,x(s),x(\lambda s))| \, ds \right] \|u\| \\
&+ (r_0 - \|q\|_\infty) \, |\sigma(t_n) - \sigma(t)| \, \|u\| \, .
\end{aligned}$$

Since $t_n \to t$, so, $(t_n,s,x(s),x(\lambda s)) \to (t,s,x(s),x(\lambda s))$, for all $s \in J$. Taking into account (H_5) the hypothesis, we obtain

$$p(t_n,s,x(s),x(\lambda s)) \to p(t,s,x(s),x(\lambda s)) \text{ in } \mathbb{R}.$$

Moreover, the use of assumption (H_6) leads to

$$|p(t_n,s,x(s),x(\lambda s)| - |p(t,s,x(s),x(\lambda s)| \leq 2(r_0 - \|q\|_\infty),$$

for all $t,s \in J, \lambda \in (0,1)$. Consider

$$\begin{cases} \varphi : J & \to \mathbb{R} \\ \quad\quad s & \to \varphi(s) = 2(r_0 - \|q\|_\infty) \end{cases} .$$

Clearly $\varphi \in L^1(J)$. Therefore, from the dominated convergence theorem and assumption (H_2), we obtain

$$(Bx)(t_n) \to (Bx)(t) \text{ in } X.$$

It follows that $Bx \in C(J,X)$.

Next, we prove B is sequentially weakly continuous on M. Let x_n be any sequence in M weakly converging to a point x in M. So, from assumptions $(H_5) - (H_6)$ and the dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^1 p(t,s,x_n(\lambda s))ds = \int_0^1 p(t,s,x(s),x(\lambda s)),$$

which implies

$$\lim_{n\to\infty}\left(q(t)+\int_0^1 p(t,s,x_n(s),x_n(\lambda s))ds\right).u=\left(q(t)+\int_0^1 p(t,s,x(s),x(\lambda s))ds\right).u.$$

Hence

$$Bx_n(t)\to Bx(t)\quad \text{in } X.$$

Since $(Bx_n)_n$ is bounded by $r_0\|u\|$, then $Bx_n \rightharpoonup Bx$ We conclude that B is sequentially weakly continuous on M.

Now, we show $B(M)$ is relatively weakly compact.

Step 1: By definition,

$$B(M):=\{B(x),\ \| x \|_\infty\leq r_0\}.$$

For all $t\in J$, we have

$$B(M)(t):=\{B(x)(t),\ \| x \|_\infty\leq r_0\}.$$

We claim that $B(M)(t)$ is sequentially weakly relatively compact in X. To see this, let x_n be any sequence in M, we have $(Bx_n)(t)=r_n(t).u$, where

$$r_n(t)=q(t)+\int_0^1 p(t,s,x_n(s),x_n(\lambda s))ds.$$

Since $|r_n(t)|\leq r_0$ and $(r_n(t))$ is a real sequence, so, there is a renamed subsequence such that

$$r_n(t)\to r(t)\quad \text{in } \mathbb{R},$$

which implies

$$r_n(t).u\to r(t).u\quad \text{in } X,$$

and, consequently

$$(Bx_n)(t)\to (q(t)+r(t)).u\quad \text{in } X.$$

then, $B(M)(t)$ is sequentially relatively weakly compact in X.

Step 2: We prove that $B(M)$ is weakly equicontinuous on J . If we take $\varepsilon > 0$; $x \in M$; $x^* \in X^*$; $t,t' \in J$ such that $t \leq t'$ and $t' - t \leq \varepsilon$. Then,

$$\begin{aligned}
\left|x^*\left((Bx)(t) - (Bx)(t')\right)\right| &= \left|\int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds - \int_0^{\sigma(t')} p(t',s,x(s),x(\lambda s))ds\right| \|x^*(u)\| \\
&\leq \left[\int_0^{\sigma(t)} \left|p(t,s,x(s),x(\lambda s)) - p(t',s,x(s),x(\lambda s))\right| ds\right] \|x^*(u)\| \\
&+ \left[\int_{\sigma(t)}^{\sigma(t')} \left|p(t',s,x(s),x(\lambda s))\right| ds\right] \|x^*(u)\| \\
&\leq \left[\omega(p,\varepsilon) + (r_0 - \|q\|_\infty)(\omega(\sigma,\varepsilon))\right] \|x^*(u)\|
\end{aligned}$$

where

$$\omega(p,\varepsilon) = \sup\left\{\left|p(t,s,x,y) - p(t',s,x,y)\right| : t,t',s \in J; \left|t - t'\right| \leq \varepsilon; x,y \in B_{r_0}\right\},$$
$$\omega(\sigma,\varepsilon) = \sup\left\{\left|\sigma(t) - \sigma(t')\right| : t,t' \in J; \left|t - t'\right| \leq \varepsilon\right\}$$

.

Taking into account the hypothesis (H_5) and in view of the uniform continuity of the function σ on the set J , it follows that $\omega(p,\varepsilon) \to 0$ and $\omega(\sigma,\varepsilon) \to 0$ as $\varepsilon \to 0$. An application of the Arzela-Ascoli Theorem 1.6 , we conclude that $B(M)$ is sequentially weakly relatively compact in X. Again an application result of Eberlein-Šmulian theorem yields that $B(M)$ is relatively weakly compact.

(iii) From assumption $(H_4)(b)$, A is regular on $C(J,X)$.

(iv) We show that $\left(\frac{I-C}{A}\right)^{-1}$ is sequentially weakly continuous on $B(M)$.

To see this, let $x, y \in C(J,X)$ such that

$$\left(\frac{I-C}{A}\right)(x) = y,$$

or, equivalently

$$\frac{(1-a)x}{Tx} = y.$$

Since $\| a \|_{\infty} < 1$, so, $(1-a)^{-1}$ exists on $C(J,X)$, then

$$\left(\frac{I}{T}\right)(x) = (1-a)^{-1}y.$$

This implies, from assumption $(H_4)(c)$, that

$$x = \left(\frac{I}{T}\right)^{-1}(1-a)^{-1}y.$$

Thus

$$\left(\frac{I-C}{A}\right)^{-1}(x) = \left(\frac{I}{T}\right)^{-1}(1-a)^{-1}(x),$$

for all $x \in C(J,X)$. Now, let x_n be a weakly convergent sequence of $B(M)$ to a point x in $B(M)$, then

$$(1-a)^{-1}x_n \rightharpoonup (1-a)^{-1}x,$$

and so, it follows from assumption $(H_4)(d)$ that

$$\left(\frac{I}{T}\right)^{-1}(1-a)^{-1}x_n \rightharpoonup \left(\frac{I}{T}\right)^{-1}(1-a)^{-1}x,$$

we conclude that

$$\left(\frac{I-C}{A}\right)^{-1}(x_n) \rightharpoonup \left(\frac{I-C}{A}\right)^{-1}(x).$$

(v) Finally, it remains to prove the hypothesis (v) of Theorem 4.16.

To see this, let $x \in C(J,X)$ and $y \in M$ such that

$$x = AxBy + Cx,$$

or, equivalently for all $t \in J$,

$$x(t) = a(t)x(t) + (Tx)(t)(By)(t).$$

But, for all $t \in J$, we have

$$\begin{aligned}
\| x(t) \| &\leq \| x(t) - a(t)x(t) \| + \| a(t)x(t) \| . \\
&\leq \|(Tx)(t)\| \, r_0 \|u\| + \|a\|_{\infty}\|x(t)\| \\
&\leq \left(1 - \frac{\| a \|_{\infty}}{r_0}\right) r_0 + \| a \|_{\infty} \| x(t) \| \\
&\leq \frac{r_0 - \| a \|_{\infty}}{1 - \| a \|_{\infty}} \\
&\leq r_0.
\end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\| x \|_{\infty} \leq r_0.$$

We conclude that the operators A, B and C satisfy all the requirements of Theorem 4.16. So the FIE (4.6) has a solution in the space $C(J, X)$. □

Conclusion

Troughs this work we have treated some new fixed point theorems of Krasnoselskii under weak topology of Banach spaces, for the sum and the product of two operators. Particularly, contraction and expansive mapping, then we make as applications some existence results for some nonlinear integral equations in Banach spaces endowed with their weak topologies. The structure of the space and the properties of the map are very important to obtain a fixed point, for this reason we proposed some conditions for the space.

In the near future, we aim at investigating the fixed point theorems of Krasnoselskii in generalized Banach space under weak topology and it's applications.

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