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كلية العلوم والتكنولوجيا
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Ministry of Higher Education and Scientific Research
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Dedications

....I dedicate this work

To those who are unmatched in the universe...To those whome Allah oredred us to setting down beside theme and treated them kindly...To those who have sacrificed and done a lot for free...To those who have stood by me all these years inspire me to achieve my goals...Mommy and Daddy...The most precious persons in my life, to theme I dedicate this work with love.

TO

My most beautiful destinies, my support and the spirit of my life, my brothers and my sisters.

TO

My precious grandmother, my dear brothers' wives, my dear uncles and aunts.

TO

The joy and light of my life: Mohammed Abdelmoubine, Ahmed Yassine, Mohammed Abdelwadood, Abdullah Anas.

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My wonderful trail mates, my friends, all in his name.

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My distinguished teachers and all my friends from the Math class of 2022-2023.

★ ★ ★

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Abstract

This treatise concentrates on fixed point theorems in a locally convex space X, taking into account the concept of Φ_Λ^τ-measure of noncompactness and the so-called τ-Krein-$\check{S}$mulian property, where τ represents a weaker Hausdorff locally convex topology. In application, we examine the existence of solutions for certain nonlinear integral equations. Initially, we present several definitions and auxiliary results that will be utilized later. Following this, we provide a version of fixed point theorems for ws-compact and Hausdorff weakly sequentially continuous maps in a locally convex space, as well as in an angelic space. In conclusion, we propose some existence results to solve certain nonlinear integral equations in the Lebesgue space L^1 and in the Fréchet space L^1_{loc}.

Keywords: Locally convex space, Φ_Λ^τ-measure of noncompactness, τ-Krein-$\check{S}$mulian property, Weakly sequentially continuous, Angelic space.

Résumé

Ce mémoire se concentre sur les théorèmes de point fixe dans un espace localement convexe X, en considérant le concept de Φ_Λ^τ-mesure de non-compacité et la propriété τ-Krein $\check{S}$mulian, où τ est la topologie localement convexe Hausdorff la plus faible. Comme application, nous étudions l'existence de solutions pour certaines équations intégrales non linéaires. Premièrement, nous présentons quelques définitions et résultats auxiliaires qui seront utilisés ultérieurement. Deuxièmement, nous proposons des versions de théorèmes de points fixes pour des applications ws-compactes et faiblement séquentiellement continues de Hausdorff dans l'espace localement convexe et dans l'espace angélique. Enfin, nous proposons des résultats d'existence pour résoudre certaines équations intégrales non linéaires dans l'espace Lebesgue L^1 et dans l'espace Fréchet L^1_{loc}.

Mots clés: Espace localement convexe, Φ_Λ^τ-mesure de non compacité, Propriété de τ-Krein $\check{S}$mulian, Faiblement séquentiellement continues, Espace angélique.

ملخص

تركز هذه المذكرة على نظريات النقطة الثابتة في فضاء محدب محليًا X مع الأخذ في الاعتبار مفهوم Φ_Λ^τ ـ مقياس عدم التراص وما يسمى بخاصية τ ـ كرين سميليان حيث τ أضعف طوبولوجيا منفصلة محدبة محليًا. كتطبيق، ندرس وجود حلول لبعض المعادلات التكاملية غير الخطية. أولاً، نقدم بعض التعاريف والنتائج المساعدة التي سيتم استخدامها لاحقًا. ثانيًا، نعطي نسخة من نظريات النقاط الثابتة لتطبيقات ws ـ متراصّة و لتطبيقات مستمرة بشكل متتالي ضعيف في الفضاء المحدب محليًا وفي الفضاء الملائكي. أخيرًا، نقدم بعض نتائج الوجود لحل بعض المعادلات التكاملية غير الخطية في فضاء لوبيغ L^1 وفي فضاء فريشي L^1_{loc} .

الكلمات المفتاحية: فضاء محدب محلياً، Φ_Λ^τ ـ مقياس عدم التراص، خاصية τ ـ كرين سميليان، مستمرة بشكل متتالي ضعيف، الفضاء الملائكي.

Contents

Acronyms

(X, T)	Topological space.
$d(x, y)$	The distance between x and y.
(X, d)	Metric space.
$(X, \|.\|)$	Norm vector space.
$\overline{M}$	The closure of M.
$B(a, r)$	The open ball of center a and radius r.
$B_{\mathbb{R}^n}$	The ball in $\mathbb{R}^n$.
B_1	The unit ball in an infinite dimensional Hilbert space.
X'	Dual topology of X.
$\sigma(X, X')$	The weak topology.
$\mathcal{B}_X$	The family of all nonempty and bounded subsets of X.
$\mathfrak{B}_X$	The subfamily consisting of all relatively compact sets.
$\mu(.)$	Measure of noncompactness in Banach space.
$ker\mu$	The kernel of the measure of noncompactness μ.
$\overline{conv}(M)$	The closed convex hull of M with respect to the strong topology.
$\delta(M)$	Diameter of M.
Λ	Indexed family.
(X, τ)	Hausdorff locally convex topological space where τ is a weaker Hausdorff locally convex topology.
ϕ_{p_τ}	Φ-measure of noncompactness in locally convex space.
$\overline{conv}^\tau(M)$	The closed convex hull of M with respect to the topology τ.
L^P	The vector space of classes of functions whose exponent power P is integrable in the sense of Lebesgue, where $p > 0$.
$L^1_{loc}(\mathbb{R}^+)$	The space of all real measurable functions that are locally Lebesgue integrable on $\mathbb{R}^+$.

Introduction

One of the most dynamic area of research of the last 60 years, with a lot of applications in various fields of pure and applied mathematics, as well as, in physical, economic or life sciences, is without doubt the **fixed point theory**. A fixed point theorem is a result saying that a function A will have at least one fixed point (a point x for which $A(x) = x$), under some conditions on A that can be stated in general terms.

Most of the natural phenomena of real life are expressed mathematically in the form of different nonlinear equations. Many questions related to existence and uniqueness of solution of certain types of equation can be reduced to the question of existence and uniqueness of a fixed point defined on a metric and topological spaces. The fixed point theory is at the heart of nonlinear analysis since it provides the tools necessary to have existence theorems in many different non linear problems.

The first fixed point result was stated by Brouwer in 1912 for finite dimensional spaces. In 1930, Schauder has improved and generalized the problem of Brower and insists on the continuous map on a convex and compact set. Also, a very interesting useful result in fixed point theory is due to Banach knows as the Banach contraction principle which guarantee the existence and uniqueness of fixed point. On the other hand, in many problems of analysis, we encounter such a format T $=A+B$, where A is compact (completely continuous), and B is a contraction in some sense, and T itself has neither of these properties. Thus neither the Schauder fixed point theorem nor the Banach fixed point theorem applies directly in this case, and it becomes desirable to develop fixed point theorems for such situations. An early theorem of this type was given by Krasnoselskii [22].

Compactness play an important role in fixed point theory in Banach space as Schauder fixed point theorem. But there are problems if the operators are not compact. For solve this problems, it was suggested to define a function on the family of bounded sets of a metric space that measure the noncompactness of a set which maps any bounded set in a more compact set. Such a function is called a measure of noncompactness (in short, MNC). The first who introduce this concept is Kuratowski [23] in 1930 on the family of all nonempty and bounded subsets of a metric space with real nonnegative values and proved a theorem which ensures the existence of a fixed point of the so-called μ-contraction operators (where μ is Kuratowskii's measure of noncompactness) and generalizes both the classical Schauder fixed point theorem and a special variant of Banach contraction principle. The second one is Hausdorff measure of noncompactness (or Ball measure of noncompactness) were introduced by Goldenstein, Gohberg and Markus in 1965. These two measures of noncompactness are satisfied in some spaces and also difficult to prove. So many recherchers have worked hard to generalize the notion of measure of noncompactnesss. As Banaś and Goebel in 1979 introduced an axiomatic approach for measure of noncompactness in a metric space X (Definition 1.30).

The measure of noncompactness play an important role in fixed point theory. In 1955, G. Darbo generalized the theory of Schauder using Kuratowski measure of noncompactness for Lipschitz maps. Later on, Sadoveskii's fixed point theorem stated a more general fixed point result than Darbo's theorem for condensing mapping. As we have just mentioned, Darbo and Sadoveski's results use strongly the concept of measure of noncompactness.

Another important measure of noncompactness in a Banach space endowed with the weak topology introduced by De Blasi in 1977 who proved the analogousness of Sadovskii's fixed point theorem for the weak topology. It is always so hard to build certain formulas to express the De Blasi measure in a practical form for application. For this purpose, in 1988, Banaś and Rivero defined an axiomatic measure of weak noncompactness (in short, MWNC) on the family of all nonempty bounded sets into the real positive values which satisfies conditions (1)-(5) of a measure of noncompactness relative to the weak topology.

The question arises whether the results of the fixed point theories discussed above can be retained in a larger space than Banach space with the lowest conditions?. yes, can be proposed a great number of results as far as the theory of fixed point in locally convex spaces (in short, LCSs) which is define by the origine has a neighborhood basis consisting of convex sets. Or can be defined with a family of seminorms, and a topology can be defined in terms of that family. In 1935, Tychonoff extended Schauder fixed point theorem in LCSs. In 2013, Ben Amar and Banaś introduced the concept of the family of MNC in LCS and prove some version of fixed point theorems. Such as, Arino, Gautier and Penot fixed point theorem for sequentially weakly continuous maps and weakly compact convex sets in metrizable locally convex space, the proof of their theorem is reducing the result of Schauder-Tychonoff's using Eberlin-Šmulian's theorem (Theorem 1.2) that make (weakly) compact, (weakly) relatively compact and (weakly) sequentially compact coincide, but the metrization is not available in many spaces. To solve this problems appeared the angelic space that make the classes of compact coincide.

The aim of this memoir is study the existance of fixed point theory in angelic locally convex space involving its weak Hausdorff locally convex topology to find some results and solution for some nonlinear integral equations.

This work consists of three chapters. The first chapter contains some basic concepts and definitions used throughout this memoir like metric space, complet space, Banach space, contraction mapping, weak topology, measure of (weak) noncompactess, locally convex space, angelic space and some fixed point theorems in Banach space and locally convex space.

In the second chapter, there are three section. In section 1, we present some fixed point theorem for ws-compact maps (an operator which maps relatively weakly compact sets into relatively strongly compact ones for continuous maps) using the Krein-Šmulian property. In section 2, we present some version of fixed point theorems for weakly sequentially continuous maps in locally convex space . In section 3, we stated and proved a generalisation of some fixed point theorems in angelic spaces for τ sequentially continuous maps where τ is a weaker Hausdorff locally convex topology using the τ-KS property (Definition 1.42) such as Arino, Gautier, and Penot fixed point theorem, krasnoselskii fixed point theorem, Darbo fixed point theorem, and Sadoveskii fixed point theorem using the family of measur of noncompactness.

In the third chapter, we apply the results obtaine in section 3 of the last chapter to study the existence of solution for the following nonlinear integral equation

$$x(t) = a(t) + \int_0^1 h(t,s)f(s, x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds, \quad t \in [0,1],$$

in the Lebesgue space L^1, where a, h, f, u and φ are given functions satisfying certain conditions.

And the solvability of the nonlinear integral equation

$$x(t) = a(t) + \int_0^1 h(t,s)f(s, x(s))ds + g(t, x(t)), \quad t \in [0,1],$$

in the Fréchet space $L^1_{loc}(\mathbb{R}^+)$.

Chapter 1

Preliminaries

In this chapter, we present all the notations and definitions that we need in writing this memoir.

1.1 Topological vector space

Definition 1.1. We call **topological space** a couple (X, T) where X is a set and T a family of parts of X verifying:

1. $\emptyset \in T, \ X \in T$,
2. A finite intersection of elements of T belongs to T i.e
$$(\{A_i \in T, \ i = 1, 2, .., n\} \implies \cap_{i=1}^{n} A_i \in T),$$
3. Any union of elements of T belongs to T i.e
$$(\{A_i \in T, \ i \in I\} \implies \cup_{i \in I} A_i \in T).$$

We call T the topology on X.

Example. Let $X = \{1, 2, 3\}$ be a set

- $T = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$ is a topology on X.
- If $T = \{\emptyset, X\}$. We call T the trivial topology.

- If $T = P(X)$, the family of all subsets of X. We call T the discrete topology.

Definition 1.2 (Hausdorff spaces). A topological space X is Hausdorff if for any $x, y \in X$ with $x \neq y$ there exists open sets U containing x and V containing y such that $U \cap V = \emptyset$.

Topologies can be constructed using distances.

Definition 1.3. A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following axioms for all points $x, y, z \in X$:

1. $d(x, y) = 0 \Longleftrightarrow x = y$,
2. $d(x, y) = d(y, x), \quad \forall x, y \in X$, (Symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z), \quad \forall x, y, z \in X$. (triangle inequality)

Remark 1.1. A **metric space** is an ordered pair (X, d) where X is a set and d is a metric on X.

Example.
- The real numbers with the distance function $d(x, y) = |y - x|$ given by the absolute difference form a metric space.
- $X = \mathbb{R}^n$ with
$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
is a metric space. Indeed,

$$\begin{aligned} d(x, y) &= \sqrt{\sum_{i=1}^{n} (x_i + z_i - z_i - y_i)^2} \\ &= \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2 + \sum_{i=1}^{n} (z_i - y_i)^2 + 2 \sum_{i=1}^{n} (x_i - z_i)(z_i - y_i)}, \end{aligned}$$

by Holder's inequality, we have

$$\begin{aligned} d(x,y) &\leq \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2 + \sum_{i=1}^{n}(z_i - y_i)^2 + 2\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{\frac{1}{2}}} \\ &= \sqrt{\left(\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{\frac{1}{2}}\right)^2} \\ &= \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{\frac{1}{2}} \\ &= \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} + \sqrt{\sum_{i=1}^{n}(z_i - y_i)^2} \\ &= d(x,z) + d(z,y). \end{aligned}$$

Definition 1.4. Let (X, d) be a metric space, $a \in X$ et $r > 0$, the set

$$B(a, r) = \{x \in X,\ d(x, a) < r\},$$

is called the **open ball** with center a and radius r.

Definition 1.5. Let $M \subset X$, we say that M an **open set** if $\forall x \in M$, $\exists \varepsilon > 0$ such as $B(x, \varepsilon) \subset M$.

Definition 1.6. A subset M of X is **closed set** if $M \backslash X$ is open.

Definition 1.7. A subset M of a metric space (X, d) is **bounded** if there exists $r > 0$ such that for all x and y in M, we have $d(x, y) < r$.

Definition 1.8. In a metric space (X, d), a set V is a **neighbourhood** of a point x if there exists an open ball with center x and radius $r > 0$ such that

$$B(x, r) = \{y \in X : d(y, x) < r\}$$

is contained in V.

Definition 1.9. Let (X, d) and (Y, d') two metric spaces, $A : X \to Y$ and $a \in X$. We say that the map A is **continuous** at a if:

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in X \ \ d(x, a) < \eta \implies d'(Ax, Aa) < \varepsilon.$$

Remark 1.2. A map A is continuous on X if and only if it is continuous at any point a of X.

Theorem 1.1. Let X, Y two metric spaces. A function $A : X \to Y$ is continuous at $a \in X$ if and only if for every neighborhood $V \subset Y$ of $A(a)$, the inverse image $A^{-1}(V) \subset X$ is a neighborhood of a.

Corollary 1.1. [Characterization of a continuous map] Let X, Y two metric spaces. The three conditions below are equivalent:

1. $A : X \to Y$ is continuous at every point of X.
2. For each open set O in Y, $A^{-1}(O)$ is open in X.
3. For each closed set F in Y, $A^{-1}(F)$ is closed in X.

Definition 1.10. Recall that a map $A : X \to X$ is said to be a **contraction** map, if $d(Ax, Ay) \leq kd(x, y)$, where X is a metric space; $x, y \in X$ and $0 \leq k < 1$.

Remark 1.3. Every contraction map is a continuous map, but a continuous map need not be a contraction map.

Definition 1.11. In metric space (X, d), a sequence (x_n) is said to be **converges** to x and denote $x_n \to x$ in X means that $d(x_n, x) \to 0$ i.e

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N \;\; d(x_n, x) < \varepsilon.$$

Definition 1.12 (Closure of a set). Let X be a topological space and $M \subset X$. We say that a point $a \in X$ is adherent to M if there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset M$ converging to a. The set of points adherent to M is called the closure of M and we note $\overline{M}$.

Definition 1.13. A sequence (x_n) of a metric space (X, d) is called a **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall m \geq N \;\; d(x_n, x_m) < \varepsilon.$$

.

Definition 1.14 (Complete metric space). A metric space (X, d) is said to be complete if every Cauchy sequence converges in X.

Remark 1.4. In general; every convergent sequence is a Cauchy sequence, but the converse is not true.

Example.

- The space $\mathbb{Q}$ of rational numbers, with the standard metric given by the absolute value of the difference, is not complete. Indeed,

  Let $\bar{x}$ be an irrational number, and for each $n \in \mathbb{N}$ let x_n be a rational number in the interval $[\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n}]$. Then (x_n) is a Cauchy sequence of rational numbers that converges to the irrational number $\bar{x}$.

- The real interval $X =]0, 1]$ with the usual metric is not a complete space. Because the sequence $x_n = \frac{1}{n}$ is Cauchy but does not converge to an element of $]0, 1]$.

Definition 1.15. Let X be a vector space. A norm on X is a map $\|.\| : X \to [0, \infty)$ that satisfies the following three properties:

(i) $\|x\| = 0$ if and only if $x = 0$, (Definiteness)

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and any scalar λ, (Homogeneity)

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Triangle inequality)

A normed space is a pair $(X, \|.\|)$, where X is a vector space and $\|.\|$ is a norm on X.

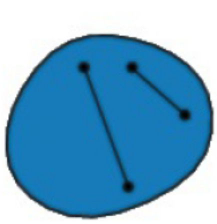
Definition 1.16. A **Banach space** is a complete normed space $(X, \|.\|)$.

Example. A normed space $(\mathbb{K}, \|.\|)$ is a simple example of a Banach space. where ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

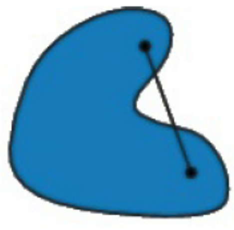
Definition 1.17 (Convex set). A set M is convex if the line segment between any two points in M lies in M, i.e., $\forall x_1, x_2 \in M, \forall \theta \in [0, 1]$

$$\theta x_1 + (1 - \theta)x_2 \in M.$$

Definition 1.18. Let M be a subset of a normed space X. The **closed convex hull** of M denoted by $\overline{conv}(M)$, is defined as the smallest closed and convex subset of X that contains M.



(a) Convex set



(b) Not convex set

Definition 1.19 (**Compact space**). A topological space (X, T) is said to be compact if and only if Hausdorff and every open cover of X has a finite subcover.

Definition 1.20. A subset M of a topological space X is compact if the topological subspace M of X is compact.

Definition 1.21. Let X be a topological space and $M \subset X$

(i) M is said to be **countably compact** if every countable open covering of M contains a finite sub-covering.

(ii) M is said to be **relatively compact**, if its closure $\overline{M}$ in X is compact.

(iii) M is said to be **sequentially compact**, if every sequence in M has a subsequence which converges to a point in M.

(iv) M is said to be **relatively countably compact**, if any sequence in M has a cluster point in X.

(v) M is said to be **relatively sequentially compact**, if any sequence in M has a subsequence which converges to a point in X.

Remark 1.5.
- Compact $\Rightarrow$ Sequentially compact $\Rightarrow$ Countably compact.
- Compact $\Rightarrow$ Relatively compact.

Definition 1.22. Let X be a normed space, an operator $A : X \to Y$ is said to be completely continuous if it satisfies the following condition:

(i) A is continuous.

(ii) A maps bounded subsets to relative compact sets.

We may also define a compact operator by the following definition:

Definition 1.23. Let M be a subset of a Banach space X and let $A : M \to X$ an operator. If A is continuous and $A(M)$ is contained in a compact subset of X, then we say that A is a compact operator.

1.2 Weak topology

The topology induced by a norm on a vector space is a very strong topology in the sense that it has many open sets. This brings advantages to the functions whose domain is such a space because for them is easy to be continuous but it brings disadvantages to compactness because the richness of open sets makes it difficult for a set to be compact.

Compactness is depended on open sets i.e. is depended on the nature of topology. The weaker topology is the more compact sets we have, this fact motivates us to search for a topology defined on a normed space X which is the weakest topology among all topologies which we can define on X, so we can have the biggest class of compact sets. The topology which gives us the desired result is the weak topology defined on X. This topology helps us to characterize properties of topological spaces with infinite dimensions and provides simple means to check their nature.

Definition 1.24. Let $(X, \|.\|)$ be a normed space. The **weak topology** on X is the weakest topology (the topology with the fewest open sets) such that all elements of X' (the topological dual of X) remain continuous.

One may call subsets of a topological vector space weakly closed (respectively, weakly compact, etc.) if they are closed (respectively, compact, etc.) with respect to the weak topology.

Proposition 1.1. [14] The weak topology $\sigma(X, X')$ is Hausdorff.

Theorem 1.2 (Eberlein–Šmulian Theorem [25]). Let M be a subset of a Banach space X. The following assertions are equivalent:

(i) M is (relatively) weakly compact,

(ii) M is (relatively) weakly sequentially compact,

(iii) M is (relatively) weakly countably compact.

Definition 1.25. If $(x_n)_{n\in N}$ is a sequence of elements of a topological vector space X and x is an element of X, then $x_n \rightharpoonup x$ if, and only if, $\varphi(x_n) \to \varphi(x)$ for each $\varphi \in X'$. Here, $\rightharpoonup$ denotes the weak convergence and $\to$ denotes the strong convergence in X, respectively.

Definition 1.26. A function $A : X \to Y$ is called weakly continuous if it is continuous with respect to the weak topologies of X and Y.

Definition 1.27. Let X be a Banach space. An operator $A : X \to Y$ is said to be weakly sequentially continuous on X if, for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$.

Definition 1.28. Let X and Y two topological vector spaces. A mapping $A : M \subseteq X \to Y$ is said to ws-compact if it is continuous and maps relatively weakly compact sets of M into relatively strongly compact ones of Y.

Theorem 1.3 (Krein-Šmulian theorem [17]). The closed, convex hull of weakly compact subset of a Banach space is weakly compact.

1.3 Elementary fixed point theorems in Banach space

Definition 1.29. Let X be a nonempty set and let $A : X \to X$ be a function, we say that $x \in X$ a fixed point of X if and only if $Ax = x$.

Example.

- The mapping $x \to x^2$ on $\mathbb{R}$ has two fixed points; 0 and 1.
- A rotation of the plane has alone fixed point, is the center of rotation.
- A translation $x \to x + a$ in $\mathbb{R}$ has no fixed points.

1.3.1 Brouwer fixed point theorem

The first famous Brouwer fixed point theorem was given in 1912.

Theorem 1.4 (**Brouwer fixed point theorem** [15]). Let $A : B_{\mathbb{R}^n} \to B_{\mathbb{R}^n}$ be a continuous function and $B_{\mathbb{R}^n}$ is a ball in $\mathbb{R}^n$, then A has a fixed point.

In application, this theorem simply guarantees the existence of a solution, but gives no information about the uniqueness and determination of the solution. Also, this result is not true in infinite dimensional spaces. For example, if B_1 is a unit ball in an infinite dimensional Hilbert space and $A : B_1 \to B_1$ is a continuous function, then A need not have a fixed point. This was given by Kakutani in 1941 [20] .

1.3.2 Schauder fixed point theorem

The first fixed point theorem in an infinite dimensional Banach space was given by Schauder in 1930. This theorem has applications in approximation theory, game theory and other scientific area like engineering, economics and optimization theory. The theorem is stated below:

Theorem 1.5 (**Schauder fixed point theorem** [26]). If M is a compact, convex subset of a Banach space X and $A : M \to M$ is a continuous function, then A has a fixed point .

The compactness condition on M is a very strong one and most of the problems in analysis do not have compact setting. It is natural to prove the theorem by relaxing the condition of compactness. Schauder proved the following theorem [26]:

Theorem 1.6. If M is a closed bounded convex subset of a Banach space X and $A : M \to M$ is continuous map such that $A(M)$ is compact, then A has a fixed point.

1.3.3 Banach contraction principle

The method of successive approximation introduced by Liouville in 1837 and systematically developed by Picard in 1890 culminated in formulation by Banach known as the

Banach contraction principle (BCP) is a very useful result in fixed point theory stated as below:

Theorem 1.7 (**BCP** [10]). If X is a complete metric space and $A : X \to X$ is a contraction map, then A has a unique fixed point or $Ax = x$ has a unique solution.

Proof. **(i) Existence:** on choose $x_0 \in X$ and define $x_{n+1} = Ax_n$, $n = 0, 1, 2, ...$

step 1: We show that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

We have by recurrence

$$\begin{aligned} d(x_{m+1}, x_m) &= d(Ax_m, Ax_{m-1}) \\ &\leq kd(x_m, x_{m-1}) \\ &= kd(Ax_{m-1}, Ax_{m-2}) \\ &\leq k^2 d(x_{m-1}, x_{m-2}) \\ &\vdots \\ &\leq k^m d(x_1, x_0). \end{aligned}$$

Hence by the triangle inequality we get (for $n \geq m$) that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-1}, x_n) \\ &\leq (k^m + k^{m+1} + \cdots + k^{n-1}) d(x_0, x_1) \\ &= k^m \frac{1 - k^{n-m}}{1 - k} d(x_0, x_1), \end{aligned}$$

since $0 < k < 1$ we have $1 - k^{n-m} < 1$ and consequently

$$d(x_m, x_n) \leq \frac{k^m}{1 - k} d(x_0, x_1).$$

Therefore $d(x_m, x_n) \to 0$ when $m \to \infty$, this expresses the fact that $(x_n)_n$ is a Cauchy sequence in X, and since X is a complete space, there exists $x \in X$ such that $x_n \to x$.

step 2: Its limit is a fixed point of X.

We consider the distance $d(x, Ax)$. From the triangle inequality and Definition 1.10, we get

$$\begin{aligned} d(x, Ax) &\leq d(x, x_m) + d(x_m, Ax) \\ &= d(x, x_m) + d(Ax_{m-1}, Ax) \\ &\leq d(x, x_m) + kd(x_{m-1}, x), \end{aligned}$$

and since $x_n \to x$ it is clear that we can make this distance as small as we choosing m sufficiently large. We conclude that

$$d(x, Ax) = 0 \implies Ax = x.$$

(ii)Uniqueness: Suppose there are two fixed points such as $x = Ax$ and $y = Ay$:

$$d(x, y) = d(Ax, Ay) \leq kd(x, y),$$

as $0 < k < 1$ so $d(x, y) = 0 \implies x = y$.

□

1.3.4 Krasnoselskii's fixed point theorem

In many problems of analysis, we encounter such a format T $=A + B$, where A is compact(completely continuous), and B is a contraction in some sense, and T itself has neither of these properties. Thus neither the Schauder fixed point theorem nor the Banach fixed point theorem applies directly in this case, and it becomes desirable to develop fixed point theorems for such situations. An early theorem of this type was given by Krasnoselskii.

Theorem 1.8 (**Krasnoselskii's theorem(1955)** [22]). Let $(X, \|.\|)$ be a Banach space, and let M is a nonempty, convex and closed subset of a Banach space X.
We suppose that: $A, B : M \to X$ operators satisfying:

(i) $Ax + By \in M, \quad$ for all $x, y \in M$.

(ii) A is continuous and $A(M)$ is contained in a compact set.

(iii) B is a contraction of constant $k < 1$.

So it exists $\tilde{x} \in M, \;\; A\tilde{x} + B\tilde{x} = \tilde{x}$.

Note: If $A = 0$ this theorem coincides with Banach's theorem, and if $B = 0$ it coincides with Schauder's theorem.

Proof. By condition (iii) we have

$$\begin{aligned}
\|(I-B)(x)-(I-B)(y)\| &= \|(x-y)+(Bx-By)\| \\
&\leq \|x-y\| + \|Bx-By\| \\
&\leq \|x-y\| + k\|x-y\| \\
&\leq (1+k)\|x-y\|,
\end{aligned}$$

on the other hand

$$\begin{aligned}
\|(I-B)(x)-(I-B)(y)\| &= \|(x-y)-(Bx-By)\| \\
&\geq \|(x-y)\| - \|Bx-By\| \\
&\geq \|x-y\| - k\|x-y\| \\
&\geq (1-k)\|x-y\|,
\end{aligned}$$

therefore

$$(1-k)\|x-y\| \leq \|(I-B)(x)-(I-B)(y)\| \leq (1+k)\|x-y\|,$$

this inequality shows that $(I-B) : M \to (I-B)M$ is continuous and bijective. So $(I-B)^{-1}$ exist and is continuous .
Let's put $U = (I-B)^{-1}A$. It is clear that U is a compact operator since it is composition of continuous operator with compact operator. Under the Schauder theorem, U admits a fixed point i.e: it exists $x \in M$ such as $(I-B)^{-1}Ax = x$. This is equivalent to saying that $Ax + Bx = x$. □

1.4 Measure of (weak) noncompactness in Banach space

Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. If X is a Banach space with the norm $\|.\|$. Let M is a subset of X, then the symbols $\overline{M}$ and $\overline{conv}(M)$ stand for the strong closure and convex closure of M, respectively.

Next, let us denote by $\mathcal{B}_X$ the family of all nonempty and bounded subsets of X and by $\mathfrak{B}_X, \mathfrak{W}_X$ are the subsets of $\mathcal{B}_X$ consisting of all relatively compact and weakly compact subsets of X respectively.

Definition 1.30. [13] A mapping $\mu : \mathcal{B}_X \to \mathbb{R}^+$ will be called a measure of noncompactness in X if it satisfies the following conditions:

1. The family $ker\mu = \{M \in \mathcal{B}_X, \mu(M) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{B}_X$.
2. $M \subset N \implies \mu(M) \leq \mu(N)$.
3. $\mu(\overline{M}) = \mu(\overline{conv}(M)) = \mu(M)$.
4. $\mu(\lambda M + (1-\lambda)N) \leq \lambda\mu(M) + (1-\lambda)\mu(N)$, for $\lambda \in [0, 1]$.
5. If (M_n) is a sequence of closed sets from $\mathcal{B}_X$ such that $M_{n+1} \subset M_n$ for $n = 1, 2, ...$ and if $\lim_{n\to\infty} \mu(M_n) = 0$, then the set $M_\infty = \cap_{n=1}^{\infty} M_n$ is nonempty.

The family $ker\mu$ described in 1 is said to be the kernel of the measure of noncompactness μ .

Observe that the set M_∞ from the axiom 5 is a member of the family $ker\mu$. Indeed, from the inequality $\mu(M_\infty) \leq \mu(M_n)$ being satisfied for all $n = 1, 2, ...$ we derive that $\mu(M_\infty) = 0$ which means that $M_\infty \in ker\mu$.

Example. [Kuratowski measure] Let X be a metric space and M a nonempty and bounded subset of X.

The Kuratowski noncompactness measure of the set M, denoted $\alpha(M)$ is the lower of the numbers $d > 0$, such that M admits a finite covering by sets of diameter less than d, i.e.

$$\alpha(M) = \inf\{d > 0, \;\; M = \bigcup_{i=1}^{n} M_i \;\text{ tel que }\; \delta(M_i) \leq d\}.$$

Definition 1.31. We say that a measure of noncompactness μ has the maximum property (or it is semi-additive) if

$$\mu(A \cup B) = \max\{\mu(A), \mu(B)\}.$$

Definition 1.32. A function $\varpi : \mathcal{B}_X \to \mathbb{R}^+$ is said to be a measure of weak noncompactness in X if it is subject the following conditions

1. The family $ker\varpi = \{M \in \mathcal{B}_X, \varpi(M) = 0\}$ is nonempty and $ker\varpi \subset \mathfrak{W}_X$.

2. $M \subset N \implies \varpi(M) \leq \varpi(N)$.

3. $\varpi(\overline{conv}(M)) = \varpi(M)$.
   $ker(\varpi)$ is called the kernel of the measure of weak noncompactness ϖ.
   A consequence of the definition above is that $\varpi(conv(M)) = \varpi(M)$,
   $$\varpi(\overline{M}^w) = \varpi(\overline{M}) = \varpi(M)$$
   where $\overline{M}^w$ means the weak closure of M.
   When $ker(\varpi) = \mathfrak{W}_X$ the measure of weak noncompactness is called full. Quite often, the measure of weak noncompactness also satisfies:

4. the maximum property, this is, for every $x \in X$, $\varpi(M \cup \{x\})) = \varpi(M)$.

5. $\varpi(\lambda M + (1-\lambda)N) \leq \lambda\varpi(M) + (1-\lambda)\varpi(N)$ for $\lambda \in [0, 1]$.

6. If (M_n) is a sequence of closed sets from $\mathcal{B}_X$ such that $M_{n+1} \subset M_n$ for $n = 1, 2, ...$ and if $\lim\limits_{n\to\infty} \varpi(M_n) = 0$, then the set $M_\infty = \cap_{n=1}^{\infty} M_n$ is nonempty.

It is clear that if a measure of weak noncompactness satisfies property (6), M_∞ belongs to $ker(\varpi)$ because $M_\infty \subseteq M_n$ for each $n \in N$ and $\lim_{n\to\infty} \varpi(M_n) = 0$.

Example. The measure of weak noncompactness of De Blasi in Banach space is defined in the following way:

$$\beta(X) = \inf\{r > 0 : \text{there exists a set } W \in \mathfrak{W}_X \text{ such that } M \subset W + \overline{B}_r\},$$

for all $M \in \mathcal{B}_X$ with $\overline{B}_r$ is the closed ball centered at 0 with radius r.

Definition 1.33. Let X be a Banach space and consider $\mu(.)$ a measure of (weak) noncompactness on X. If M is a nonempty subset of X and $A : M \to M$ is a mapping,

(a) Given $k \in [0, 1)$, the mapping A is called μ-contractive if $\mu(A(N)) \leq k\mu(N)$ for all bounded subset N of M.

(b) The mapping A is called μ-condensing if $\mu(A(N)) < \mu(N)$ for all bounded subset N of M with $\mu(N) > 0$.

1.5 Some fixed point theorems in Banach space using MNC

1.5.1 Darbo's fixed point theorem

The following fixed point theorem is a version of classical fixed point for Lipschitz maps in the context of measure of noncompactness.

Theorem 1.9 (**Darbo's fixed point theorem** [1]). Let M be a nonempty, bounded, closed and convex subset of a Banach space X and let $A : M \to M$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that

$$\mu(A(N)) \leq k\mu(N) \tag{1.1}$$

for any nonempty subset N of M, where μ is a measure of noncompactness defined in X. Then A has a fixed point in the set M.

Proof. We consider a decreasing sequence $(M_n)_n$ defined by

$$M_0 = M \ \text{ and } \ M_n = Conv(A(M_{n-1})), \text{ for all } n \geq 1.$$

If it exists a natural number n_0 such that $\mu(M_{n_0}) = 0$, so M_{n_0} is relatively compact and since $A(M_{n_0}) \subseteq M_{n_0}$. By Schauder fixed point theorem 1.5, A admits a fixed point.
On the other hand, suppose that $\mu(M_n) > 0$ for all $n \geq 0$.
Using Eq. (1.1) we obtain

$$\begin{aligned} \mu(M_{n+1}) &= \mu(Conv(A(M_n))) \\ &= \mu(A(M_n)) \leq k\mu(M_n). \end{aligned} \tag{1.2}$$

Since $k < 1$, so

$$\mu(M_{n+1}) < \mu(M_n), \quad \forall n \geq 0.$$

As a result, $(\mu(M_n))_n$ is a decreasing sequence of positive real numbers.
Then there exists $r > 0$ such that

$$\lim_{n\to\infty} \mu(M_n) = r. \tag{1.3}$$

So, by Eq. (1.2) we get

$$\lim_{n\to\infty} \mu(M_{n+1}) \leq \lim_{n\to\infty} k\mu(M_n),$$

so

$$0 \leq r \leq kr,$$

implies that $r = 0$. So

$$\lim_{n\to\infty} \mu(M_n = 0).$$

Now, since

$$M_{n+1} \subseteq M_n \text{ and } A(M_n) \subseteq M_n.$$

According to property 5 of measure of noncompactness, we will conclude that $M_\infty = \cap_{n=1}^{\infty} M_n$ is a nonempty, closed, convex subset of M and invariant under the operator A, and therefore it belongs to $ker\mu$. Therefore, by Schauder's theorem 1.5, A admits a fixed point in M. □

1.5.2 Sadoveskii's fixed point theorem

Theorem 1.10 (Sadoveskii's fixet point theorem 1967 [28]). Let M be a nonempty, bounded, closed and convex subset of a Banach space X, and let $A : M \to M$ be a continuous operator. If A is a μ-condensing, where μ is a measure of noncompactness which has the maximum property, then A has at least one fixed point, and:

$$\mu(Fix(A)) = 0.$$

Proof. Let us choose a point $x^* \in M$ and denote by $\mathcal{S}$ the class of all closed and convex subsets K of M such that $m \in K$ and $A(K) \subset K$; that is:

$$\mathcal{S} = \{K \subset M, K \text{ is closed, convex and } A\text{-invariant with } x^* \in K\}.$$

Also set

$$B = \bigcap_{K \in \mathcal{S}} K,$$

and

$$C = \overline{conv}(A(B) \cup \{x^*\}).$$

Obviously $\mathcal{S} \notin \emptyset$ as $x^* \in \mathcal{S}$ and $B \notin \emptyset$ as $x^* \in B$.
Furthermore, we have:

$$A(B) = A(\bigcap_{K \in \mathcal{S}} K) \subset \bigcap_{K \in \mathcal{S}} A(K) \subset \bigcap_{K \in \mathcal{S}} K = B,$$

and so, we have

$$A : B \to B$$

Now we want to show that $B = C$. Indeed, since $x^* \in B$, $A(B) \subset B$ and B is closed and convex; it follows that $C = \overline{conv}(A(B) \cup \{x^*\}) \subset B$. This implies $A(C) \subset A(B) \subset C$ and so, $C \in \mathcal{S}$, and hence $B \subset C$. Therefore the properties of μ now imply that

$$\mu(B) = \mu(C) = \mu(A(B) \cup \{x^*\}) = \max\{\mu(A(B)), \mu(\{x^*\})\} = \mu(A(B)).$$

Since A is μ-condensing, it follows that $\mu(B) = 0$ so, B is compact. Obviously B is also convex. Thus from Schauder fixed point theorem there is a fixed point for the mapping $A : M \to M$.
And since we have $A(Fix(A)) = Fix(A)$, and μ is condensing then $\mu(Fix(A)) = 0$. □

1.6 Locally convex spaces

1.6.1 Basic tools in LCS

Definition 1.34. A topological vector space X is locally convex if there is a basis of neighborhoods in X consisting of convex sets.

In applications, it is often useful to define a locally convex space using a seminorm system.

Definition 1.35 (Seminorm). A seminorm is a function on a vector space X, denoted $|.|_p$ $(p \in \Lambda)$, such that the following conditions hold for all v and w in X, and any scalar λ.

1. $|v|_p \geq 0$, (Nonnegativity)
2. $|\lambda v|_p = |\lambda||v|_p$, (Absolute homogeneity)
3. $|v + w|_p \leq |v|_p + |w|_p$. (Subadditivity)

Remark 1.6.
- A semi-normed space is a vector space endowed with a nonempty family of seminorms.
- The family of seminorms is denoted by $\{|.|_p\}_{p\in\Lambda}$.

Example.
- $|(v, w)|_p = |v|$ on $\mathbb{R}^2$ is a seminorm, indeed
  1. $|(v, w)|_p = |v| \geq 0$ always, for all $(v, w) \in \mathbb{R}^2$,
  2. $|\lambda(v, w)|_p = |(\lambda v, \lambda w)|_p = |\lambda v| = |\lambda||v| = |\lambda||(v, w)|_p$, for any scalar λ, and $(v, w) \in \mathbb{R}^2$,
  3. $|(v, w) + (x, y)|_p = |(v + x, w + y)|_p = |v + x| \leq |v| + |x| = |(v, w)|_p + |(x, y)|_p$, for all $(v, w), (x, y) \in \mathbb{R}^2$.

Definition 1.36. A locally convex space is defined to be a vector space X along with a family of seminorms $\{|.|_p\}_{p\in\Lambda}$ on X.

As an example of locally convex topological space we define a Fréchet space.

Definition 1.37. A Fréchet space is a complete Hausdorff topological vector space whose topology may be given by a countable family of seminorms. it is a generalization of Banach space.

Remark 1.7. As with other topological vector spaces, a locally convex space (LCS, in short) is often assumed to be Hausdorff.

Definition 1.38. [16] Let X be a Hausdorff locally convex topological vector space. A mapping $A : X \to X$ is said to be a $\{|.|_p\}_{p\in\Lambda}$-contraction if for each $p \in \Lambda$, there exists $\alpha_p \in [0, 1)$ such that for all $x_1, x_2 \in X$

$$|Ax_1 - Ax_2|_p \leq \alpha_p |x_1 - x_2|_p.$$

Definition 1.39. Let X be a Hausdorff locally convex topological vector space, we say that X satisfies the **Krein-Šmulian property** if the closed convex hull of each weakly compact set is weakly compact.

1.6.2 Weak topology in LCS

We define another weak topology note τ where τ is a weaker Hausdorff locally convex topology of X.

Definition 1.40. Let M be a nonempty subset of X. An operator $A : M \to X$ is said to be τ-closed on M if for each sequence $(x_n)_n \subset M$ such that $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$, then $x \in M$ and $y = Ax$.

Definition 1.41. [12] Let M be a nonempty subset of X, $A : M \to X$ be an operator. We say that A is τ-sequentially continuous on M if for each sequence $(x_n)_n \subset M$ with $x_n \xrightarrow{\tau} x$ and $x \in M$ we have that $Ax_n \xrightarrow{\tau} Ax$.

Remark 1.8. Clearly, every τ-sequentially continuous operator is τ-closed, but the converse is not true.

Remark 1.9.
- We denote by $\xrightarrow{\tau}$ the convergence in (X, τ) and by $\to$ the convergence in $(X, \{|.|\}_{p\in\Lambda})$.
- We mean by τ-compact, (resp. τ-closed) sets, compact, (resp. closed) sets with respect to the topology τ.

Definition 1.42. Let X be a Hausdorff locally convex topological vector space, we say that X has the **τ-Krein-Šmulian property**(τ-KS, in short) if the closed convex hull of a τ-compact set is τ-compact.

1.6.3 Some fixed point theorems in LCS

Now we provide an important result proved by Cain and Nashed in Hausdorff locally convex topological vector spaces [16].

Theorem 1.11. Let X be a Hausdorff locally convex topological vector space. M is a sequentially complete subset of X and the mapping $A : M \to M$ is a $\{|.|_p\}_{p\in\Lambda}$-contraction. Then, A has a unique fixed point $x \in M$, and $A^n\bar{x} \to x$ for every $\bar{x} \in M$.

Proof. Let $\bar{x} \in M$, and let $\mathcal{U}$ be the neighborhood system of the origin obtained from $\{|.|_p\}_{p\in\Lambda}$. Then for any given $U \in \mathcal{U}$, there exist a finite number of seminorms in $\{|.|_p\}_{p\in\Lambda}$, say $|.|_{p_1}, |.|_{p_2}, ..., |.|_{p_n}$ and $r_i > 0$, $i = 1, ..., n$ such that $U = \bigcap_1^n r_i V(|.|_{p_i})$ be given, where $V(|.|_p) = \{x : |x|_p < 1\}$. For any $y \in M$ and $k \geq 1$, we have

$$|A^k y - y|_{p_i} \leq (1 - \alpha_{p_i})^{-1} |Ay - y|_{p_i}, \quad i = 1, 2, \cdots, n.$$

Choose m sufficiently large to insure that

$$\alpha_{p_i}^m (1 - \alpha_{p_i})^{-1} |A\bar{x} - \bar{x}|_{p_i} \leq r_i \text{ for } i = 1, 2, \cdots, n.$$

Then for $y = A^m x$, we have

$$\begin{aligned} |A^{m+k}\bar{x} - A^m\bar{x}|_{p_i} &\leq (1 - \alpha_{p_i})^{-1} |A^{m+1}\bar{x} - A^m\bar{x}|_{p_i} \\ &\leq \alpha_{p_i}^m (1 - \alpha_{p_i})^{-1} |A\bar{x} - \bar{x}|_{p_i} \leq r_i. \end{aligned}$$

Thus $(A^k\bar{x})$ is a Cauchy sequence in M and so converges to a point x in M. Clearly $Ax = x$, and uniqueness of the fixed point follows as usual since X is Hausdorff. □

The theorem 1.6 was generalized to locally convex topological vector spaces by Tychonoff in 1935.

Theorem 1.12 (Tychonoff theorem [29]). If M is a nonempty compact convex subset of a locally convex topological vector space X and $A : M \to M$ is a continuous map, then A has a fixed point.

1.6.4 Measure of (weak) noncompactness in LCS

Now, let us consider the following axiomatic definition of a family of measures of noncompactness in a Hausdorff locally convex vector space.

Definition 1.43. A family of function $\phi_{p_\tau} : \mathcal{B}(X) \to \mathbb{R}^+, (p \in \Lambda)$ is said to be a Φ^τ_Λ-measures of noncampactness in X (Φ^τ_Λ-MNC, in short) if for each $p \in \Lambda$, it satisfies the following condition:

(i) $\phi_{p_\tau}(\overline{conv}^\tau(M)) \leq \phi_{p_\tau}(M)$ for each $M \in \mathcal{B}(X)$, where $\overline{conv}^\tau(M)$ is the closure of the convex hull of M in (X, τ),

(ii) $M_1 \subseteq M_2 \implies \phi_{p_\tau}(M_1) \leq \phi_{p_\tau}(M_2)$, where M_1, $M_2 \in \mathcal{B}(X)$,

(iii) $\phi_{p_\tau}(\{x\} \cup M) = \phi_{p_\tau}(M)$ for any $x \in X$ and $M \in \mathcal{B}(X)$,

(iv) $\phi_{p_\tau}(M) = 0$ implies M is relatively τ-compact in X, and

(v) if $(M_n)_n$ is a sequence of τ-closed sets of $\mathcal{B}(X)$ such that $M_{n+1} \subset M_n$, $n = 1, 2, ...$ and $\lim_{n\to\infty} \phi_{p_\tau}(M_n) = 0$ for each $p \in \Lambda$, then $M_\infty = \cap_{n=1}^\infty M_n$ is nonempty relatively τ-compact subset of X.

The family Φ^τ_Λ-MNC is called:

(vi) Positively homogeneous, if for each $p \in \Lambda$, $\phi_{p_\tau}(\lambda M) = \lambda\phi_{p_\tau}(M)$, $\lambda > 0$, where $M \in \mathcal{B}(X)$.

(vii) Subadditive, if for each $p \in \Lambda$, $\phi_{p_\tau}(M_1 + M_2) \leq \phi_{p_\tau}(M_1) + \phi_{p_\tau}(M_2)$, where $M_1, M_1 \in \mathcal{B}(X)$.

Definition 1.44. Let M be a nonempty subset of a Hausdorff locally convex vector space X and let $\Phi^\tau_\Lambda := \{\phi_{p_\tau}, p \in \Lambda\}$ be a family of Φ^τ_Λ-MNC in X. An operator $A : M \to M$ is said to be a Φ^τ_Λ-contraction if for any bounded subset S of M, $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$, there exists a constant $\beta_p \in [0, 1)$ such that $\phi_{p_\tau}(A(S)) \leq \beta_p\phi_{p_\tau}(S)$. The operator A is called Φ^τ_Λ-condensing if for any bounded subset S of M, $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$ such that $\phi_{p_\tau}(S) > 0$, $\phi_{p_\tau}(A(S)) < \phi_{p_\tau}(S)$.

Definition 1.45. Let X satisfy the Krein-$\check{S}$mulian property. A family of functions $\omega_\rho : \mathcal{B}(X) \to \mathbb{R}^+(\rho \in \Lambda)$ is said to be a family of the measures of weak noncompactness of X if this family satisfies the following conditions:

1. The family $\ker(\omega_\rho) := \{M \in \mathcal{B}(X) : \omega_\rho(M) = 0 \text{ for all } \rho \in \Lambda\}$ is nonempty and $\ker(\omega_\rho)$ is contained in the subfamily consisting of all relatively weakly compact sets of X,
2. $N \subseteq M \Rightarrow \omega_\rho(N) \leq \omega_\rho(M)$ for each $\rho \in \Lambda$, where $M, N \in \mathcal{B}(X)$,
3. $\omega_\rho(\overline{\text{conv}}(M)) = \omega_\rho(M)$ for each $\rho \in \Lambda$, where $\overline{\text{conv}}(M)$ is the closed convex hull of $M \in \mathcal{B}(X)$,
4. $\omega_\rho(\lambda M + (1-\lambda)N) \leq \lambda\omega_\rho(M) + (1-\lambda)\omega_\rho(N)$ for each $\rho \in \Lambda, \lambda \in [0,1]$ and $M, N \in \mathcal{B}(X)$,
5. if $(M_n)_{n=1}^{\infty}$ is a decreasing sequence of nonempty, bounded, and weakly closed subsets of X with $\lim_{n\to\infty} \omega_\rho(M_n) = 0$ for each $\rho \in \Lambda$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker(\omega_\rho)$ described in (1) is called the kernel of the measure of weak noncompactness ω_ρ. Note that the intersection M_∞ from (5) belongs to $\ker(\omega_\rho)$ since we have $\omega_\rho(M_\infty) \leq \omega_\rho(M_n)$ for each $\rho \in \Lambda$ and all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \omega_\rho(M_n) = 0$.

Example. The family of measures of weak noncompactness in the Fréchet space $L^1_{loc}(\mathbb{R}^+)$ may be defined by:

$$\omega_T(\Omega) = \lim_{\varepsilon\to 0} \left\{ \sup_{x\in\Omega} \left\{ \sup \left[\int_D |x(t)|dt : D \subset [0,T], T > 0, m(D) \leq \varepsilon \right] \right\} \right\} \tag{1.4}$$

for all bounded subset Ω of $L^1_{loc}(\mathbb{R}^+)$, and $m(.)$ denote the Lebesgue measure.

1.6.5 Angelic space

The term "angelic space" was introduced by Fremlin.

Definition 1.46. [18] A topological (Hausdorff) space X is called angelic (or has countably determined compactness) if for every relatively countably compact subset M of X, the following holds:

(i) M is relatively compact.

(ii) For each $x \in \overline{M}$, there is a sequence in M which converges to x.

Example.
- Any metric space is angelic.
- Banach spaces X with the weak topology.
- If X is an angelic space, then X endowed with any stronger regular topology is also angelic.

Theorem 1.13. [4] If X is an angelic space, and $M \subset X$, then the following assertions are equivalent:

1. M is countably compact.
2. M is sequentially compact.
3. M is compact.

Remark 1.10. **(i)** All metrizable locally convex spaces endowed whith τ are angelic.

(ii) If X is anglic, then any τ-sequentially continuous map on a τcompact set is τ-continuous.

Chapter 2

Fixed Point Theory Under Hausdorff Weak Topology

Throughout this chapter, X will denote a Hausdorff locally convex topological vector space, and $\{|.|_p\}_{p\in\Lambda}$ a family of seminorms which generates the topology of X. In the first section, let the weak topology of X. And in the second section, let τ is weaker Hausdorff locally convex vector topology of X.

2.1 Fixed point theory for ws-compact maps

Theorem 2.1. [30] Let M be a nonempty, closed, and convex subset of X, and let the Krein-Šmulian property be satisfied. Suppose that $A : M \to M$ is ws-compact such that $A(M)$ is relatively weakly compact, then A has at least one fixed point.

Proof. Let $\mathcal{N} := \overline{conv}(A(M))$. Since M is closed and convex satisfying $A(M) \subseteq M$, since the $\mathcal{N} = \overline{conv}(A(M)) \subseteq \overline{conv}(M) = M$ and therefore $A(\mathcal{N}) \subseteq A(M) \subseteq \mathcal{N}$.
It is clear that $\mathcal{N}$ is weakly compact according to the relatively weak compactness of $A(M)$ and the Krein-Šmulian property. Moreover, $A(\mathcal{N})$ is relatively compact since A is ws-compact. Now applying the Schauder-Tychonoff fixed point theorem 1.12 for $A : \mathcal{N} \to \mathcal{N}$, we conclude that A has at least one fixed point $x \in \mathcal{N} \subseteq M$ such that $Ax = x$. $\square$

2.2 Fixed point theory for weakly sequentially continuous maps

Theorem 2.2 (**Arino, Gautier, and Penot theorem(1984)** [7]**).** Let X be a metrizable, locally convex topological vector space, and let M be a weakly compact, and convex subset of X. Then, any weakly sequentially continuous map $A : M \to M$ has at least a fixed point.

Proof. It is sufficiant to prove that A is weakly continuous, so that the Schauder-Tychonoff's fixed point theorem can be applied. Now, for each weakly closed subset E of X, $A^{-1}(E)$ is sequentially closed in M, hence weakly compact by using Eberlin-Šmulian's theorem(Theorem 1.2), and $A^{-1}(E)$ is weakly closed. Hence, A is weakly continuous. □

Theorem 2.3. [2] Let X be a Banach space, M be a nonempty closed convex subset of X, and $A : M \to M$ a sequentially weakly continuous map. If $A(M)$ is relatively weakly compact, then A has a fixed point in M.

Proof. Let $\mathcal{N} = \overline{conv}(A(M))$, the closed convex hull of $A(M)$. Because $A(M)$ is relatively weakly compact, then $\mathcal{N}$ is a weakly compact convex subset of X. On the other hand, $A(\mathcal{N}) \subset A(M) \subset \overline{conv}(A(M)) = \mathcal{N}$, i.e., A maps $\mathcal{N}$ into itself. Because A is sequentially weakly continuous, it follows using Theorem2.2 that A has at least one fixed point in $\mathcal{N}$. □

The next result is the analogue of Darbo's fixed point result for strongly contractive maps.

Theorem 2.4. [24] Let M be a nonempty, bounded, convex, closed set in a Banacb space X. Assume $A : M \to M$ is weakly sequentially continuous and μ-contractive. Then A has a fixed point.

Proof. Let

$$S_1 = M \text{ and } S_{n+1} = \overline{conv}(A(S_n)), \quad n = 1, 2...$$

Notice

$$\mu(S_2) = \mu(\overline{conv}(A(S_1))) = \mu(A(S_1)) \leq k\mu(S_1) \text{ and } S_2 \subseteq \overline{conv}(M) = M = S_1$$

It is easy to see (via induction) that

$$S_{n+1} \subseteq S_n \text{ and } \mu(S_{n+1}) \leq k^n \mu(S_1), \text{ for } n = 1, 2, ...$$

Since $0 < k < 1$, we have $\mu(S_n) \to 0$ as $n \to \infty$. Therefore, $\cap_{n=1}^{\infty} S_n = S_\infty$, is nonempty. Also, S_∞, is weakly closed and convex, since each S_n, is (recall a convex subset of a locally convex space is closed iff it is weakly closed). Also, since $\mu(S_\infty) \leq \mu(S_n) \to 0$ as $n \to \infty$, we have $\mu(S_\infty) = 0$.
Thus, S_∞, is weakly compact. Also, since

$$A(S_n) \subseteq A(S_{n-1}) \subseteq \overline{conv}(A(S_{n-1})) = S_n, \text{ for all } n,$$

we have $A : S_\infty \to S_\infty$. Theorem 2.2 implies that A has a fixed point in $S_\infty \subseteq S_1 = M$. □

Theorem 2.5. [3] Let M be a nonempty, convex closed set in a Banach space X. Assume $A : M \to M$ is a weakly sequentially continuous map and condensing with respect to μ. In addition, suppose that $A(M)$ is bounded. Then, A has a fixed point.

Proof. Let $x_0 \in M$. We consider the family $\mathcal{S}$ of all closed bounded convex subsets D of M such that $x_0 \in D$ and $A(D) \subset D$. Obviously $\mathcal{S}$ is nonempty, since $\overline{conv}(A(M) \cup \{x_0\}) \in \mathcal{S}$. We denote $K = \bigcap_{D \in \mathcal{S}} D$. We have that K is closed convex and $x_0 \in K$. If $x \in K$, then $Ax \in D$ for all $D \in \mathcal{S}$ and hence $A(K) \subset K$. Therefore, we have that $K \in \mathcal{S}$. We will prove that K is weakly compact. Denoting by $\mathcal{N} = \overline{conv}(A(K) \cup \{x_0\})$, we have $\mathcal{N} \subset K$, which implies that $A(\mathcal{N}) \subset A(K) \subset \mathcal{N}$. Therefore $\mathcal{N} \in \mathcal{S}$, $K \subset \mathcal{N}$. Hence $K = \mathcal{N}$. Since K is weakly closed, it suffices to show that K is relatively weakly compact. If $\mu(K) > 0$, we obtain

$$\mu(K) = \mu(\overline{conv}(A(K) \cup \{x_0\})) \leq \mu(A(K)) < \mu(K)$$

which is a contradiction. Hence, $\mu(K) = 0$ and so K is relatively weakly compact. Now, A is a weakly sequentially continuous of K into itself. From Theorem 2.2, A has a fixed point in $K \subset M$. □

2.3 Fixed point theorems in angelic spaces for τ-sequentially continuous maps

Now, we state a generalization of Theorem 2.2

Theorem 2.6. Let (X, τ) be an angelic space and has the τ-KS property. Let M be a nonempty, convex, and τ-compact subset of X. If $A : M \to M$ is a τ-sequentially continuous operator, then, A has a fixed point in M.

Proof. As M is a τ-compact and A is τ-sequentially continuous operator on a τ-compact subset of angelic space, then A is a τ-continuous. By Tychonoff fixed point theorem 1.12, A has a fixed point in M. □

This next theorem is a generalization of Theorem 2.3

Theorem 2.7. Assume that (X, τ) is angelic and has the τ-KS property. Let M be a nonempty, τ-closed convex subset of X and $A : M \to M$ be a sequentially continuous operator. If $A(M)$ is relatively τ-compact, then A has a fixed point in M.

Proof. Let $\mathcal{N} = \overline{conv}^{\tau}(A(M))$. Clearly, $A(\mathcal{N}) \subset \mathcal{N}$. Since $A(M)$ is relatively τ-compact and X satisfy the τ-KS property, then $\mathcal{N}$ is relatively τ-compact. By Theorem 2.6, there exist $x \in \mathcal{N}$ such that $Ax = x$. Which achieves the proof. □

As a consequence of the above theorem we can formulate the following results.

Corollary 2.1. Suppose that (X, τ) is an angelic space and has the τ-KS property. Let M be a nonempty, τ-closed convex subset of X. Let $A : M \to M$ be a τ-closed operator such that $A(M)$ is relatively τ-compact. Then, A has a fixed point in M.

Proof. Let $\mathcal{N} = \overline{conv}^{\tau}(A(M))$. By the same arguments used in the proof of Theorem 2.7, $\mathcal{N}$ is relatively τ-compact and $A(\mathcal{N}) \subset \mathcal{N}$. Now, we prove that A is τ-sequentially continuous. For this purpose, let $(x_n)_n$ be a sequence in $\mathcal{N}$, such that $x_n \xrightarrow{\tau} x$. Since $(Ax_n)_n \subset A(\mathcal{N}) \subset \mathcal{N}$, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $Ax_{n_k} \xrightarrow{\tau} y$. Taking into account that A is τ-closed, and $x_{n_k} \xrightarrow{\tau} x$, we get $y = Ax$. Next, we will show that

$$Ax_n \xrightarrow{\tau} Ax.$$

Suppose the contrary, then there exists a τ-neighborhood V^{τ} of Ax and a subsequence $(x_{n_i})_i$ of $(x_n)_n$ such that $Ax_{n_i} \notin V^{\tau}$ for all i. Moreover, $x_{n_i} \xrightarrow{\tau} x$, then arguing as before, there exists a subsequence $(x_{n_{i_k}})_k$ of $(x_{n_i})_i$ such that

$$Ax_{n_{i_k}} \xrightarrow{\tau} Ax,$$

which is absurd, since $Ax_{n_{i_k}} \notin V^\tau$ for all k. This prove that A is τ-sequentially continuous. Then, by Theorem 2.7, we deduce that A has a fixed point in M.

□

Now, we present a new variant of Darbo's fixed point theorem for τ-sequentially continuous maps involving a family of Φ_Λ^τ-MNC.

Corollary 2.2. Assume that (X, τ) is angelic and has the τ-KS property. Let M be a nonempty, bounded, τ-closed, and convex subset of X and $\Phi_\Lambda^\tau = \{\phi_{p_\tau},\ p \in \Lambda\}$ be a family of Φ_Λ^τ-MNC in X. Let $A : M \to M$ be a τ-sequentially continuous operator. If A is Φ_Λ^τ-contractive, then A has a fixed point in M.

Proof. Let us define the sequence $(M_n)_n$ such that $M_1 = M$ and $M_{n+1} = \overline{conv}^\tau(A(M_n))$. Clearly, the sequence $(M_n)_n$ consists of nonempty τ-closed, bounded, convex, and decreasing subsets of M. Let $p \in \Lambda$, using the property (i) of Definition 1.43 we obtain

$$\begin{aligned}\phi_{p_\tau}(M_2) &= \phi_{p_\tau}(\overline{conv}^\tau(A(M_1))) \\ &\leq \phi_{p_\tau}(A(M_1)).\end{aligned}$$

Further, since A is Φ_Λ^τ-contractive, there exists $\beta_p \in [0, 1)$ such that

$$\phi_{p_\tau}(M_2) \leq \beta_p \phi_{p_\tau}(M_1).$$

Proceeding by induction we get

$$\phi_{p_\tau}(M_n) \leq \beta_p^{n-1} \phi_{p_\tau}(M),$$

and therefore $\lim_{n\to\infty} \phi_{p_\tau}(M_n) = 0$. By using the property (v) of Φ_Λ^τ-MNC, we infer that $M_\infty = \cap_{n=1}^\infty M_n$ is nonempty, τ-closed, convex, and relatively τ-compact subset of M. Moreover,

$$A(M_\infty) = A(\cap_{n=1}^\infty M_n) \subset \cap_{n=1}^\infty A(M_n) \subset M_\infty.$$

Accordingly, $A(M_\infty)$ is relatively τ-compact. Now, the use of Theorem 2.7 concludes the proof. □

Note that Corollary generalizes Theorem 2.4.

The next result is a generalization of Theorem 1.10 for strong condensing map

Corollary 2.3. Assume that (X,τ) is angelic and has the τ-KS property. Let M be a nonempty, τ-closed, bounded, and convex subset of X and let $\Phi_\Lambda^\tau = \{\phi_{p_\tau},\ p \in \Lambda\}$ be a Φ_Λ^τ-MNC in X. Let $A : M \to M$ be a τ-sequentially continuous operator. If A is Φ_Λ^τ-condensing, then A has a fixed point in M.

Proof. Let $x_0 \in M$, we define the set

$$\mathcal{S} = \{D \subset X;\ x_0 \in D \subset M,\ D \text{ is bounded, convex, and } A(D) \subset D\}.$$

Clearly, $M \in \mathcal{S}$, then $\mathcal{S}$ is nonempty. Denote by $C = \bigcap_{D\in\mathcal{S}} D$. Obviously $x_0 \in C$ and C is a bounded convex subset of X, and $A(C) \subset C$. It follows that

$$\overline{conv}^\tau\{A(C) \cup \{x_0\}\} \subset C. \tag{2.1}$$

Therefore,

$$A(\overline{conv}^\tau\{A(C) \cup \{x_0\}\}) \subset A(C) \subset \overline{conv}^\tau\{A(C) \cup \{x_0\}\}.$$

Hence,

$$\overline{conv}^\tau\{A(C \cup \{x_0\})\} \in \mathcal{S}.$$

Consequently,

$$C \subset \overline{conv}^\tau\{A(C) \cup \{x_0\}\}. \tag{2.2}$$

When combining Eqs. (2.1) and (2.2), we get $\overline{conv}^\tau\{A(C) \cup \{x_0\}\} = C$. Using the properties of Φ_Λ^τ-MNC , we have

$$\phi_{p_\tau}(C) = \phi_{p_\tau}(\overline{conv}^\tau\{A(C) \cup \{x_0\}\}) \leq \phi_{p_\tau}(A(C) \cup \{x_0\}) = \phi_{p_\tau}(A(C)).$$

By the Φ_Λ^τ-condensibility of A, we obtain $\phi_{p_\tau}(C) = 0$, then C is relatively τ-compact and consequently $A(C)$ is relatively τ-compact. So by Theorem 2.7, we deduce that there exists $x \in C$ such that $Ax = x$. □

Remark 2.1. As an easy consequence of Corollary 2.3, we may recapture Theorem 2.5.

Basing on Theorem 2.7, we prove the following fixed point theorem for the sum of a τ-sequentially continuous and a $\{|.|_p\}_{p\in\Lambda}$-contraction mapping.

Theorem 2.8. Let X be a sequentially complete Hausdorff locally convex topological vector space. Assume that (X, τ) is angelic and has the τ-KS property. Suppose that M is a nonempty, bounded, τ-closed, and convex subset of X. Let $\{\phi_{p_\tau},\ p \in \Lambda\}$ be a Φ_Λ^τ-MNC in X. Consider $A : M \to X$ and $B : X \to X$ two operators such that for all $p \in \Lambda$:

(i) A is τ-sequentially continuous,

(ii) B is a $\{|.|_p\}_{p\in\Lambda}$-contraction and is τ-closed,

(iii) there exists $\lambda_p \in [0, 1)$ such that $\phi_{p_\tau}(A(S) + B(S)) \leq \lambda_p \phi_{p_\tau}(S)$ for all $S \subset M$, and

(iv) $(x = Bx + Ay,\ y \in M) \implies x \in M$.

Then, there exists $x \in M$ such that $x = Ax + Bx$.

Proof. Let y be a fixed in M, the map which assigns to each $x \in X$ the value $Bx + Ay$ defines a $\{|.|_p\}_{p\in\Lambda}$-contraction from X into X for all $p \in \Lambda$. So, by Theorem 1.11, the equation $x = Bx + Ay$ has a unique solution $x \in X$. By assumption (iv), it follows that $x \in M$. Hence, $x = (I - B)^{-1}Ay \in M$ which, accordingly implies the inclusion

$$(I - B)^{-1}A(M) \subset M. \tag{2.3}$$

Now, define the sequence $(M_n)_n$ of subsets of M by:

$$M_1 = M \text{ and } M_{n+1} = \overline{conv}^\tau((I - B)^{-1}A(M_n)). \tag{2.4}$$

We claim that the sequence $(M_n)_n$ satisfies the conditions of property (v) of Φ_Λ^τ-MNC. Indeed, it is clear that the sequence $(M_n)_n$ consists of nonempty τ-closed, convex and bounded subsets of M. By Eq. (2.3) one sees that it is also decreasing. Now, using Eq. (2.4) and the following equality

$$(I - B)^{-1}A = A + B(I - B)^{-1}A, \tag{2.5}$$

we obtain

$$(I - B)^{-1}A(M_n) \subseteq A(M_n) + B\overline{conv}^\tau((I - B)^{-1}A(M_n)) \subseteq A(M_n) + B(M_n). \tag{2.6}$$

kipping in mind Eq. (2.6), and the properties (i) and (ii) of Definition 1.43, we obtain

$$\begin{aligned}\phi_{p_\tau}(M_{n+1}) &= \phi_{p_\tau}(\overline{conv}^\tau(I-B)^{-1}A(M_n)) \\ &\leq \phi_{p_\tau}((I-B)^{-1}A(M_n)) \\ &\leq \phi_{p_\tau}(A(M_n)+B(M_n)).\end{aligned}$$

Further, by assumption(iii), we deduce that

$$\phi_{p_\tau}(M_{n+1}) \leq \lambda_p \phi_{p_\tau}(M_n).$$

Proceeding by induction we get

$$\phi_{p_\tau}(M_n) \leq \lambda_p^{n-1}\phi_{p_\tau}(M),$$

and therefore $\lim_{n\to\infty}\phi_{p_\tau}(M_n)=0$. Now, applying the property (v) of Φ_Λ^τ-MNC we infer that $M_\infty = \cap_{n=1}^{\infty} M_n$ is nonempty, τ-closed, convex, and relatively τ-compact subset of M. on the other hand one can easily verify that $(I-B)^{-1}A(M_n) \subset M_n$ for all n, thus we obtain $(I-B)^{-1}A(M_\infty) \subset M_\infty$. Consequently, $(I-B)^{-1}A(M_\infty)$ is relatively τ-compact. Next, let us show that $(I-B)^{-1}A : M_\infty \to M_\infty$ is τ-sequentially continuous. To this purpose, let $(x_n)_n$ be a sequence in M_∞ such that $x_n \xrightarrow{\tau} x$ in M_∞. Since

$$((I-B)^{-1}Ax_n)_n \subset (I-B)^{-1}A(M_\infty),$$

there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that

$$(I-B)^{-1}Ax_{n_k} \xrightarrow{\tau} y \tag{2.7}$$

Going back to Eq. (2.5), using the τ-sequential continuity of the operator A and Eq. (2.7), it follows that

$$B(I-B)^{-1}Ax_{n_k} \xrightarrow{\tau} y - Ax. \tag{2.8}$$

Together with Eqs. (2.7), (2.8), and assumption (ii), we infer that $By = y - Ax$, hence $y = (I-B)^{-1}Ax$. Now, we claim that

$$(I-B)^{-1}Ax_n \xrightarrow{\tau} (I-B)^{-1}Ax.$$

Suppose that contrary, then exists a τ-neighborhood V^τ of $(I-B)^{-1}Ax$ and a subsequence $(x_{n_j})_j$ of $(x_n)_n$ such that $(I-B)^{-1}Ax_{n_j} \notin V^\tau$ for all j. Moreover, $x_{n_j} \xrightarrow{\tau} x$, then arguing as before, we can extract a subsequence $(x_{n_{j_k}})_k$ of $(x_{n_j})_j$ such that

$$(I-B)^{-1}Ax_{n_{j_k}} \xrightarrow{\tau} (I-B)^{-1}Ax,$$

which is absurd, since $(I-B)^{-1}Ax_{n_{j_k}} \notin V^\tau$ for all k.
Finally, $(I-B)^{-1}A$ is τ-sequentially continuous. Now, the use of Theorem 2.7 yields a point $x \in M$ such that $x = Ax + Bx$. □

Note that condition (iii) of Theorem 2.8, can be weakend. For this purpose let us recall the following definition.

Definition 2.1. [19] Let Q be the class of function $\gamma : \mathbb{R}^+ \to [0,1)$ which satisfies the following condition: $\gamma(t_n) \to 1$ implies $t_n \to 0$.

Theorem 2.9. Let X be a sequentially complete Hausdorff locally convex topological vector space. Assume that (X,τ) is angelic and has the τ-KS property. Suppose that M is a nonempty, bounded, τ-closed, and convex subset of X. Consider $A : M \to X$ and $B : X \to X$ be two operators such that for all $p \in \Lambda$:

(i) A is τ-sequentially continuous,

(ii) B is $|.|_{p_{p\in\Lambda}}$-contraction and is τ-closed,

(iii) $\phi_{p_\tau}(A(S)+B(S)) \leq \gamma(\phi_{p_\tau}(S))\phi_{p_\tau}(S)$, for all $S \subset M, \gamma \in Q, p \in \Lambda$, and

(iv) $(x = Bx + Ay,\ y \in M) \implies x \in M$.

Then, there exists $x \in M$ such that $x = Ax + Bx$.

Proof. In view of the proof of Theorem 2.8, it is sufficient to establish that the sequence $(M_n)_n$ defined in Eq. (2.4), satisfies the condition of property (v) of Definition 1.43. Using properties of Φ^τ_Λ-MNC and Eq. (2.6) we have

$$\begin{aligned}\phi_{p_\tau}(M_{n+1}) &= \phi_{p_\tau}(\overline{conv}^\tau((I-B)^{-1}A(M_n))) \\ &\leq \phi_{p_\tau}((I-B)^{-1}A(M_n)) \\ &\leq \phi_{p_\tau}(A(M_n)+B(M_n)).\end{aligned} \tag{2.9}$$

By assumptions (iii) and Eq. (2.9), we get

$$\begin{aligned}\phi_{p_\tau}(M_{n+1}) &\leq \gamma(\phi_{p_\tau}(M_n))\phi_{p_\tau}(M_n)\\ &\leq \phi_{p_\tau}(M_n),\end{aligned} \tag{2.10}$$

this amplies that $(\phi_{p_\tau}(M_n)_n)$ is a positive decreasing sequence of real numbers. Thus, there is an $r \geq 0$ such that $\lim_{n\to\infty} \phi_{p_\tau}(M_n) = r$. Now, we prove that $r = 0$. To this end, suppose that $r \neq 0$ and $\phi_{p_\tau}(M_n) \neq 0$ for $n \geq 1$. By Eq. (2.10), we obtain

$$\frac{\phi_{p_\tau}(M_{n+1})}{\phi_{p_\tau}(M_n)} \leq \gamma(\phi_{p_\tau}(M_n)) < 1,$$

therefore, $\lim_{n\to\infty} \gamma(\phi_{p_\tau}(M_n)) = 1$. Since $\gamma \in Q$, we get $\lim_{n\to\infty} \phi_{p_\tau}(M_n) = 0$. Using property (v) of Φ_Λ^τ-MNC, we deduce that $M_\infty = \cap_{n=1}^{\infty} M_n$ is nonempty, τ-closed, convex, and relatively τ-compact subset of M.

If there exists $k \geq 0$ such that $\phi_{p_\tau}(M_k) = 0$, this implies that M_k is relatively τ-compact. Arguing as in the proof of Theorem 2.8, we deduce that the operator $(I - B)^{-1}A : M_k \to M_k$ is τ-sequentially continuous, and $(I - B)^{-1}A(M_k)$ is relatively τ-compact. Then, Theorem 2.7 can be applied and this completes the proof. □

Chapter 3

Application to Nonlinear Functional Integral Equation

In this chapter we are going to use our findings proved in the last chapter to present some existence results for a nonlinear integral equation in the Lebesgue space $L^1(I)$ [9] and in the Fréchet space $L^1_{loc}(\mathbb{R}^+)$ [8]. At the beginning of each section in this chapter we provide some auxiliary notations and results which will be useful in our investigations.

3.1 Application to nonlinear integral equation in the Lebesgue space $L^1(I)$

In this section, we will study the solvability for the following nonlinear integral equation

$$x(t) = a(t) + \int_0^1 h(t,s)f(s,x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds, \quad t \in [0,1], \tag{3.1}$$

in the Lebesgue space L^1, where a, h, f, u and φ are given functions satisfying certain conditions. Let $I = [0,1]$ be a compact interval in $\mathbb{R}$ and denote by $L^1 = L^1(I)$ the space of Lebesgue integrable real functions on the interval I with the norm

$$\|.\| = \int_0^1 |x(t)|dt.$$

Let $\xi = \xi(I)$ be the set of all real functions, Lebesgue measurable on I and denote by $m(M)$ the Lebesgue measure of a measurable subset M of $\mathbb{R}$. We define in ξ the metric

ρ by the formula

$$\rho(x,y) = \inf\{a + m(\{s \in I : \|x(s) - y(s)|) \geq a\}) : a > 0\},$$

so, ξ becomes a complete metric space. The compactness in such a space is called "compactness in measure" and such sets have very nice properties when considered as subsets of L^p-spaces of integrable functions ($p \geq 1$).

It is known that the convergence in measure coincides with the convergence generated by the metric ρ, but the convergence in measure of a sequence $(x_n)_n$ in L^1 does not imply the weak convergence of $(x_n)_n$ and conversely. While, we have the following results [11].

Lemma 3.1. A sequence $(x_n)_n$ in L^1 converges strongly to $x \in L^1$ (i.e, converges in norm) if, and only if, $(x_n)_n$ converges in measure to x and is weakly compact.

Lemma 3.2. Let M be a bounded subset of the space L^1 consisting of all functions being a.e. nondecreasing (nonincreasing) on I. Then, M is compact in measure.

The following Lemma states the case when a class of weakly sequentially continuous operators coincides with the class of continuous ones in L^1.

Lemma 3.3. Let M be a bounded subset of the Lebesgue space L^1 which is compact in measure. If $A : M \to L^1$ is continuous, then it is also weakly sequentially continuous.

Now, we give some basic results concerning the superposition operator [6].

Definition 3.1. Assume that the function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions i.e., it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every function $x(t)$ being measurable on I, we may assign the function

$$F(x(t)) = f(t, x(t)), \quad t \in I.$$

The operator F is such a way is called the superposition (Nemytskii) operator generated by the function f.

Theorem 3.1. [6] Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. Then, the superposition operator F generated by f transforms the space L^1 into itself if, and only if,

$$|f(t,x)| \leq a(t) + b|x|,$$

for $t \in I$ and $x \in \mathbb{R}$, where $a(.)$ is a function from the space L^1 and $b > 0$. Moreover, the operator F is continuous on the space L^1.

An important nonlinear integral equation is the Urysohn equation defined in Eq. (3.2) by

$$U(x(t)) = \int_I u(t, s, x(s))ds, \tag{3.2}$$

where $u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. A particular case of the Urysohn equation is the Hammerstein integral equation Eq. (3.3)

$$H(x(t)) = \int_I h(t, s)x(s)ds, \tag{3.3}$$

whith $h : I \times I \to \mathbb{R}$. Now, we provide some conditions for continuity of integral operators U and H in the following results [21].

Theorem 3.2. [21] Let the function $u(t, s, x)$ and $k(t, s, x)$ satisfy the Carathéodory conditions. Let

$$|u(t, s, x)| \leq k(t, s, x), \quad t, s \in I, \quad x \in \mathbb{R}$$

and suppose that the integral operator

$$K(x(t)) = \int_I k(t, s, x(s))ds$$

act from L^p to L^q, $q > 0$, and is continuous. Then, the integral operator U also acts from the space L^p to the space L^q and is continuous.

Theorem 3.3. [21] let $h : I \times I \to \mathbb{R}$ be measurable with respect to both variables. Let the linear integral operator H with kernel $h(., .)$ map L^p into L^q. Then, the integral operator H is continuous.

Due to the works of Appell and De Pascale [5], we can express the De Blasi measure of weak noncompactness $\omega(.)$ in L^1 by the formula:

$$\omega(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |x(t)|dt : D \subset I, m(D) \leq \varepsilon \right] \right\} \right\}.$$

Now, we will discuss the solvability of Eq. (3.1) under the following assumptions:

$(\mathcal{H}_1)$ a is continuous and is decreasing on I.

($\mathcal{H}_2$) $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. There exists $l > 0$ such that

$$|f(t,x) - f(t,y)| \leq l|x-y|, \quad \text{for each} \quad t \in I, \quad x, y \in \mathbb{R}, \quad \text{and} \quad f(t,0) \in L^1.$$

($\mathcal{H}_3$) **(i)** $h : I \times I \to \mathbb{R}^+$ be measurable with respect to both variables and such that the integral operator H with the kernel $h(t,s)$ defined on L^1 by the formula

$$H(x(t)) = \int_0^1 h(t,s)x(s)ds. \tag{3.4}$$

(ii) The function $t \longmapsto h(t,s)$ is a.e. nondecreasing on I for almost all $t \in I$,

(iii) There exists a function $p \in L^1$ such that

$$h(t,s) \leq p(t),$$

($\mathcal{H}_4$) **(i)** $u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions, i.e.,u is measurable with respect to (t,s) for any $x \in \mathbb{R}$ and is continuous in x for almost all $(t,s) \in I \times I$.

(ii) The function $t \longmapsto u(t,s,x)$ is a.e. nondecreasing on I for almost all $s \in I$ and for each $x \in \mathbb{R}$.

(iii) $|u(t,s,x)| \leq k(t,s)(q(t) + b_1|x|)$ for $(t,s) \in I \times I$ and for $x \in \mathbb{R}$, where $q \in L^1$ and $b_1 \geq 0$.

(iv) $k : I \times I \to \mathbb{R}^+$ is measurable and such that the linear operator K generated by k maps L^1 into itself.

($\mathcal{H}_5$) $\varphi : [0,1] \to [0,1]$ is increasing, absolutely continuous such that there is a constant $b_2 > 0$ such that $\varphi'(t) \geq b_2$ for almost all $t \in I$.

($\mathcal{H}_6$) $\left(\frac{l}{b_2}\|p\| + b_1\|K\|\right) < 1$, where $\|K\|$ and $\|p\|$ denote the norm of the operator K and the function p respectively.

Theorem 3.4. Assume that $(\mathcal{H}_1) - (\mathcal{H}_6)$ hold true. Then, Eq. (3.1) has at least one solution in L^1.

Proof. We can rewrite Eq. (3.1) in the form $x(t) = Bx(t) + Ax(t)$, where

$$Bx(t) = a(t) + \int_0^1 h(t,s)f(t, x(\varphi(s)))ds,$$

and

$$Ax(t) = \int_0^1 u(t, s, x(s))ds.$$

We will prove that the operators A and B satisfy conditions of Theorem 2.8. Firstly, observe that the operator B can be written as

$$Bx(t) = a(t) + H(F(x(\varphi(t)))),$$

where F is the superposition operator generated by f, and H is the integral operator defined in Eq. (3.4). In view of assumption $(\mathcal{H}_2)$, we have

$$\begin{aligned} |f(t,x)| &\leq |f(t,x) - f(t,0)| + |f(t,0)| \\ &\leq |f(t,0)| + l(x). \end{aligned} \tag{3.5}$$

From Eq. (3.5) and Theorem 3.1, it follows that F transforms L^1 into itself and is continuous. Moreover, by $(\mathcal{H}_3)$(i)and Theorem 3.3, H is continuous on L^1. This fact together with assumption $(\mathcal{H}_1)$, yield that the operator B acts from L^1 to L^1 and is continuous. Now to see that B is a contraction, let $x, y \in L^1$, then

$$\begin{aligned} \|Bx - By\| &= \int_0^1 \left| \int_0^1 h(t,s)(f(t, x(\varphi(s))) - f(t, y(\varphi(s))))ds \right| dt \\ &\leq \int_0^1 \int_0^1 h(t,s)|f(t, x(\varphi(s))) - f(t, y(\varphi(s)))|dsdt \\ &\leq l \int_0^1 \int_0^1 p(t)|x(\varphi(s)) - y(\varphi(s))dsdt \\ &\leq l \int_0^1 \int_0^1 p(t)|x(\varphi(s)) - y(\varphi(s))| \frac{\varphi'(t)}{b_2} dsdt \\ &\leq \frac{l}{b_2} \int_0^1 p(t) \int_0^1 |x(v) - y(v)|dvdt. \end{aligned}$$

Thus,

$$\|Bx - By\| \leq \frac{l}{b_2} \|p\| \|x - y\|. \tag{3.6}$$

Using Eq. (3.6) and assumption $(\mathcal{H}_6)$ we deduce that B is a contraction. Further, by assumption $(\mathcal{H}_4)$ and Theorem 3.2 we infer that the operator A transforms L^1 into itself and is continuous. Now, let N_r denotes the subset of L^1 consisting of all functions $x = x(t)$ being a.e. nondecreasing on I such that $\|x\| \leq r$, where r is a nonnegative constant that will be defined later. Firstly, we prove that the operator $A + B$ maps N_r into itself. By assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_3)$(ii), $(\mathcal{H}_4)$(ii) and $(\mathcal{H}_5)$ we obtain that the operator $A+B$ is a.e. nondecreasing. Let $x \in N_r$, then

$$\begin{aligned}
\|Ax+Bx\| &= \int_0^1 |Ax(t)+Bx(t)|dt \\
&= \int_0^1 \left|a(t) + \int_0^1 h(t,s)f(t,x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds\right| dt \\
&\leq \int_0^1 \left(|a(t)| + \int_0^1 h(t,s)|f(t,x(\varphi(s)))|ds + \int_0^1 |u(t,s,x(s))|ds\right) dt \\
&\leq \|a\| + \int_0^1 \left(\int_0^1 p(t)(|f(t,0)| + l|x(\varphi(s))|)ds + \int_0^1 k(t,s)(q(t)+b_1|x(s)|)ds\right) dt \\
&\leq \|a\| + \int_0^1\int_0^1 p(t)|f(t,0)|dsdt + l\int_0^1\int_0^1 p(t)|x(\varphi(s))|\frac{\varphi'(t)}{b_2}dsdt \\
&+ \int_0^1\int_0^1 k(t,s)q(t)dsdt + b_1\int_0^1\int_0^1 k(t,s)|x(s)|dsdt \\
&\leq \|a\| + \|p\|\|f(t,0)\| + \frac{l}{b_2}\|p\|\|x\| + \|K\|\|q\| + b_1\|K\|\|x\| \\
&= \alpha + \beta\|x\|,
\end{aligned}$$

where $\alpha = \|p\|\|f(t,0)\| + \|K\|\|q\|$ and $\beta = \left(\frac{l}{b_2}\|p\| + b_1\|K\|\right)$. Let $r = \frac{\alpha}{1-\beta} > 0$, so for $x \in N_r$ we have

$$\|Ax+Bx\| \leq \alpha + \beta\|x\| = r. \tag{3.7}$$

Thus, the operator $A+B$ maps N_r into itself. Obviously, N_r is nonempty, bounded, and convex subset of L^1, and by Lemma 3.2 it is compact in measure. Next, we show that N_r is closed.For this purpose, let $(x_n)_n$ be a sequence in N_r converging to x, so $\|x_n - x\| \to 0$ then Lemma 3.1 implies that the sequence $(x_n)_n$ converges in measure to x and there exists a subsequence $(x_{n_j})_j$ which converges a.e. to x on I. Let us prove that the function x is nondecreasing. To see that let $t_1, t_2 \in I$ such that $t_1 \leq t_2$ we have $x_{n_j}(t_1) \leq x_{n_j}(t_2)$ for every j,

$$\begin{aligned}
\|x(t_1) - x(t_2)\| &= \|x(t_1) - x_{n_j}(t_1) + x_{n_j}(t_1) - x_{n_j}(t_2) + x_{n_j}(t_2) - x(t_2)\| \\
&\leq \|x(t_1) - x_{n_j}(t_1)\| + \|x_{n_j}(t_2) - x(t_2)\|.
\end{aligned}$$

Since $(x_{n_j})_j$ is convergent, then by Eq. (2.3), and for each $\varepsilon > 0$ we get, $x(t_1) - x(t_2) \leq \varepsilon$, which proves that $x(t_1) \leq x(t_2)$, hence N_r is closed.

By Eq. (3.7), and assumption $\mathcal{H}_1$, we can see that the operators A and B converts N_r into itself continuously. Keeping in mind Lemma 3.3, we conclude that the operators A and B are sequentially weakly continuous.

Now, we will verify assumption (iii) of Theorem 2.8. For this purpose let $S \subset N_r$ and $x \in S$. Further, let $\varepsilon > 0$ and take a measurable subset $D \subset [0,1]$ such that $m(D) \leq \varepsilon$ and $\varphi(D) \subset D$. We obtain that

$$\begin{aligned}
\int_D |Ax(t)+Bx(t)|dt &\leq \int_D \left|a(t) + \int_0^1 h(t,s)f(t,x(\varphi(s)))ds + \int_0^1 u(t,s,x(s))ds\right| dt \\
&\leq \|a\|_{L^1(D)} + \int_D \left(\int_0^1 h(t,s)|f(t,x(\varphi(s)))ds + \int_0^1 |u(t,s,x(s))|ds\right) dt \\
&\leq \|a\|_{L^1(D)} + \int_D \int_0^1 p(t)(|f(t,0)| + l|x(\varphi(s))|)dsdt \\
&+ \int_D \int_0^1 k(t,s)(q(t) + b_1|x(s)|)dsdt \\
&\leq \|a\|_{L^1(D)} + \|p\|_{L^1(D)}\|f(t,0)\|_{L^1(D)} + l\|p\|_{L^1(D)} \int_D \int_0^1 |x(\varphi(s))|\frac{\varphi'(t)}{b_2}dsdt \\
&+ \int_D \int_0^1 k(t,s)q(t)dsdt + b_1 \int_D \int_0^1 k(t,s)|x(s)|dsdt \\
&\leq \|a\|_{L^1(D)} + \|p\|_{L^1(D)}\|f(t,0)\|_{L^1(D)} + \frac{l}{b_2}\|p\|_{L^1(D)} \int_{\varphi(D)} |x(v|)dv \\
&+ \|K\|_{L^1(D)}\|q\|_{L^1(D)} + b_1\|K\|_{L^1(D)} \int_D |x(t)|dt.
\end{aligned}$$

This implies that,

$$\omega(A(S)+B(S)) \leq \left(\frac{l}{b_2}\|p\|_{L^1(D)} + b_1\|K\|_{L^1(D)}\right)\omega(S),$$

where $\omega(.)$ is the De Blasi measure of weak noncompactness. Hence, all the hypothesse of Theorem 2.8 are satisfied. Then, there exists $x \in N_r$ such that $Ax + Bx = x$ which is a solution for Eq. (3.1). □

3.2 Application to nonlinear integral equation in the Fréchet space $L^1_{loc}(\mathbb{R}^+)$

In this section we are going to use the De Blasi measure of weak noncompactness in L^1_{loc} (Eq. (1.4)), and fixed point theorems to present some existence results and solution of the following equation

$$x(t) = a(t) + \int_0^1 h(t,s)f(s,x(s))ds + g(t,x(t)), \quad t \in [0,1], \tag{3.8}$$

in $L^1_{loc} := L^1_{loc}(\mathbb{R}^+)$, the space of all real measurable functions $x : \mathbb{R}^+ \to \mathbb{R}$ that are locally Lebesgue integrable on $\mathbb{R}^+$, in other words, $\|x\|_T < \infty$ for $T > 0$, where

$$\|x\|_T = \int_0^T |x(t)|dt, \quad t \in [0,T] \tag{3.9}$$

For our further purposes, we need the following result proved like Theorem 2.8.

Theorem 3.5. Let X be a Fréchet space. Suppose that M is a nonempty, bounded, weakly closed, and convex subset of X. Consider $A : M \to X$ and $B : X \to X$ two operators such that for all $p \in \Lambda$:

(i) A is weakly sequentially continuous,

(ii) B is a $\{|\cdot|_p\}_{p\in\Lambda}$-contraction and is weakly closed,

(iii) there exists $\lambda_p \in [0,1)$ such that $\omega_p(A(S) + B(S)) \leq \lambda_p \omega_p(S)$ for all $S \subset M$, and

(iv) $(x = Bx + Ay, y \in M) \Longrightarrow x \in M$.

Then, there exists $x \in M$ such that $x = Ax + Bx$.

The family of seminorms defined in Eq. (3.9) defines a metrizable topology in L^1_{loc}. This latter becomes a Fréchet space with the distance

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1 + \|x-y\|_n}.$$

We denote by $L^1 := L^1([0,T])$ the Banach space consisting of all real functions defined and Lebesgue integrable on the interval $[0,T]$, $T > 0$, and let $\pi_T : L^1_{loc} \to L^1, \pi_T(x) = x|_{[0,T]}$ refers to the restriction of the function $x \in L^1_{loc}$ to the interval $[0,T]$. Now, we recall some characterizations in the topology of L^1_{loc} space.

Now, let us give some basic results concerning the superposition(Nemytskii) operator [27].

Definition 3.2. Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. We say that f has Carathéodory conditions if it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in \mathbb{R}^+$. Then, to every function $x(\cdot)$ being measurable on $\mathbb{R}^+$, we may assign the function

$$\mathcal{N}_f(x)(t) = f(t, x(t)), t \in \mathbb{R}^+$$

where $\mathcal{N}_f$ denotes the Nemytskii operator generated by the function f.

Theorem 3.6. Assume that $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathédory conditions. Then, the operator $\mathcal{N}_f$ is weakly sequentially continuous on L^1 if, and only if,

$$f(t, x) = \alpha(t) + \beta(t)x$$

where $\alpha \in L^1(\mathbb{R}^+)$and $\beta \in L^\infty(\mathbb{R}^+)$.

An important nonlinear integral equation is the Hammerstein integral equation Eq. (3.10)

$$H(x(t)) = \int_0^t h(t, s)x(s)ds, t \geq 0, \tag{3.10}$$

with $h : \Delta \to \mathbb{R}$, where $\Delta = \{(t, s) : 0 \leq s \leq t\}$.

Lemma 3.4. Let $h : \Delta \to \mathbb{R}$ be measurable with respect to both variables. Let the linear integral operator H with kernel $h(., .)$, map L^1_{loc} into itself. Then, the integral operator H is continuous.

Now, we will discuss the solvability of Eq. (3.8) under the following assumptions :

($\mathcal{H}_1$) $a \in L^1_{loc}$.

($\mathcal{H}_2$) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ has Carathéodory conditions and there exist two functions $\alpha_1 \in L^1$ and $\beta_1 \in L^\infty$ such that:

$$f(t, x) = \alpha_1(t) + \beta_1(t)x.$$

($\mathcal{H}_3$) $h : \Delta \to \mathbb{R}$ be measurable with respect to both variables and such that the integral operator H with the kernel $h(t, s)$ defined in Eq. (3.10), transforms L^1_{loc} into itself.

($\mathcal{H}_4$) $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is sequentially weakly continuous and for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, we have

$$|g(t, x)| \leq \beta_2(t)|x|$$

where β_2 is locally essentially bounded nonnegative function on $\mathbb{R}^+$.

($\mathcal{H}_5$) $\beta_1(T)\|H\| + \beta_2(T) < 1$, for all $T > 0$, where $\beta_i(T) :=$ ess $\sup_{t\in[0,T]} \beta_i(t), i = 1, 2,$ and $\|H\|$ stands for the norm of the linear operator H in the Banach space L^1.

Theorem 3.7. Assume that $(\mathcal{H}_1) - (\mathcal{H}_5)$ hold true. Then, Eq. (3.8) has at least one solution in L^1_{loc}.

We can rewrite Eq. (3.8) in the form $x(t) = Bx(t) + Ax(t)$, where

$$\begin{aligned} Bx(t) &= a(t) + \int_0^t h(t,s)f(t,x(s))ds, \quad t \geq 0, \text{ and} \\ Ax(t) &= g(t, x(t)). \end{aligned}$$

We will prove that the operators A and B satisfy conditions of Theorem 3.5. Now, we define the subset M of L^1_{loc}

$$M = \{x \in L^1_{loc}, \|x\|_T \leq R(T) \ \text{, for all } \ T > 0\},$$

where the function $R : \mathbb{R}^+ \to \mathbb{R}^+$ defined as follows $R(T) = \dfrac{L(T)}{1 - G(T)}$, such that

$$\begin{aligned} L(T) &= \|a\|_T + \|H\| \, \|\alpha_1\|_T \, . \\ G(T) &= \beta_1(T)\|H\| \end{aligned}$$

Obviously, M is nonempty, bounded, convex, and weakly closed subset of L^1_{loc}.

Proof. The proof will be in three steps.

Step 1: Let $x \in L^1_{loc}$ and $T > 0$, using assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$, we obtain that $Bx \in L^1_{loc}$.

Now we prove that B is $\{\| \cdot \|_T\}_{T>0}$-contraction for each $T > 0$. Let $x, y \in L^1_{loc}$, and $T > 0$, by assumptions $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$, we get the estimate

$$\begin{aligned} \|Bx - By\|_T &= \int_0^T \left| \int_0^t h(t,s)(f(t,x(s)) - f(t,y(s)))ds \right| dt \\ &\leq \int_0^T \int_0^t |h(t,s)\beta_1(s)(x(s) - y(s))| \, dsdt, \end{aligned}$$

thus,

$$\|Bx - By\|_T \leq \beta_1(T)\|H\|\|x - y\|_T. \tag{3.11}$$

Using Eq. (3.11) and assumption $(\mathcal{H}_5)$, we deduce that B is a $\{\|\cdot\|_T\}_{T>0}$-contraction for each $T>0$.

Now, we claim that B is weakly sequentially continuous. We only need to prove that $\pi_T B : L^1 \to L^1$ is weakly sequentially continuous. To this end let $(x_n)_n$ be a sequence such that $x_n \xrightarrow{w} x$ in L^1, so by assumption $(\mathcal{H}_2)$ and Theorem 3.6, we obtain

$$\mathcal{N}_f x_n \xrightarrow{w} \mathcal{N}_f x, \text{ for all } t \in [0,T]$$

where $\xrightarrow{w}$ is the symbol of the convergence in a Fréchet space X with respect to the weak topology.

And from assumption $(\mathcal{H}_3)$ and Lemma 3.4, we deduce that H is linear continuous operator, thus

$$H\mathcal{N}_f x_n \xrightarrow{w} H\mathcal{N}_f x.$$

We conclude that the operator $\pi_T B$ is sequentially weakly continuous in L^1. This prove the claim.

Step 2: Now, we will verify assumption (iii) of Theorem 3.5. Firstly, the operator $A : M \to L^1_{loc}$ is well defined and sequentially weakly continuous on M. Let us take a nonempty subset S of the set M, $x \in S$, and $T>0$. Further, let $\varepsilon > 0$ and let take a measurable subset $D \subset [0,T]$ such that $m(D) \leq \varepsilon$. We obtain that

$$\begin{aligned}
\int_D |Ax(t)+Bx(t)|dt &\leq \|a\|_D + \int_D \int_0^T |h(t,s)\left(\alpha_1(t)+\beta_1(t)x(s)\right)| \, dsdt \\
&+ \int_D |g(t,x(t))|dt \\
&\leq \|a\|_D + \|\alpha_1\|_D \, \|H\| + \beta_1(T)\|H\| \int_D |x(t)|dt \\
&+ \beta_2(T) \int_D |x(t)|dt
\end{aligned}$$

This implies that,

$$\omega_T(A(S)+B(S)) \leq \left(\beta_1(T)\|H\| + \beta_2(T)\right) \omega_T(S)$$

In virtue of $(\mathcal{H}_5)$, assumption (iii) of Theorem 3.5 hold.

Step 3: Let x, $y \in M$, then

$$\begin{aligned}\int_0^T |Ax(t)+By(t)|dt &\leq \int_0^T |a(t)|dt + \int_0^T \int_0^t |h(t,s)\left(\alpha_1(t)+\beta_1(t)x(s)\right)|dsdt \\ &+ \int_0^T |g(t,x(t))|dt \\ &\leq \|a\|_T + \|\alpha_1\|_T \|H\| + \beta_1(T)\|H\|_T \int_0^T |x(t)|dt \\ &+ \beta_2(T)\int_0^T |x(t)|dt\end{aligned}$$

Hence,

$$\|Ax+By\|_T \leq L(T) + G(T)R(T) = R(T),$$

which imply that $A(M)+B(M) \subset M$.

Finally, all the hypotheses of Theorem 3.5 are satisfied. Then, there exists $x \in M$ such that $Ax + Bx = x$ which is a solution for Eq. (3.8).

□

Conclusion and perspective

Fixed point theory plays a critical role in nonlinear analysis, particularly in finding solutions to various types of equations.

In our work, we focused on fixed point theorems in locally convex spaces equipped with a Hausdorff weak topology. This was accomplished using certain definitions, properties, and theories. Subsequently, we applied our results to determine the existence of solutions for some nonlinear integral equations in the Lebesgue space L^1 and the Fréchet space L^1_{loc} under specified conditions.

Looking forward, we aim to explore the theories of the fixed point for the multivalued operator in locally convex spaces and their applications in the near future.

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