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# THESIS

In view of obtaining the grade of

# Doctor in Mathematics

**Speciality : Applied Mathematics**

Presented by

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# On Some Qualitative Properties of Solutions For Higher Order Neutral Differential Equations



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# Dedication

This work is dedicated to the memory of my dear father (may his soul rest in peace), whose guidance and love continue to inspire me. It is also dedicated to my family whoe unwavering support and encouragement made this achievement possible.

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# Liste of Publications

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(2) F. Abdellaoui, M. Rahmane, *Boundedness and Square Integrability in Neutral Differential Equations With Delay And Exponential Stability in Nonlinear Differential Equations of Third Order*, Applications and Applied Mathematics: An International Journal (AAM), Vol. 20, Issue No. 1, Article 12, 2025.

(3) M. Rahmane, F. Abdellaoui, *Uniform Stability, Boundedness and Square Integrability for Non-Autonomous Third- Order Neutral Differential Equations with Delay*, Proyecciones Journal of Mathematics, Vol. 43, Issue No. 4, pp. 827-847, 2024.

# Contents

# List of Symbols

| | |
|---|---|
| ODE : | Ordinary Differential Equations |
| IVP : | Initial Value Problem |
| NFDE : | Neutral Functional Differential Equation |
| sup : | least upper bound |
| inf : | greatest lower bound |
| $\mathcal{K}$ : | Class $\mathcal{K}$ Function, $\varphi : [0,a) \longrightarrow [0,+\infty)$ a continuous function is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\varphi(0) = 0$. |
| $\mathcal{K}_\infty$ : | Class $\mathcal{K}_\infty$ Function, $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ a continuous function is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing and $\varphi(u) \longrightarrow +\infty$ as $u \longrightarrow +\infty$. |
| $\mathcal{KL}$ : | Class $\mathcal{KL}$ Function, $\psi : [0,a) \times [0,+\infty) \longrightarrow [0,+\infty)$ a continuous function is said to belong to class $\mathcal{KL}$ if for each fixed $s$, $\psi(u,s)$ is class $\mathcal{K}$ with respect to $u$, and for each fixed $u$, $\psi(u,s)$ is decreasing with respect to $s$ and $\psi(u,s) \longrightarrow 0$ as $s \longrightarrow +\infty$. |
| max : | Maximum |
| min : | Minimum |

# Introduction

Differential equations (DEs) are crucial in numerous scientific and engineering fields because they effectively represent complex systems and natural phenomena. They are essential in describing how physical quantities change over time or space, making them fundamental in many fields. For instance, in physics. DEs are used to model the motion of particles and the propagation of heat. In biology, they help in understanding population dynamics and the spread of diseases. In economics DEs are employed to model market dynamics and economic growth. Their versatility and wide-ranging applications underscore their importance in both theoretical research and practical problem solving, see [43], [93].

The study of differential equations began with the birth of calculus, which dates back to the 1660s. It should be noted that differential equations are largely related to the qualitative behavior of solutions such as stability, instability, convergence, boundedness, etc. These are very important problems in the theory and applications of differential equations, see [46], [15], [56].

Neutral differential equations constitute an important class of functional differential equations. Unlike traditional differential equations, neutral equations allow for the inclusion of delays that depend not only on the current state but also on past states of the system. This feature makes them particularly suitable for describing phenomena such as biological processes with memory effects, chemical reactions with transport delays see [13, 60].

Delay differential equations (DDEs) are a type of differential equation. Delay equations have been introduced to model phenomena in which there is a temporal mixture between the action on the system and the system's response to that action. For example, in the process of birth in the biological populations (cells, bacteria, etc.). In [27], the authors, introduce an epidemic model based on delay equations. Many phenomena encountered in physics, biology, chemistry, etc, have found in the theory of delay equations a good means of modeling (a more realistic means than in the case of ordinary differential equations). Since the 1940s, the theory of delay equations has known a great development, in particular we find Bellman and Cooke [21] (1963), Hale [38] (1977), [5, 6, 11, 87].

In recent years, much attention has been devoted to the qualitative properties of such equations, including existence and uniqueness of solutions, boundedness and stability. These properties are fundamental for understanding the long-term dynamics of the systems and for ensuring the reliability of mathematical models used in applications. Various analytical tools, such as fixed-point theorems, Lyapunov functionals and integral inequalities have been developed to tackle these issues see [13, 59]. The stability and the boundedness of solutions constitute one of the most burning problems in control theory, dynamical systems, systems with delay. Ezeilo [5–7], Tejumola [83] and Tunç [87] . Among the recently proposed works, many authors use the Lyapunov method to discuss the qualitative behavior of solutions to such equations.

• In 1959, Ezeilo [28] studied the boundedness of solutions to the differential equation

$$x''' + ax'' + bx' + f(x) = p(t).$$

• In 1968, the same author [29] introduced new conditions to establish that all solutions to the equation.

$$x''' + ax'' + f(x)x' + g(x) = p(t),$$

are uniformly ultimately bounded.

• In 1971, Nakashima [50] proposed asymptotic stability conditions for the solutions of the following equations

$$x''' + ax'' + g(x)x' + h(x) = e(t,x,x',x''),$$

and

$$x''' + p(t)x'' + q(t)g(x') + h(x) = e(t,x,x',x'').$$

Let us then mention Hara's results:

- Hara [41] studied the uniform asymptotic behavior of the solutions of the differential equation of the form

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0,$$

- The same author [41] also considered the following differential equations

$$x''' + a(t)x'' + b(t)x' + c(t)x = e(t),$$

$$x''' + a(t)x'' + b(t)x' + c(t)h(x) = e(t),$$

and

$$x''' + a(t)f(x,x')x'' + b(t)g(x,x')x' + c(t)h(x) = e(t),$$

and introduction of the conditions for which all the solutions of the The above equations are uniformly bounded and tend to zero as $t \to +\infty$ .

- In 1992, Zhu [99] showed the stability of the zero solutions of the two delay equations

$$x''' + ax'' + bx' + f(x(t-r)) = p(t), \tag{1}$$

and

$$x''' + ax'' + \phi(x'(t-r)) + f(x) = p(t).$$

It establishes sufficient conditions to guarantee the uniformity of the bounds, the ultimate uniformity of the solution bounds, and the existence of periodic solutions to equation (1).

- In 1999, Mehri and Shadman [49] considered the following third-order nonlinear differential equation:

$$x''' + a(t)f(x'') + b(t)g(x') + c(t)h(x) = 0,$$

and established sufficient conditions for its solutions to be bounded.

- In 2005, Sadek [80] considered the following equation:

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-r)) = 0.$$

- In 2007, Tunc [86],

$$x''' + a_1x'' + f_2(x'(t-r(t)))x' + a_3x = p(t,x,x',x(t-r(t)),x'(t-r(t)),x'').$$

• In 2007, Zhang Li Juan and Si Li Geng [98]

$$x''' + g(x')x'' + f(x,x')x' + h(x) = 0.$$

• In 2008, Yao and Meng [94]

$$x''' + \phi(x'') + g(x(t-r(t)),x'(t-r(t))) + f(x(t-r(t))) = 0,$$

And

$$x''' + \phi(x,x')x'' + g(x(t-r(t)),x'(t-r(t)))x' + f(x(t-r(t))) = 0.$$

• In 2009, Omeike [54] examined the asymptotic stability and boundedness of the solutions of the nonlinear differential delay equation:

$$x''' + a(t)x'' + b(t)g(x') + c(t)h(x(t-r)) = p(t).$$

He discussed the stability of the solutions to this equation when $p(t) = 0$ and the boundedness of the solutions when $p(t) = 0$.

• In 2010,

- Ademola [5] obtained sufficient conditions that ensure the boundedness of the solutions of differential equations of the form

$$x''' + f(x'') + g(x') + h(x) = p(t,x,x',x'').$$

- Afuwape and Omeike [11], proved under certain assumptions that all solutions of the equation:

$$x''' + h(x')x'' + g(x'(t-r(t))) + f(x(t-r(t))) = p(t,x,x',x(t-r(t)),x'(t-r(t)),x''),$$

are asymptotically stable for $p(.) = 0$ and bounded for $p(.) \neq 0$.

- Tun¸ c [88] examined the asymptotic behavior of the solutions of the following equation:

$$x'''+a(t)x''+b(t)g_1(x'(t-r(t)))+g_2(x')+h(x(t-r(t))) = p(t,x,x',x(t-r(t)),x'(t-r(t)),x'').$$

• In 2015, Remili and Beldjerd [68] studied the uniform asymptotic behavior of the solutions, and established sufficient conditions that ensure the boundedness of the solutions of differential equations of the form

$$\phi(x(t))x'(t)'' + a(t)\psi(x'(t))x''(t) + b(t)f(x'(t)) + c(t)h(x(t-r(t)))$$
$$= e(t,x(t),x(t-r(t)),x'(t),x'(t-r(t)),x''(t)),$$

for $e(.) = 0$ and $e(.) \neq 0$ respectively.

• In 2016, Remili and Rahmane [72] examined the boundedness and the Square integrability of the solutions of the equation

$$x'''' + a(t)\big(p(x(t))x''(t)\big)' + b(t)\big(q(x(t))x'(t)\big)' + c(t)f(x(t))x'(t) + d(t)h(x(t)) = e(t),$$

• In 2019, Oudjedi, Lekhmissi and Remili [58], in this article, sufficient conditions are obtained of the boundedness and the square integrability of the neutral equation of the form

$$x(t) + \beta x(t-r)''' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = h(t),$$

• Recently, in 2022, Graef, Beldjerd and Remili [37] have established new conditions for the stability, boundedness, and square integrability of solutions to a class of third-order neutral delay differential equation :

$$(x''(t) + \Omega(x''(t-r)))' + \Psi(x(t))\,x''(t) + \Phi(x(t))\,x'(t) + h(x(t-\sigma)) = 0.$$

• In 2023, Fatmi, Remili and Rahmane [31] give sufficient conditions for the stability and square integrability of solutions of the nonlinear fourth order differential equations.

$$\big(x'''(t) + \rho x'''(t-r(t))\big)' + a(t)\big(\phi(x(t))x''(t)\big)' + b(t)\big(g(x(t))x'(t)\big)' + c(t)f(x(t))x'(t)$$

$$+d(t)h(x(t-r(t))) = \psi(t,x(t),x'(t),x''(t),x'''(t))$$

for $\psi(.) = 0$ and $\psi(.) \neq 0$ respectively.

• Very recently, in 2025, Ayman M. Mahmoud, [48] extends and generalizes previous findings on third-order neutral differential equations and providing new qualitative results by investigating the stability, boundedness, and asymptotic stability of the solutions for the third order neutral delayed differential equation

$$\left[\eta\left(t\right)\left(x\left(t\right)+\zeta x\left(t-\sigma\right)\right)'\right]''+\vartheta\left(t\right)x''\left(t\right)+\varphi\left(t\right)x'\left(t-\tau\right)+\psi\left(t\right)R\left(x\left(t-\tau\right)\right)$$
$$=Q\left(t,x\left(t\right),x\left(t-\sigma\right),x'\left(t\right),x'\left(t-\sigma\right),x''\left(t\right)\right)$$

for all $t\geq t_1=t_0+\rho$, where $\rho=\sup\{\sigma,\tau\}$ and $\zeta$ is a constant with $0\leq\zeta\leq1$ and $\sigma,\tau\geq0$. The functions $\eta\left(t\right)$, $\vartheta\left(t\right)$, $\varphi\left(t\right)$, $\psi\left(t\right)$, $R\left(x\right)$ and $Q\left(.\right)$ are continuous depending only on the arguments shown.

Thus, motivated by the work cited above on the stability of nonlinear non-autonomous delay systems, we propose to study the stability, boundedness and ultimate boundedness of some differential equations that generalize certain equations those cited above.

This thesis focuses on some qualitative properties of solutions for higher-order neutral differential equations, aiming to establish sufficient conditions for boundedness, stability, and square integrability of solutions. We also investigate oscillatory and asymptotic behaviors under different structural assumptions, extending and generalizing many existing results in the literature.

The thesis is organized as follows :

• Chapter 1 presents the necessary mathematical concepts and theory concerning stability and boundedness of solutions for ordinary differential equations, delay differential equations and neutral differential equations.

• Chapter 2 presents our contributions to the qualitative study of certain third-order nonlinear delay differential equations of neutral type, where we have used the Lyapunov's second method (direct method) in each contribution. This chapter is composed of three sections:

1: First section deals with the uniform asymptotic stability, the boundedness and the square integrability of solutions for a given neutral delay differential equation.

2: Second section establishes some sufficient conditions for the boundedness, the square integrability and also the uniform asymptotic stability of solutions for certain neutral differential equation.

3: Third section contains the proofs of two new results concerning sufficient conditions that guarantee the boundedness and the square integrability of solutions for a considered third order neutral differential equation with delay.

• Chapter 3 extends the analysis to fourth-order nonlinear neutral differential equations and provides new theoretical results.

• Chapter 4 investigates some qualitative properties of solutions to certain classes of third, fourth order and integro-differential equations of neutral type

Finally, we summarize the main contributions and outline possible directions for future research.

Chapter 1

# Preliminaires

The primary objective of this chapter is to provide a solid theoretical foundation and establish the mathematical framework necessary for the subsequent analysis of higher-order differential equations. Before diving into the specific complexities of delay and neutral differential equations, it is essential to revisit the fundamental principles governing Ordinary Differential Equations (ODEs). This chapter serves as a comprehensive review of the standard introductory concepts, focusing on the structural properties of solutions. Furthermore, we introduce the core definitions and theorems related to stability theory and boundedness, which are the central themes of this work. By systematically presenting the basics of ODEs alongside the specialized theory of delay systems, this section ensures a clear understanding of the transition from memoryless systems to those whose evolution depends on their past states. The topics covered in this chapter include:

Fundamental Definitions: An overview of n-th order ODEs and their functional relationships.

Stability Concepts: Defining different types of stability (Lyapunov, Asymptotic, etc.).

Delay Systems: An introduction to the notation and structure of delay and neutral differential equations.

## 1.1 Basics of Ordinary Differential Equations

This section establishes the fundamental groundwork for the study of Ordinary Differential Equations (ODEs) by revisiting their core definitions and structural classifications. It emphasizes the importance of understanding $n^{th}$ order equations, distinguishing between implicit and explicit forms, and defining the functional relationship between independent variables and unknown functions with their derivatives. These elementary concepts serve as the essential building blocks for exploring more complex systems, particularly those involving time delays, and provide the necessary mathematical language to analyze advanced qualitative properties and stability criteria throughout the work.

**Definition 1.1.** [14] (ODE). An nth order ordinary differential equation (ODE) is a functional relationship taking the form

$$F\Big(t,x(t),\frac{d}{dt}x(t),\frac{d^2}{dt^2}x(t),...,\frac{d^n}{dt^n}x(t)\Big) = 0$$

that involves an independent variable $t \in I \subset \mathbb{R}$, an unknown function $x(t) \in D \subset \mathbb{R}^n$ of the independent variable, its derivative and derivatives of order up to $n$. For simplicity, the time dependence of $x$ is often omitted, and we in general write equations as

$$F(t,x,x',x'',...,x^{(n)}) = 0 \tag{1.1}$$

where $x^{(n)}$ denotes the nth order derivative of $x$. An equation such as (1.1) is said to be in general (or implicit) form.

An equation is said to be in normal (or explicit) form when it is written as

$$x^{(n)} = f(t,x,x',x'',...,x^{(n-1)})$$

Note that it is not always possible to write a differential equation in normal form, as it can be impossible to solve $F(t, x, ..., x^{(n)}) = 0$ in terms of $x^{(n)}$.

**Definition 1.2.** [14] (First-order ODE) Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a given function, where $I \subset \mathbb{R}$ is an interval. A first-order ordinary differential equation in normal form is expressed as :

$$x' = f(t, x) \tag{1.2}$$

Note that the theory developed here holds usually for nth order equations.
The function $f$ is assumed continuous and real valued on a set $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$.

**Definition 1.3.** [14] (Initial value problem). An initial value problem IVP for equation (1.2) is given by

$$\begin{cases} x' = f(t,x), \\ x(t_0) = x_0 \end{cases} \tag{1.3}$$

where $f$ is continuous and real valued on a set $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$, with $(t_0, x_0) \in \mathcal{U}$.

## 1.2 Existence and Uniquness

**Theorem 1.1.** *[26] Let $f$ be continuous on $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ and let $(t_0,x_0) \in \mathcal{U}$. Then IVP has a solution.*

**Definition 1.4.** [26] A function $f : \mathcal{U} \to \mathbb{R}^n$ satisfy a local Lipschitz condition with respect to $x$ at point $(t_0,x_0)$ if there is a neighborhood $N$ of $(t_0,x_0)$ and a constant $K$ such that $(t,x_1)$ and $(t,x_2) \in N$ imply $\| f(t,x_1) - f(t,x_2) \| \leq K \| x_1 - x_2 \|$.

**Theorem 1.2.** *[26] Let $f$ be continuous on $\mathcal{U}$ and let $(t_0,x_0) \in \mathcal{U}$. If $f$ satisfied a local Lipschitz condition with respect to $x$ at $(t_0,x_0)$, then IVP has a unique solution.*

## 1.3 Stability and Boundedness

The investigation of the qualitative behavior of solutions is a central theme in the theory of differential equations. Since finding explicit analytical solutions for nonlinear systems is often mathematically challenging, it is essential to determine the characteristics of these solutions over a long-term horizon. This section focuses on the fundamental concepts of stability and boundedness, which serve as the primary tools for characterizing the evolution and constraints of the system's state variables. The analysis presented here is deeply rooted in Lyapunov's Direct Method. This robust approach allows for the assessment of stability properties through the construction of scalar functions, known as Lyapunov functions, without requiring direct knowledge of the solutions themselves. We will outline the various categories of stability ranging from simple stability to asymptotic stability and establish the criteria for ensure that solutions remain within predefined bounds. These preliminary results are indispensable for the subsequent chapters, where we extend these methodologies to evaluate the stability of fourth-order equations with multiple delays.

### 1.3.1 Boundedness

**Definition 1.5.** [46] A solution $x(t,t_0,x_0)$ of the system (1.3) is bounded, if there exists a $\beta > 0$ such that $\| x(t,t_0,x_0) \| < \beta$ for all $t \geq t_0$, where $\beta$ may depend on each solution.

**Definition 1.6.** [46] The solutions of the system (1.3) are

(i) Uniformly bounded if there exists a positive constant $c$, independent of $t_0 \geq 0$, and for every $a \in ]0,c[$, there is $\beta = \beta(a) > 0$, independent of $t_0$, such that

$$\| x(t_0) \| \leq a \Longrightarrow \| x(t) \| \leq \beta, \ \ \forall t \geq t_0 \tag{1.4}$$

(ii) Globally uniformly bounded if (1.3) holds for arbitrarily large $a$.

(iii) Uniformly ultimately bounded with ultimate bound $b$ if there exist positive constant band $c$, independent of $t_0 \geq 0$, and for every $a \in ]0,c[$, there is $T = T(a,b) \geq 0$, independent of $t_0$, such that

$$\| x(t_0) \| \leq a \Longrightarrow \| x(t) \| \leq b, \ \ \forall t \geq t_0 + T \tag{1.5}$$

(iv) Globally uniformly ultimately bounded if (1.3) holds for arbitrarily large $a$.

### 1.3.2 Stability

**Definition 1.7.** [46] A point $x^*$ is an equilibrium point of the system (1.3) if $f(t,x^*) = 0$ for all $t \geq 0$.

**Definition 1.8.** [46] The equilibrium point $x^* = 0$ of the system (1.3) is

(i) Stable if :

$$\forall \varepsilon > 0, \delta = \delta(\varepsilon,t_0) > 0 : \ \| x(t_0) \| < \delta \Rightarrow \| x(t) \| < \varepsilon, \ \ \forall t \geq t_0 \geq 0.$$

(ii) Uniformly stable if :

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \ \| x(t_0) \| < \delta \Rightarrow \| x(t) \| < \varepsilon, \ \ \forall t \geq t_0 \geq 0.$$

(iii) Unstable if 'it is not stable.

(iv) Asymptotically stable if it is stable and if there exists $\delta_1 = \delta_1(t_0) > 0$ such that

$$\| x(t_0) \| \leq \delta_1 \Longrightarrow \lim_{t \to +\infty} x(t) = 0.$$

The next lemma gives equivalent, more transparent, definitions of uniform stability and uniform asymptotic stability by using class $\mathcal{K}$ and class $\mathcal{KL}$ functions.

## Class $\mathcal{K}$ and Class $\mathcal{KL}$ Functions

**Definition 1.9.** [46] A continuous function $\varphi : [0,a) \longrightarrow [0,+\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\varphi(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\varphi(u) \longrightarrow +\infty$ as $u \longrightarrow +\infty$.

**Definition 1.10.** [46] A continuous function $\psi : [0,a) \times [0,+\infty) \longrightarrow [0,+\infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed $s$, the mapping $\psi(u,s)$ belongs to class $\mathcal{K}$ with respect to $u$ and, for each fixed $u$, the mapping $\psi(u,s)$ is decreasing with respect to $s$ and $\psi(u,s) \longrightarrow 0$ as $s \longrightarrow +\infty$.

**Lemma 1.1.** *[46] The equilibrium point $x^* = 0$ of the system (1.3) is*

*(i) Uniformly stable if and only if there exists a class $\mathcal{K}$ function $\varphi$ and a positive constant $c$, independent of $t_0$, such that:*

$$\| x(t) \| \leq \varphi(\| x(t_0) \|), \ \ \forall t \geq t_0 \geq 0, \ \ \forall \| x(t_0) \| < c.$$

*(ii) Uniformly asymptotically stable if and only if there exists a function $\psi$ of class $\mathcal{KL}$ and a positive constant $c$ independent of $t_0$ such that:*

$$\| x(t) \| \leq \psi\Big( \| x(t_0) \| \, , \, t - t_0\Big), \; \forall t \geq t_0 \geq 0, \, \forall \, \| x(t_0) \| < c.$$

*(iii) Globally uniformly asymptotically stable if and only if inequality (1.3) is satisfied for any initial state $x(t_0)$.*

**Definition 1.11.** [46] The equilibrium point $x^* = 0$ of the system (1.3) is

(i) Exponentially stable if there exist positive constants $c$, $k$ and $\lambda$ such that:

$$\| x(t) \| \leq k \| x(t_0) \| \, e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0, \;\; \forall \, \| x(t_0) \| < c.$$

(ii) Globally exponentially stable if the previous condition is verified for any initial state $x(t_0)$.

**Definition 1.12.** [46] We consider the system (1.3). Let $D \subset \mathbb{R}^n$ be a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuous and differentiable function on $D$. The function $V$ is said to be positive semi definite if :

(i) $V(t,0) = 0$, $\forall t \in \mathbb{R}_+$.

(ii) $V(t,x) \geq 0$, $\forall t \in \mathbb{R}_+$, $\forall x \in D - \{0\}$.

**Definition 1.13.** [46] Let $D \subset \mathbb{R}^n$ be a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuous function that is differentiable on $D$. The function $V$ is said to be positive definite if:

(i) $V(t,0) = 0$, $\forall t \in \mathbb{R}_+$.

(ii) There exists a positive definite function $W_0$ such that:

$$W_0(x) \leq V(t,x), \, \forall t \in \mathbb{R}_+, \, \forall x \in D.$$

**Definition 1.14.** [46] Let $D \subset \mathbb{R}^n$ be a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuous and differentiable function on $D$. The function $V$ is said to be decrescent if there exists a positive definite function $W_1$ such that: $V(t,x) \leq W_1(x)$, $\forall t \in \mathbb{R}_+$, $\forall x \in D$.

**Definition 1.15.** [46] Let $D \subset \mathbb{R}^n$ be a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuous and differentiable function on $D$. The function $V$ is said to be radially unbounded if : $V(t,x) \to +\infty$ when $\| x \| \to +\infty$.

**Remark 1.1.** [46] If $V$ is a positive definite function on $D$ then there exists a positive definite function $W_0$ such that $W_0(x) \leq V(t,x)$, $\forall x \in D$. which implies the existence of a function $\varphi_0$ of class $\mathcal{K}$ such that

$$\varphi_0(\| x \|) \leq W_0(x) \leq V(t,x), \quad \forall x \in B_l \subset D.$$

If moreover $V$ is a decrescent function on $D$ then there exists a positive definite function $W_1$ such that $V(t,x) \leq W_1(x)$, $\forall x \in D$, which implies the existence of a function $\varphi_1$ of class $\mathcal{K}$ such that

$$V(t,x) \leq W_1(x) \leq \varphi_1(\| x \|), \;\; \forall x \in B_l \subset D.$$

So $V$ is a positive definite, decrescent function on $D$ if and only if there exist functions $\varphi_0$ and $\varphi_1$ of class $\mathcal{K}$ such that

$$\varphi_0(\| x \|) \leq V(t,x) \leq \varphi_1(\| x \|), \;\; \forall x \in B_l \subset D.$$

$V$ is a positive definite, decrescent and radially unbounded function on $D$ if and only if the functions $\varphi_0$ and $\varphi_1$ are of class $\mathcal{K}_\infty$.

**Definition 1.16.** [46] (Lyapunov function) Let $D$ be a neighborhood of 0 and $V : \mathbb{R}_+ \times D \to \mathbb{R}$ a continuous and differentiable function on $D$.

1- We say that $V$ is a broad-sense Lyapunov function at 0 if it verifies the following two properties:

(i) $V$ is positive definite.

(ii) $\dot{V}(t,x) \leq 0$, $\forall (t,x) \in \mathbb{R}^+ \times D$.

2- We say that $V$ is a strict Lyapunov function at 0, if it verifies the following two properties:

(i) $V$ is positive definite.

(ii) $\dot{V}(t,x) < 0$, $\forall (t,x) \in \mathbb{R}^+ \times D - \{0\}$.

**Theorem 1.3.** *[46] Let $x^* = 0$ be an equilibrium point of the system (1.3) and $D$ a domain that contains the origin. Let $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuously differentiable function on $D$ such that*

$$W_0(x) \leq V(t,x) \leq W_1(x),$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0,$$

*for all $t \geq 0$ and $x \in D$, where $W_0$ and $W_1$ are continuous positive defined. Then $x^* = 0$ is uniformly stable.*

**Theorem 1.4.** *[46] Let $x^* = 0$ be an equilibrium point of the system (1.3) and $D$ a domain that contains the origin. Let $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuously differentiable function on $D$ such that*

$$W_0(x) \leq V(t,x) \leq W_1(x),$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_2(x),$$

*for all $t \geq 0$ and $x \in D$, where $W_0$, $W_1$ and $W_2$ are continuous positive defined. Then $x^* = 0$ is uniformly asymptotically stable. If $D = \mathbb{R}^n$, then $x^* = 0$ is globally uniformly asymptotically stable.*

**Theorem 1.5.** *[26] Let $x^* = 0$ be an equilibrium point of the system (1.3) and $D$ a domain that contains the origin. Let $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be a continuously differentiable function on $D$ such that*

$$W_1(x) \leq V(t,x) \leq W_2(x),$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) + M, \quad \text{for } M > 0$$

*for all $t \geq 0$ and $x \in D$, where $W_1$, $W_2$ and $W_3$ are class $\mathcal{K}$ functions. Then the zero solution of (1.3) is uniform bounded and uniform ultimate bounded.*

**Lemma 1.2.** *[22] ( Gronwall's lemma) Let $k$ be a non negative constant and let $f$ and $g$ be continuous non negative functions on some interval $a \leq t \leq b$ satisfy the inequality*

$$f(t) \leq k + \int_a^t f(s)g(s)ds \quad \text{for all} \quad a \leq t \leq b$$

*then*

$$f(t) \leq k \exp\Big( \int_a^t g(s)ds \Big) \quad \textit{for all} \;\; a \leq t \leq b$$

## 1.4 Delay Differential Equations

While the classical theory of ordinary differential equations assumes that the future state of a system depends solely on its current condition, many real-world processes exhibit a "memory" effect. This necessitates the study of Delay Differential Equations (DDEs), where the derivative of the state variable at any given time depends on its values at previous times. This section introduces the fundamental mathematical framework and notations required to analyze such systems. To rigorously define the solution space for DDEs, we introduce the concept of the phase space, typically represented as the Banach space of continuous functions $C([-r, 0], \mathbb{R}^n)$. Unlike ODEs, where the initial condition is a single point, the initial condition for a delay system is a function defined over a past interval. This transition from finite-dimensional to infinite-dimensional spaces is crucial for understanding the stability and qualitative properties of the fourth-order neutral delay differential equations that will be addressed in the subsequent chapters of this thesis. In the following, we provide the necessary preliminary definitions, including the history function $x_t$ and the associated norms, to establish a consistent language for our analysis.

For $x \in \mathbb{R}^n$, $\parallel . \parallel$ is any norm. For $r > 0$ and $H > 0$, we define

$C_H := \Big\{ \phi \in C([-r,0], \mathbb{R}^n) : \parallel \phi \parallel_C \leq H \Big\}$ with $(C, \;\; \parallel . \parallel_C)$ is the Banach space of continuous functions $\phi : [-r,0] \to \mathbb{R}^n$ and $\parallel . \parallel_C$ is the norm on C defined by

$$\forall \phi \in C, \;\; \parallel \phi \parallel_C = \sup_{\theta \in [-r, \; 0]} \parallel \phi(\theta) \parallel .$$

Let $x : [t_0 - r, \; t_0 + A] \to \mathbb{R}^n$ continuous where $t_0 \geq 0$ and $A > 0$. For $t$ fixed in $[t_0, \; t_0 + A]$, we define the function :

$$x_t = x(t + \theta), \quad -r \leq \theta \leq 0.$$

$x_t \in C([-r, \; 0], \; \mathbb{R}^n)$ it is the restriction of $x$ to the interval $[t - r, \; t]$ translated to $[-r, \; 0]$.

Let us now consider

$$\dot{x} = f(t,\, x_t) \tag{1.6}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous application, locally Lipschitzian in its second argument and such that $f(t{,}0) = 0$ for all $t \in \mathbb{R}$. Moreover, $f$ satisfy the condition :

$$\forall H_1 < H,\, \exists L(H_1) > 0,\ \parallel \phi \parallel_C < H_1 \Rightarrow \parallel f(t{,}\phi) \parallel_C < L(H_1)$$

A function $x(t)$ is said to be a solution of (1.6) if it is defined and continuous on $[t_0 - r,\, t_0 + A]$ verifies $x(t) = \phi(t)$ on $[t_0 - r,\, t_0]$, is differentiable on $[t_0,\, t_0 + A]$ and satisfy (1.6) on $[t_0,\, t_0 + A]$.

**Definition 1.17.** (Yoshizawa [96] ). A function $x(t_0{,}\phi)$ is said to be a solution of the system (1.6) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \geq 0$, if there exists a constant $A > 0$ such that $x(t_0,\, \phi)$ is a function from $[t_0 - r,\, t_0 + A]$ to $\mathbb{R}^n$ with the properties :

(i) $x_t(t_0,\, \phi) \in C_H$ for $t_0 \leq t \leq t_0 + A$,

(ii) $x_{t_0}(t_0,\, \phi) = \phi$,

(iii) $x(t_0,\, \phi)$ satisfies (1.6) for $t_0 \leq t \leq t_0 + A$.

$x(t,\, t_0,\, \phi)$ is the value of $x(t_0,\, \phi)$ at point $t$.

**Theorem 1.6.** *[39] Suppose that the function $f$ is continuous. Then for all $\phi \in C$, the equation (1.6) has at least one solution. Moreover, if the function $f$ is locally Lipschitzian with respect to $x_t$, then the solution is unique.*

**Theorem 1.7.** *[26] Let $V(t{,}x_t) : [t_0, +\infty[\times C_H \to R_+$ be a continuous functional satisfying a local Lipschitz condition, and $V(t{,}0) = 0$, $\forall t \geq t_0$ such that :*

*(i)* $W_0(\parallel x(t) \parallel) \leq V(t{,}x_t) \leq W_1(\parallel x(t) \parallel) + W_2(\parallel x_t \parallel_2)$, *where* $\parallel x_t \parallel_2 = \left[\sum_{i=1}^{n} \int_{t-r}^{t} x_i^2(s)ds\right]^{\frac{1}{2}}$

*(ii)* $V'_{(1.6)}(t{,}x_t) \leq -W_3(\parallel x(t) \parallel)$,

*for all $t \geq t_0$ and $x \in C_H$, where $W_0$, $W_1$, $W_2$ and $W_3$ are class $\mathcal{K}$ functions. Then the zero solution of (1.6) is uniformly asymptotically stable.*

**Theorem 1.8.** *[26] Let $V(t,x_t) : [t_0, +\infty[\times C_H \to R_+$ be a continuous functional satisfying a local Lipschitz condition with $H = +\infty$, such that :*

*(i)* $W_0(\| x(t) \|) \leq V(t,x_t) \leq W_1(\| x(t) \|) + W_2\Big(\int_{t-\alpha}^{t} W_3(\| x(s) \|)ds\Big)$

*(ii)* $V'_{(1.6)}(t,x_t) \leq -W_3(\| x(t) \|) + M$*, for* $M > 0$

*for all $t \geq t_0$ and $x \in C_H$, where $W_0$, $W_1$, $W_2$ and $W_3$ are class $\mathcal{K}$ functions. Then the solutions of (1.6) are uniform bounded and uniform ultimate bounded.*

## 1.5 Neutral Functional Differential Equations

After establishing the framework for delay differential equations, we now progress to a more specific and complex class known as Neutral Functional Differential Equations (NFDEs). The defining characteristic of neutral equations, which distinguishes them from retarded delay equations, is that the derivative of the state variable at the current time t depends not only on the past values of the state but also on the past values of the derivative itself. Mathematically, this relationship is expressed through the operator $G(t, x_t)$, representing the neutral part of the system. Such equations arise frequently in modeling phenomena where signal propagation speeds are finite, such as in electrical networks, aeroelasticity, and complex mechanical systems. In this section, we provide the rigorous definitions of NFDEs and their solutions within the context of continuous and differentiable functions. These foundations are essential for investigating the qualitative properties of the fourth-order neutral delay systems that constitute the core contribution of this thesis.

**Definition 1.18.** [38] Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $G : \Omega \to \mathbb{R}^n$ are given continuous functions, with $G$ is continuously differentiable. The relation

$$\frac{d}{dt}G(t,x_t) = f(t,x_t) \tag{1.7}$$

is called neutral functional differential equation NFDE$(G, f)$.

**Definition 1.19.** [38] For a given NFDE$(G, f)$, a function $x$ is said to be a solution of the NFDE$(G, f)$ if there are a $t_0 \in \mathbb{R},\ A > 0$, such that

$$x \in C([t_0 - r, t_0 + A),\ \mathbb{R}^n),\ (t, x_t) \in \Omega,\ t \in [t_0, t_0 + A),$$

$G(t, x_t)$ is continuously differentiable and satisfied equation (1.7) on $[t_0, t_0 + A)$. For a given $t_0 \in \mathbb{R}$, $\phi \in C$, and $(t_0, \phi) \in \Omega$, we say $x(t_0, \phi,G,f)$ is a solution of equation (1.7) with initial value $\phi$ at $t_0$ or simply a solution through $(t_0, \phi)$ if there is an $A > 0$ such that $x(t_0, \phi,G,f)$ is a solution of equation (1.7) on $[t_0 - r, t_0 + A)$ and $x(t_0, \phi,G,f) = \phi$.

**Theorem 1.9.** *[38] (Existence). If $\Omega$ is an open set in $\mathbb{R} \times C$ and $(t_0, \phi) \in \Omega$, then there exists a solution of the NFDE$(G,f)$ through $(t_0, \phi)$.*

**Theorem 1.10.** *[38] (Uniqueness). If $\Omega \subseteq \mathbb{R} \times C$ is open and $f : \Omega \to \mathbb{R}^n$ is Lipschitzian in $\phi$ on compact sets of $\Omega$, then, for any $(t_0, \phi) \in \Omega$, there exists a unique solution of the NFDE$(G, f)$ through $(t_0, \phi)$.*

Chapter 2

# Third-Order Differential Equations With Delay of Neutral Type

Building upon the mathematical foundations established in the previous chapter, this chapter is devoted to a detailed qualitative investigation of a specific class of third-order neutral delay differential equations. Neutral differential equations, characterized by the presence of delay in the highest-order derivative, present unique analytical challenges compared to retarded or ordinary differential equations. Such models are of significant importance due to their ability to describe complex physical phenomena where the rate of change is influenced by past states and past rates of change. The primary objective of this chapter is to derive sufficient conditions that guarantee the uniform asymptotic stability, boundedness, and square integrability of the solutions. We focus on non-autonomous systems where the presence of time-varying coefficients and nonlinear perturbations requires a sophisticated approach. By employing the Lyapunov direct method and constructing suitable functional candidates, we establish rigorous criteria that ensure the energy of the system remains controlled and converges toward equilibrium. The analysis is structured as follows:

Section 2.1: Provides a comprehensive study of the first equation, introducing the core assumptions and establishing the main stability theorems.

Asymptotic Analysis: Investigates the behavior of solutions under various parametric constraints and external perturbations.

## 2.1 Qualitative Study of a Third-Order Neutral Differential Equation with Delay

This section presents a rigorous qualitative analysis of the first third-order neutral delay differential equation discussed in this chapter. The primary objective is to establish sufficient conditions for core solution properties, including uniform asymptotic stability, boundedness, and square integrability. By utilizing the Lyapunov direct method and constructing a suitable Lyapunov functional that accounts for neutral delays and nonlinearities, the study derives parametric constraints and integral inequalities to ensure energy dissipation within the system. These theoretical results provide the necessary framework for ensuring that solutions remain bounded and converge toward equilibrium, facilitating subsequent investigations of more complex perturbed systems.

Let the following neutral delay differential equation

$$\begin{aligned}\Big[\Omega\left(x'(t)\right)x'(t)+\rho_1\Omega\big(x'(t-r)\big)x'(t-r)\Big]''+P(t)x''(t)+Q(t)x'(t)\\ +R(t)\Big[g\big(x(t)\big)+\rho_2 g\big(x(t-\ell)\big)\Big]=0\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\Big[\Omega\left(x'(t)\right)x'(t)+\rho_1\Omega\big(x'(t-r)\big)x'(t-r)\Big]''+P(t)x''(t)+Q(t)x'(t)\\ +R(t)\Big[g\big(x(t)\big)+\rho_2 g\big(x(t-\ell)\big)\Big]=\psi\big(t,x(t),x(t-\ell),x'(t),x'(t-\ell),x''(t)\big)\end{aligned}\tag{2.2}$$

for all $t\geq t_1=t_0+\sigma$ where $\sigma=\max\{r,\ell\}$ and $\max\{\rho_1,\rho_2\}=\rho<1$, $r,\ell,\rho_1$ and $\rho_2$ some positive constants and $P,Q,R\in\mathcal{C}^1(I,(0,\infty))$, $I=[t_1,\infty)$; $g\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$; $\Omega\in\mathcal{C}^2(\mathbb{R},\mathbb{R})$; $\psi\in\mathcal{C}(I\times\mathbb{R}^5,\mathbb{R})$ and $g(0)=0$.

Define a solution of (2.1) by continuous function $x:[t_x,\infty)\to\mathbb{R}$ for $t_x\geq t_1$ satisfying equation (2.1) on $[t_x,\infty)$ and such that

$$\begin{cases} x(.)\in\mathcal{C}^2([t_x,\infty),\mathbb{R}) \\ Z(.)\in\mathcal{C}^1([t_x,\infty),\mathbb{R})\end{cases}\quad \text{where}\quad Z(t)=\Big[\Omega\left(x'(t)\right)x'(t)+\rho_1\Omega\big(x'(t-r)\big)x'(t-r)\Big]'.$$

We will assume throughout this work that every considered solution of (2.1) is continuable and nontrivial.

The famous Lyapunov second method was the main device used for studying the stability by large class of authors in the literature especially when dealing with a non linear differential equations.

Equation (2.1) can be written as the system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = \dfrac{z(t)}{H(t)} \\ Z'(t) = -P(t)\dfrac{z(t)}{H(t)} - Q(t)y(t) - (1+\rho_2)R(t)g(x(t)) \\ \qquad\qquad +\rho_2 R(t)\displaystyle\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} Z(t) =& \Omega'(x'(t))x'(t)x''(t) + \Omega(x'(t))x''(t) \\ &+ \rho_1\big(\Omega'(x'(t-r))x'(t-r)x''(t-r) + \Omega(x'(t-r))x''(t-r)\big), \\ Z(t) =& y'(t)H(t) + \rho_1 y'(t-r)H(t-r) = z(t) + \rho_1 z(t-r) \end{aligned}$$

and

$$H(t) = \Omega'(y(t))y(t) + \Omega(y(t)).$$

The following notation will be adopted in the whole work

$$H(t) = \Phi(x'(t)) = \Omega'(x'(t))x'(t) + \Omega(x'(t)) \quad \text{and} \quad \theta(t) = \Phi'(x'(t))x''(t).$$

### 2.1.1 Basic Assumptions

This subsection establishes the necessary theoretical framework for the qualitative analysis of the first neutral delay differential equation (2.1). It focuses on defining a set of fundamental assumptions and mathematical constraints regarding the functions and parameters involved in the system. These assumptions are essential to ensure the validity of the stability and boundedness theorems developed throughout the chapter. By imposing specific conditions on the coefficients $P(t), Q(t)$, and $R(t)$, as well as the nonlinear components, the section provides a rigorous basis for investigating the long-term behavior of the solutions.

Suppose that there are positive constants $h_0, P_0, Q_0, R_0, h_1, P_1, Q_1, R_1, k, \delta, \delta_0, \delta_1, \alpha_1$ and $\alpha_2$ such that the following conditions which will be used on the functions that appeared in equation (2.1) are satisfied :

i) $h_0 \leq H(t) \leq h_1,\ \ P_0 \leq P(t) \leq P_1,\ \ Q_0 \leq Q(t) \leq Q_1,\ \ R_0 \leq R(t) \leq R_1$ for all $t \geq t_1$;

ii) $\delta_1 \leq \dfrac{g(x)}{x} \leq \delta_2$ for all $x \neq 0$ and $|g'(x)| \leq \delta$ for all $x \in \mathbb{R}$;

iii) $\displaystyle\int_{t_1}^{+\infty} (|R\,'(s)| + |P\,'(s)| + |Q\,'(s)|)\, ds < \alpha_1$;

iv) $\displaystyle\int_{-\infty}^{+\infty} |\Phi'(u)|\, du < \alpha_2$.

### 2.1.2 Uniform Asymptotic Stability

In this subsection, we present the central stability results for the third-order neutral delay differential equation (2.1). The primary focus is to establish sufficient conditions that guarantee the uniform asymptotic stability of the equilibrium solution. Unlike simple stability, uniform asymptotic stability ensures that all solutions starting sufficiently close to the equilibrium not only remain nearby for all future time but also converge to the equilibrium state at a rate that is independent of the initial time $t_0$. The following theorem provides a rigorous set of constraints on the system parameters, including the delay terms and the nonlinear coefficients. These conditions are derived by ensuring that the time derivative of the constructed Lyapunov functional is negative definite. The complexity of the bounds, particularly the requirements involving the minimum values of the system constants, reflects the intricate interaction between the neutral delay and the third-order dynamics. Satisfying these mathematical criteria is crucial for ensuring the reliable and predictable behavior of the system in practical applications.

For the case $\psi \equiv 0$, we have the following theorem:

**Theorem 2.1.** *In addition to conditions (i)-(iv) being fulfilled. Then every solution of*

*(2.1) is uniformly asymptotically stable, provided that*

$$\rho < \min\left\{1, \frac{2kQ_0 - 2R_1h_1\delta - 2kP_1 - k}{Q_1 + k + 2(1+k)R_1\delta\ell + (\ell k + 2h_1)R_1\delta}, \frac{2kR_0\delta_1}{R_1(\delta_2^2 + \ell k\delta)},\right.$$
$$\left.\frac{2h_0}{4P_1 + 2k + 2\ell R_1\delta h_0 + h_0(Q_0 + 2R_1 + k)}\left(\frac{P_0}{h_1} - \frac{k(2+h_0)}{2h_0}\right)\right\}$$

$$\textit{where} \quad \begin{cases} k < \min\left\{\dfrac{P_0}{3}, \ \dfrac{Q_0}{6}, \ \dfrac{Q_0^2h_0}{4P_1^2}, \ 2\dfrac{P_0h_0}{h_1\,(2+h_0)}\right\}, \\ \\ Q_0 > \max\left\{\dfrac{4\delta h_1R_1}{k}, \ \dfrac{R_1h_1\delta}{k} + P_1 + \dfrac{1}{2}\right\}. \end{cases} \tag{2.4}$$

**Proof.** The proof of this theorem depends on the properties of the continuously differentiable function $W(t,x,y,z) = W$ defined as

$$W = e^{-\frac{1}{\alpha}\int_{t_1}^{t}\varphi(s)ds}F \tag{2.5}$$

in which $\varphi(t) = |R'(t)| + |P'(t)| + |Q'(t)| + |\theta(t)|$ and

$$\begin{aligned} F = F(t,x,y,z) \ &= \ (1+\rho_2)\,kR\,(t)\,G\,(x) + (1+\rho_2)\,R\,(t)\,H\,(t)\,g\,(x)\,y + \frac{1}{2}kP\,(t)\,y^2 \\ &+ \frac{1}{2}Q\,(t)\,H\,(t)\,y^2 + kyZ + \frac{1}{2}Z^2 + kxZ + \frac{1}{2}kQ\,(t)\,x^2 + kP\,(t)\,xy \\ &+ \rho\beta_1\int_{t-r}^{t} z^2\,(s)\,ds + \beta_2\int_{-\ell}^{0}\int_{t+s}^{t} y^2\,(\zeta)\,d\zeta ds, \end{aligned} \tag{2.6}$$

where $G(x) = \int_0^x g(u)du$, $\beta_1$, $\beta_2$ and $\alpha$ are some positive constants that will be determined later in the proof.

Rewriting $F$ as follow

$$F = F_1 + F_2 + F_3 + F_4 + F_5,$$

with the notations

$$\begin{aligned} F_1 &= \frac{1}{4}Q(t)H(t)y^2 + kP(t)xy + \frac{1}{4}kQ(t)x^2, \\ F_2 &= \frac{1}{6}Z^2 + kyZ + \frac{1}{2}kP(t)y^2 + \frac{1}{6}Z^2, \\ F_3 &= \frac{1}{6}Z^2 + kxZ + \frac{1}{4}kQ(t)x^2, \\ F_4 &= (1+\rho_2)kR(t)G(x) + \frac{1}{4}Q(t)H(t)y^2 + (1+\rho_2)R(t)H(t)g(x)y \end{aligned}$$

and $$F_5 = \rho\beta_1\int_{t-r}^{t} z^2(s)\,ds + \beta_2\int_{-\ell}^{0}\int_{t+s}^{t} y^2(\zeta)\,d\zeta ds.$$

By conditions (i) and (2.4)

$$\begin{aligned} F_1 &= \frac{1}{4}Q(t)H(t)y^2 + kP(t)xy + \frac{1}{4}kQ(t)x^2 \\ &= \frac{1}{4}Q(t)H(t)\left[y^2 + \frac{4kP(t)}{Q(t)H(t)}xy + \frac{k}{H(t)}x^2\right] \\ &= \frac{1}{4}Q(t)H(t)\left[\left(y + 2\frac{kP(t)}{Q(t)H(t)}x\right)^2 + \frac{k}{H(t)}\left(1 - 4\frac{kP^2(t)}{Q^2(t)H(t)}\right)x^2\right] \\ &\geq \frac{1}{4}kQ_0\left(1 - 4\frac{kP_1^2}{Q_0^2h_0}\right)x^2 \\ &= \alpha_1 x^2, \end{aligned}$$

where $\alpha_1 = \frac{1}{4}kQ_0\left(1 - 4\frac{kP_1^2}{Q_0^2h_0}\right)$

also

$$\begin{aligned} F_2 &= \frac{1}{6}Z^2 + kyZ + \frac{1}{2}kP(t)y^2 + \frac{1}{6}Z^2 \\ &= \frac{1}{6}\left(Z^2 + 6kyZ + 3kP(t)y^2\right) + \frac{1}{6}Z^2 \\ &= \frac{1}{6}\left((Z+3ky)^2 + 3k(P(t) - 3k)y^2\right) + \frac{1}{6}Z^2 \\ &\geq \frac{1}{2}k(P_0 - 3k)y^2 + \frac{1}{6}Z^2 \\ &= \alpha_2 y^2 + \frac{1}{6}Z^2 \end{aligned}$$

where $\alpha_2 = \frac{1}{2}k\left(P_0 - 3k\right)$ and

$$\begin{aligned} F_3 &= \frac{1}{6}Z^2 + kxZ + \frac{1}{4}kQ\left(t\right)x^2 \\ &= \frac{1}{6}\left(Z^2 + 6kxZ + \frac{3}{2}kQ\left(t\right)x^2\right) \\ &= \frac{1}{2}\left(\left(Z + 3kx\right)^2 + \frac{3}{2}k\left(Q\left(t\right) - 6k\right)x^2\right) \\ &\geq \frac{3}{4}k\left(Q_0 - 6k\right)x^2 \\ &= \alpha_3 x^2 \end{aligned}$$

where $\alpha_3 = \frac{3}{4}k\left(Q_0 - 6k\right)$

Using the equality

$$2\int_0^x g'(u)g(u)du = g^2(x),$$

using conditions (i), (ii) and (2.4) we obtain

$$\begin{aligned} F_4 &= \left(1+\rho_2\right)kR\left(t\right)G\left(x\right) + \frac{1}{4}Q\left(t\right)H\left(t\right)y^2 + \left(1+\rho_2\right)R\left(t\right)H\left(t\right)g\left(x\right)y \\ &= \left(1+\rho_2\right)kR\left(t\right)\int_0^x g\left(u\right)du + \frac{1}{4}Q\left(t\right)H\left(t\right)\left[y^2 + \frac{4\left(1+\rho_2\right)R\left(t\right)g\left(x\right)y}{Q\left(t\right)}\right] \\ &= \left(1+\rho_2\right)kR\left(t\right)\int_0^x g\left(u\right)du + \frac{1}{4}Q\left(t\right)H\left(t\right)\left[\left[y + 2\frac{\left(1+\rho_2\right)R\left(t\right)g\left(x\right)}{Q\left(t\right)}\right]^2 \right. \\ &\quad \left. - \frac{4\left(1+\rho_2\right)^2R^2\left(t\right)g^2\left(x\right)}{Q^2\left(t\right)}\right], \end{aligned}$$

then

$$\begin{aligned} F_4 &\geq \left(1+\rho_2\right)kR\left(t\right)\int_0^x g\left(u\right)du - \frac{\left(1+\rho_2\right)^2H\left(t\right)R^2\left(t\right)g^2\left(x\right)}{Q\left(t\right)} \\ &\geq \left(1+\rho_2\right)kR\left(t\right)\int_0^x g\left(u\right)du - 2\delta\frac{\left(1+\rho_2\right)^2H\left(t\right)R^2\left(t\right)}{Q\left(t\right)}\int_0^x g\left(u\right)du \\ &\geq \left(1+\rho_2\right)R\left(t\right)\left(k - 2\delta\frac{\left(1+\rho_2\right)h_1R\left(t\right)}{Q\left(t\right)}\right)\int_0^x g\left(u\right)du \\ &\geq \left(1+\rho_2\right)R_0\left(k - 4\delta h_1\frac{R_1}{Q_0}\right)\int_0^x g\left(u\right)du \geq 0. \end{aligned}$$

from (ii) we have

$$g^2(x) = 2\int_0^x g'(u)g(u)du \leq 2\delta G(x)$$

$$\delta_1 x^2 \leq g^2(x) \leq 2\delta G(x),$$

so

$$G(x) \geq \frac{\delta_1}{2\delta} x^2$$

so

$$F_4 \geq \frac{\delta_1 \left(1+\rho_2\right) R_0}{2\delta} \left(k - 4\delta h_1 \frac{R_1}{Q_0}\right) x^2$$

$$F_4 \geq \alpha_4 x^2$$

where $\alpha_4 = \dfrac{\delta_1 \left(1+\rho_2\right) R_0}{2\delta} \left(k - 4\delta h_1 \dfrac{R_1}{Q_0}\right)$

Since

$$F_5 = \rho\beta_1 \int_{t-r}^{t} z^2\left(s\right) ds + \beta_2 \int_{-\ell}^{0} \int_{t+s}^{t} y^2\left(\zeta\right) d\zeta ds > 0,$$

there exists a positive constant $\lambda_0$, small enough such that

$$F \geq \lambda_0 \left[x^2(t) + y^2(t) + Z^2(t)\right]. \tag{2.7}$$

where

$$\lambda_0 = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

Since $g(0) = 0$ and $|g'(x)| \leq \delta$, we see that $|g(x)| \leq \delta |x|$.

From i), ii) and inequality $|g(x)| \leq \delta|x|$ and the fact that $2\alpha\beta \leq \alpha^2 + \beta^2$ we get

$$F \leq \lambda_1 \left(x^2 + y^2 + Z^2\right) + \rho\beta_1 \int_{t-r}^{t} z^2\left(s\right) ds + \beta_2 \int_{-\ell}^{0} \int_{t+s}^{t} y^2\left(\zeta\right) d\zeta ds, \tag{2.8}$$

where $\lambda_1 = \frac{1}{2} \max\Big\{\delta\left(1+\rho_2\right) k R_1 + \left(1+\rho_2\right) R_1 h_1 \delta_2^2 + k(1+Q_1+P_1),\ \left(1+\rho_2\right) R_1 h_1 + kP_1 + Q_1 h_1 + k + kP_1,\ 1+2k\Big\}$.

By iv) we have

$$\int_{t_1}^{t} |\theta(s)| ds = \int_{\tau_1(t)}^{\tau_2(t)} |\phi'(u)| du \leq \int_{-\infty}^{+\infty} |\phi'(u)| du < \alpha_2 < \infty, \tag{2.9}$$

where $\tau_1(t) = \min\{x(t_1),x(t)\}$ and $\tau_2(t) = \max\{x(t_1),x(t)\}$. From inequalities (iii), (2.9) and estimate $e^{-\dfrac{\alpha_1+\alpha_2}{\alpha}} \le e^{-\dfrac{1}{\alpha}\int_{t_1}^{t}\varphi(s)\,ds} \le 1$ implies

$$W \ge \lambda_2(x^2 + y^2 + Z^2), \tag{2.10}$$

where $\lambda_2 = \lambda_0 e^{-\dfrac{\alpha_1+\alpha_2}{\alpha}}$ and by (2.8) we get

$$W \le \lambda_1 \left(x^2 + y^2 + Z^2\right) + \rho\beta_1 \int_{t-r}^{t} z^2(s)\,ds + \beta_2 \int_{-\ell}^{0}\int_{t+s}^{t} y^2(\zeta)\,d\zeta ds, \tag{2.11}$$

for all $x$,$y$ and $z$, and all $t \ge t_1$.

Next, the derivative of $F$ along system (2.3) trajectories can be given by

$$\begin{aligned}
\dot{F}_{(2.3)} \;=\;& (1+\rho_2) R(t) H(t) g'(x) y^2 + \frac{k}{H(t)} z^2 + \rho_1 \frac{k}{H(t)} zz(t-r) - \frac{P(t)}{H(t)} z^2 \\
& -\rho_1 \frac{P(t)}{H(t)} zz(t-r) - \rho_1 Q(t) yz(t-r) \\
& -\rho_1 (1+\rho_2) R(t) g(x) z(t-r) - kQ(t) y^2 + kyz + \rho_1 kyz(t-r) \\
& +kP(t)y^2 - (1+\rho_2)kR(t)xg(x) \\
& +\rho_2 R(t)(Z + ky + kx) \int_{t-\ell}^{t} y(s) g'(x(s))\,ds + \rho\beta_1 z^2 - \rho\beta_1 z^2(t-r) \\
& +\beta_2 \ell y^2 - \beta_2 \int_{t-\ell}^{t} y^2(s)\,ds + \frac{\partial F}{\partial t}.
\end{aligned}$$

Rewriting $\dot{F}_{(2.3)}$ as,

$$\dot{F}_{(2.3)} = F_6 + F_7 + \frac{\partial F}{\partial t},$$

such that

$$\begin{aligned}
F_6 \;=\;& (1+\rho_2) R(t) H(t) g'(x) y^2 + \frac{k}{H(t)} z^2 + \rho_1 \frac{k}{H(t)} zz(t-r) - \frac{P(t)}{H(t)} z^2 \\
& -\rho_1 \frac{P(t)}{H(t)} zz(t-r) - \rho_1 Q(t) yz(t-r) - \rho_1 (1+\rho_2) R(t) g(x) z(t-r) \\
& -kQ(t) y^2 + kyz + \rho_1 kyz(t-r) + kP(t)y^2 - (1+\rho_2)kR(t)xg(x),
\end{aligned}$$

$$\begin{aligned}
F_7 \;=\;& \rho_2 R(t)(Z + ky + kx) \int_{t-\ell}^{t} y(s) g'(x(s))\,ds + \rho\beta_1 z^2 - \rho\beta_1 z^2(t-r) \\
& +\beta_2 \ell y^2 - \beta_2 \int_{t-\ell}^{t} y^2(s)\,ds
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F}{\partial t} &= (1+\rho_2)\,kR'(t)\,G(x) + (1+\rho_2)\,R'(t)\,H(t)\,g(x)\,y \\ &+ (1+\rho_2)\,R(t)\,\theta(t)\,g(x)\,y + \frac{1}{2}kP'(t)\,y^2 + kP'(t)\,xy \\ &+ \frac{1}{2}Q'(t)\,H(t)\,y^2 + \frac{1}{2}Q(t)\,\theta(t)\,y^2 + \frac{1}{2}kQ'(t)\,x^2,\end{aligned}$$

by conditions i), ii) and (2.4) we have

$$\begin{aligned}F_6 \leq\ & (1+\rho_2)\,R(t)\,H(t)\,g'(x)\,y^2 + \frac{k}{H(t)}z^2 + \rho_1\frac{k}{H(t)}zz(t-r) - \frac{P(t)}{H(t)}z^2 \\ & -\rho_1\frac{P(t)}{H(t)}zz(t-r) - \rho_1 Q(t)\,yz(t-r) - \rho_1(1+\rho_2)\,R(t)\,g(x)\,z(t-r) \\ & -kQ(t)\,y^2 + kyz + \rho_1 kyz(t-r) + kP(t)y^2 - (1+\rho_2)kR(t)xg(x) \\ & +\frac{1}{2}\rho\frac{P(t)}{H(t)}|zz(t-r)| - \frac{1}{2}\rho\frac{P(t)}{H(t)}|zz(t-r)| + \frac{1}{4}(\rho-\rho^2)\frac{P(t)}{H(t)}z^2(t-r)\end{aligned}$$

it follows

$$\begin{aligned}F_6 \leq\ & (1+\rho_2)\,R(t)\,H(t)\,g'(x)\,y^2 + \frac{k}{H(t)}z^2 + \rho_1\frac{k}{2H(t)}z^2 + \rho_1\frac{k}{2H(t)}z^2(t-r) \\ & -\frac{P(t)}{H(t)}z^2 + \frac{\rho_1 P(t)}{2H(t)}z^2 + \frac{\rho_1 P(t)}{2H(t)}z^2(t-r) + \frac{\rho_1 Q(t)}{2}y^2 + \frac{\rho_1 Q(t)}{2}z^2(t-r) \\ & +\frac{\rho_1}{2}(1+\rho_2)\,R(t)\,g^2(x) + \frac{\rho_1}{2}(1+\rho_2)\,R(t)\,z^2(t-r) - kQ(t)\,y^2 + \frac{1}{2}ky^2 \\ & +\frac{1}{2}kz^2 + \frac{1}{2}\rho_1 ky^2 + \frac{1}{2}\rho_1 kz^2(t-r) + kP(t)y^2 - (1+\rho_2)kR(t)xg(x) \\ & +\rho\frac{P(t)}{4H(t)}z^2 + \rho\frac{P(t)}{2H(t)}z^2(t-r) - \rho\frac{P(t)}{2H(t)}|zz(t-r)| - \frac{\rho^2 P(t)}{4H(t)}z^2(t-r) \\ \leq\ & -\left[kQ(t) - (1+\rho_2)\,R(t)\,H(t)\,g'(x) - \frac{1}{2}\rho Q(t) - kP(t) - \frac{1}{2}\rho k - \frac{1}{2}k\right]y^2 \\ & -\left[\frac{P(t)}{H(t)} - \frac{k}{H(t)} - \rho\frac{k}{2H(t)} - \rho\frac{3P(t)}{4H(t)} - \frac{1}{2}k\right]z^2 \\ & +\frac{1}{2}\rho\left[\frac{k}{H(t)} + \frac{2P(t)}{H(t)} + Q(t) + 2R(t) + k\right]z^2(t-r) \\ & +\frac{1}{2}\rho(1+\rho_2)\,R(t)\,g^2(x) - (1+\rho_2)kR(t)xg(x) \\ & -\frac{1}{2}\rho\frac{P(t)}{H(t)}|zz(t-r)| - \frac{1}{4}\rho^2\frac{P(t)}{H(t)}z^2(t-r),\end{aligned}$$

$$\begin{aligned}
F_6 &\leq -\left[kQ_0 - R_1h_1\delta - \frac{1}{2}k - kP_1 - \frac{1}{2}\rho Q_1 - \frac{1}{2}\rho k - \rho R_1h_1\delta\right] y^2 \\
&\quad -\left[\frac{P_0}{h_1} - \frac{k}{h_0} - \frac{1}{2}k - \rho\frac{k}{2h_0} - \rho\frac{3P_1}{4h_0}\right] z^2 \\
&\quad +\frac{1}{2}\rho\left[Q_0 + \frac{2P_1}{h_0} + \frac{k}{h_0} + 2R_1 + k\right] z^2\left(t-r\right) \\
&\quad -\left(1+\rho_2\right)\left[kR_0\delta_1 - \frac{1}{2}\rho R_1\delta_2^2\right] x^2 - \frac{1}{2}\rho\frac{P_0}{h_1}|zz\left(t-r\right)| - \frac{1}{4}\rho^2\frac{P_0}{h_1}z^2\left(t-r\right),
\end{aligned}$$

as well as

$$\begin{aligned}
F_7 &= \rho_2 R\left(t\right)\left(Z + ky + kx\right)\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds \\
&\quad +\beta_2\ell y^2 + \rho\beta_1 z^2 - \rho\beta_1 z^2\left(t-r\right) - \beta_2\int_{t-\ell}^{t} y^2\left(s\right)ds \\
&= \rho_2 R\left(t\right)z\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds + \rho_2\rho_1 R\left(t\right)z\left(t-r\right)\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds \\
&\quad +\rho_2 kR\left(t\right)y\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds + \rho_2 kR\left(t\right)x\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds \\
&\quad +\rho\beta_1 z^2 - \rho\beta_1 z^2\left(t-r\right) + \beta_2\ell y^2 - \beta_2\int_{t-\ell}^{t} y^2\left(s\right)ds
\end{aligned}$$

then

$$\begin{aligned}
F_7 &\leq \frac{\rho}{2}\ell R_1\delta z^2 + \frac{\rho}{2}R_1\delta\int_{t-\ell}^{t} y^2\left(s\right)ds + \frac{\rho^2}{2}\ell R_1\delta z^2\left(t-r\right) + \frac{\rho^2}{2}R_1\delta\int_{t-\ell}^{t} y^2\left(s\right)ds \\
&\quad +\frac{1}{2}\rho\ell kR_1\delta y^2 + \frac{1}{2}\rho kR_1\delta\int_{t-\ell}^{t} y^2\left(s\right)ds + \frac{1}{2}\rho kR_1\delta\int_{t-\ell}^{t} y^2\left(s\right)ds \\
&\quad +\frac{1}{2}\rho\ell kR_1\delta x^2 + \rho\beta_1 z^2 - \rho\beta_1 z^2\left(t-r\right) + \beta_2\ell y^2 - \beta_2\int_{t-\ell}^{t} y^2\left(s\right)ds \\
&\leq \left(\frac{1}{2}\rho\ell R_1\delta - \rho\beta_1\right)z^2\left(t-r\right) + \left(\frac{1}{2}\rho\ell R_1\delta + \rho\beta_1\right)z^2 + \left(\frac{1}{2}\rho\ell kR_1\delta + \beta_2\ell\right)y^2 \\
&\quad +\frac{1}{2}\rho\left(1+\rho_2\right)\ell kR_1\delta x^2.
\end{aligned}$$

By taking

$$\begin{aligned}
\beta_1 &= \frac{1}{2}\left(\ell R_1\delta + Q_0 + \frac{2P_1}{h_0} + \frac{k}{h_0} + 2R_1 + k\right), \\
\beta_2 &= \frac{1}{2}\rho\left(1+\rho+2k\right)R_1\delta,
\end{aligned}$$

we obtain

$$\begin{aligned} F_6+F_7 \quad \leq \quad & -\Bigg[kQ_0 - R_1h_1\delta - \frac{1}{2}k - kP_1 - \frac{1}{2}\rho Q_1 - \frac{1}{2}\rho k - \rho R_1h_1\delta - \frac{1}{2}\rho\ell kR_1\delta \\ & -\beta_2\ell\Bigg]y^2 - \left[\frac{P_0}{h_1} - \frac{k}{h_0} - \frac{1}{2}k - \rho\frac{P_1}{h_0} - \rho\frac{k}{2h_0} - \frac{1}{2}\rho\ell R_1\delta - \rho\beta_1\right]z^2 \\ & + \left[\frac{1}{2}\rho\left(\ell R_1\delta + Q_0 + \frac{2P_1}{h_0} + \frac{k}{h_0} + 2R_1 + k\right) - \rho\beta_1\right]z^2(t-r) \\ & -(1+\rho_2)\left[kR_0\delta_1 - \frac{1}{2}\rho R_1\left(\delta_2^2 + \ell k\delta\right)\right]x^2 \\ & -\frac{1}{2}\rho\frac{P_0}{h_1}|zz(t-r)| - \frac{1}{4}\rho^2\frac{P_0}{h_1}z^2(t-r), \end{aligned}$$

thus

$$\begin{aligned} F_6+F_7 \leq & -\left[kQ_0 - R_1h_1\delta - \frac{1}{2}k - kP_1 - \frac{1}{2}\rho Q_1 - \frac{1}{2}\rho k - \rho R_1h_1\delta\right. \\ & \left. -\frac{1}{2}\rho\ell kR_1\delta - \rho(1+k)R_1\delta\ell\right]y^2 \\ & -\left[\frac{P_0}{h_1} - \frac{k}{h_0} - \frac{1}{2}k - \rho\frac{2P_1}{h_0} - \rho\frac{k}{h_0} - \rho\ell R_1\delta - \frac{1}{2}\rho(Q_0 + 2R_1 + k)\right]z^2 \\ & -(1+\rho_2)\left[kR_0\delta_1 - \frac{1}{2}\rho R_1\left(\delta_2^2 + \ell k\delta\right)\right]x^2 \\ & -\frac{1}{2}\rho\frac{P_0}{h_1}|zz(t-r)| - \frac{1}{4}\rho^2\frac{P_0}{h_1}z^2(t-r) \\ \leq & -\varrho\left(x^2 + y^2 + z^2 + \rho^2z^2(t-r) + 2\rho|zz(t-r)|\right) \\ \leq & -\varrho\left(x^2 + y^2 + z^2 + \rho^2z^2(t-r) + 2\rho zz(t-r)\right) = -\varrho\left(x^2 + y^2 + Z^2\right), \end{aligned}$$

where

$$\begin{aligned} \varrho = \min\Bigg\{ & kQ_0 - R_1h_1\delta - \frac{1}{2}k - kP_1 - \frac{1}{2}\rho Q_1 - \frac{1}{2}\rho k - \rho R_1h_1\delta - \frac{1}{2}\rho\ell kR_1\delta - \rho(1+k)R_1\delta\ell, \\ & \frac{P_0}{h_1} - \frac{k}{h_0} - \frac{1}{2}k - \rho\frac{2P_1}{h_0} - \rho\frac{k}{h_0} - \rho\ell R_1\delta - \frac{1}{2}\rho(Q_0 + 2R_1 + k), \\ & (1+\rho_2)[kR_0\delta_1 - \frac{1}{2}\rho R_1(\delta_2^2 + \ell k\delta)]\ , \ \frac{1}{4}\frac{P_0}{h_1}\Bigg\} \end{aligned}$$

and

$$\rho < \min\left\{1, \frac{2kQ_0 - 2R_1h_1\delta - 2kP_1 - k}{Q_1 + k + 2(1+k)R_1\delta\ell + (\ell k + 2h_1)R_1\delta}, \frac{2kR_0\delta_1}{R_1(\delta_2^2 + \ell k\delta)}, \right.$$

$$\left. \frac{2h_0}{4P_1 + 2k + 2\ell R_1\delta h_0 + h_0(Q_0 + 2R_1 + k)}\Big(\frac{P_0}{h_1} - \frac{k(2+h_0)}{2h_0}\Big)\right\}$$

On the other hand, by conditions i), iii) and iv) we obtain

$$\begin{aligned}
\frac{\partial F}{\partial t} &= (1+\rho_2)\,kR'(t)\,G(x) + (1+\rho_2)\,R'(t)\,H(t)\,g(x)\,y \\
&\quad + (1+\rho_2)\,R(t)\,\theta(t)\,g(x)\,y + \frac{1}{2}kP'(t)\,y^2 + kP'(t)\,xy \\
&\quad + \frac{1}{2}Q'(t)\,H(t)\,y^2 + \frac{1}{2}Q(t)\,\theta(t)\,y^2 + \frac{1}{2}kQ'(t)\,x^2 \\
&\leq (1+\rho)\,k\,|R'(t)|\,G(x) + \frac{1}{2}(1+\rho)\,|R'(t)|\,h_1\left(g^2(x) + y^2\right) \\
&\quad + \frac{1}{2}(1+\rho)\,R_1\,|\theta(t)|\left(g^2(x) + y^2\right) + \frac{1}{2}k\,|P'(t)|\,(x^2 + 2y^2) \\
&\quad + \frac{1}{2}h_1\,|Q'(t)|\,y^2 + \frac{1}{2}Q_1\,|\theta(t)|\,y^2 + \frac{1}{2}k\,|Q'(t)|\,x^2 \\
&\leq \left[(1+\rho)\,k\delta_2^2x^2 + \frac{1}{2}(1+\rho)\,h_1\delta_2^2x^2 + \frac{1}{2}(1+\rho)\,h_1y^2\right]|R'(t)| \\
&\quad + \frac{1}{2}k\left(x^2 + 2y^2\right)\big|P'(t)\big| + \frac{1}{2}\left(kx^2 + h_1y^2\right)\big|Q'(t)\big| \\
&\quad + \left[\frac{1}{2}Q_1y^2 + \frac{1}{2}(1+\rho)\,R_1y^2 + \frac{1}{2}(1+\rho)\,R_1\delta_2^2x^2\right]|\theta(t)| \\
&\leq \alpha_3\left(x^2 + y^2 + Z^2\right)\left[|R'(t)| + |P'(t)| + |Q'(t)| + |\theta(t)|\right] \\
&\leq \alpha_3\left(x^2 + y^2 + Z^2\right)\varphi(t) \leq \frac{\alpha_3}{\lambda_0}\varphi(t)\,F \quad \text{where } \left(F \geq \lambda_0\left(x^2 + y^2 + Z^2\right)\right).
\end{aligned}$$

By taking $\frac{1}{\alpha} = \frac{\alpha_3}{\lambda_0}$ we obtain

$$\dot{F}_{(2.3)} \leq -\varrho(x^2 + y^2 + Z^2) + \frac{1}{\alpha}\varphi(t)\,F. \tag{2.12}$$

By condition iii) and inequalities (2.9 ) and (2.12 ) we have

$$\begin{aligned}
\dot{W}_{(2.3)} &= \left(\dot{F}_{(2.3)} - \frac{1}{\alpha}\varphi(t)\,F\right)e^{-\frac{1}{\alpha}\int_{t_1}^{t}\varphi(s)\,ds} && (2.13) \\
&\leq -\varrho\left(x^2 + y^2 + Z^2\right)e^{-\frac{1}{\alpha}\int_{t_1}^{t}\varphi(s)\,ds} \\
&\leq -\alpha_4\left(x^2 + y^2 + Z^2\right), && (2.14)
\end{aligned}$$

where $\alpha_4 = \varrho\, e^{-\dfrac{\alpha_1}{\alpha}}$.

We have established that the zero solution of (2.1) is uniformly asymptotically stable. This fact completes the proof of Theorem 2.1. □

### 2.1.3 Boundedness And Square Integrability

This subsection extends the qualitative analysis of the third-order neutral delay differential equation (2.2) to include boundedness and square integrability under nonlinear perturbations ($\psi \neq 0$). The primary goal is to ensure that the system's trajectories remain within finite limits, a fundamental requirement for physical reliability. Furthermore, the property of square integrability is investigated to provide insights into energy dissipation, suggesting that the solutions and their derivatives diminish such that their total "energy" over time remains finite. These results are formalized in Theorem 2.2, which establishes the specific parametric constraints and integral bounds required to guarantee these two vital properties.

For $\psi \neq 0$ we have a theorem as follows :

**Theorem 2.2.** *Assume that all the conditions of Theorem 2.1 are fulfilled and that there exist positive constants $e_1$ and $e_2$ such that :*

$$|\psi(t,x(t),x(t-\ell),x'(t),x'(t-\ell),x''(t))| \leq |e(t)| \leq e_1 \quad \text{and} \quad \int_{t_1}^{+\infty} |e(s)|ds < e_2. \tag{2.15}$$

*Then there exists a positive constant $\mu_1$ such that any solution of (2.2) satisfy*

*1 .* $|x(t)| \leq \mu_1,\ |x'(t)| \leq \mu_1,\ |x''(t)H(t) + \rho_1 x''(t-r)H(t-r)| \leq \mu_1$

*and*

*2 .* $\displaystyle\int_{t_1}^{\infty} \Big(\big(x''(s)H(s) + \rho_1 x''(s-r)H(s-r)\big)^2 + x'^2(s) + x^2(s)\Big)ds < \infty.$

**Proof.** For the case $\psi(t) \neq 0$, the equation (2.1) is equivalent to the system

$$\begin{cases} x'(t) = & y(t), \\ y'(t) = & \dfrac{z(t)}{H(t)}, \\ Z'(t) = & -P(t)\dfrac{z(t)}{H(t)} - Q(t)y(t) - (1+\rho_2)R(t)g(x(t)) \\ & +\psi(t,x(t),x(t-\ell),x'(t),x'(t-\ell),x''(t)) \\ & +\rho_2 R(t)\displaystyle\int_{t-\ell}^{t} y\left(s\right)g'\left(x\left(s\right)\right)ds. \end{cases} \tag{2.16}$$

The derivative of the function $F$ along the trajectories of system (2.16) and by (2.14 ) we get

$$\begin{aligned} \dot{F}_{(2.16)} \leq & -\varrho\left(x^2(t)+y^2(t)+Z^2(t)\right) \\ & +\left(kx(t)+ky(t)+Z(t)\right)\psi(t,x(t),x(t-\ell),x'(t),x'(t-\ell),x''(t)). \end{aligned}$$

Using condition (2.15), we have

$$\dot{F}_{(2.16)} \leq -\varrho\left(x^2(t)+y^2(t)+Z^2(t)\right)+\left(k\left|x(t)\right|+k\left|y(t)\right|+\left|Z(t)\right|\right)\left|e(t)\right|.$$

Now, the inequality $|\nu| \leq \nu^2+1$ leads to

$$\dot{F}_{(2.16)} \leq -\varrho\left(x^2(t)+y^2(t)+Z^2(t)\right)+\lambda_3|e(t)|\big(x^2(t)+y^2(t)+Z^2(t)+3\big), \tag{2.17}$$

where $\lambda_3 = \max\{k,1\}$.
Due to ( 2.7), the above inequality implies

$$\dot{F}_{(2.16)} \leq \frac{\lambda_3}{\lambda_0}|e(t)|F(t)+3\lambda_3|e(t)|. \tag{2.18}$$

Integrating both sides (2.18) from $t_1$ to $t$, we obtain

$$F(t)-F(t_1) \leq 3\lambda_3\int_{t_1}^{t}|e(s)|ds+\frac{\lambda_3}{\lambda_0}\int_{t_1}^{t}F(s)|e(s)|ds.$$

Let

$$e_3 = F(t_1)+3\lambda_3 e_2. \tag{2.19}$$

Thus

$$F(t) \leq e_3 + \frac{\lambda_3}{\lambda_0}\int_{t_1}^{t} F(s)|e(s)|ds.$$

Using Gronwall inequality and ( 2.7) we have

$$F(t) \leq e_3 \exp\left(\frac{\lambda_3}{\lambda_0}\int_{t_1}^{t} |e(s)|ds\right) \leq e_4, \tag{2.20}$$

where $e_4 = e_3 \exp\left(\frac{\lambda_3}{\lambda_0} e_2\right)$. This result implies that there exists a constant $\mu_1$ such that

$$|x(t)| \leq \mu_1,\ |x'(t)| \leq \mu_1,\ |x''(t)H(t) + \rho_1 x''(t-r)H(t-r)| \leq \mu_1, \quad \text{for all} \ \ t \geq t_1.$$

Our next result concerns the square integrability of solutions of equation (2.2).

Define $V(t)$ as

$$V(t) = F(t) + \tau\int_{t_1}^{t}(x^2(s) + y^2(s) + Z^2(s))ds \quad \text{for all } t \geq t_1, \tag{2.21}$$

in which $\tau > 0$ is a constant to be specified later. By differentiating $V(t)$ and using (2.18) we obtain

$$V'(t) \leq (\tau - \varrho)\left(x^2(t) + y^2(t) + Z^2(t)\right) + \left(\frac{\lambda_3}{\lambda_0}F(t) + 3\lambda_3\right)|e(t)|.$$

If we choose $\tau - \varrho < 0$, then from (2.20) we get

$$V'(t) \leq \lambda_4|e(t)|, \tag{2.22}$$

where $\lambda_4 = \frac{\lambda_3}{\lambda_0}e_4 + 3\lambda_3$.
Integrating (2.22) from $t_1$ to $t$ and using the condition of Theorem 2.2 we obtain

$$V(t) - V(t_1) = \int_{t_1}^{t} V'(s)ds \leq \lambda_4 e_2.$$

By (2.20) and equality $V(t_1) = F(t_1)$ it is clear that

$$V(t) \leq \lambda_4 e_2 + e_3 - 3\lambda_3 e_2.$$

We conclude by (2.21) that

$$\int_{t_1}^{t}(x^2(s)+y^2(s)+Z^2(s))ds<\frac{\lambda_4 e_2+e_3-3\lambda_3 e_2}{\tau},$$

which means the existence of positive constant $\mu_2$ such that

$$\int_{t_1}^{t}x^2(s)ds\le\mu_2,\quad \int_{t_1}^{t}y^2(s)ds\le\mu_2 \text{ and } \int_{t_1}^{t}Z^2(s)ds\le\mu_2. \qquad \square$$

Consequently,

$$\begin{aligned} &\int_{t_1}^{t}x^2(s)ds\le\mu_2,\int_{t_1}^{t}x'^2(s)ds\le\mu_2 \qquad (2.23)\\ \text{and}\quad &\int_{t_1}^{t}\left(x''(s)H(s)+\rho_1 x''(s-r)H(s-r)\right)^2 ds\le\mu_2. \end{aligned}$$

This completes the proof of Theorem 2.2.

### 2.1.4 Example

We consider the following third order non-autonomous delay neutral differential equation

$$\begin{aligned} &\left[\left(2+\frac{x'(t)}{1+x'^2(t)}\right)x'(t)+\frac{2}{100}\left(2+\frac{x'(t-0.5)}{1+x'^2(t-0.5)}\right)x'(t-0.5)\right]''\\ &+\left(7+\frac{5\ln(3+\sin t)}{3(1+\cosh t)}\right)x''(t)+\left(\frac{33}{2}+\frac{1}{1+te^{-t^2}}\right)x'(t)\\ &+\left(\frac{1}{2}+\frac{1}{1+\cosh t}\right)\left(x(t)+\frac{2x(t)}{3+\ln(1+x(t)+x^2(t))}\right.\\ &\left.+\frac{3}{100}\left(x(t-0.3)+\frac{2x(t-0.3)}{3+\ln(1+x(t-0.3)+x^2(t-0.3))}\right)\right)\\ =\;&\frac{2+e^{-x''^2(t)}-\ln\left(\frac{1+t+t^2}{1+t^2}\right)+\cos(x(t-0.3))-\sin(x'(t-0.3))}{1+t+t^2+x^2(t)+x'^2(t)}, \end{aligned}$$

For all $t\ge t_1=t_0+0.5$. It is esay to see that:

$$\Omega\left(x'(t)\right) = 2+\frac{x'(t)}{1+x'^2(t)}$$

$$(i) \qquad 7=P_0\le P(t)=7+\frac{5\ln(3+\sin t)}{3(1+\cosh t)}\le 8=P_1$$

$$17 = Q_0\le Q(t)=\frac{33}{2}+\frac{1}{1+te^{-t^2}}\le\frac{37}{2}=Q_1$$

$$\frac{1}{2}=R_0\le R(t)=\frac{1}{2}+\frac{1}{1+\cosh t}\le 1=R_1.$$

$$(ii) \qquad 1=\delta_1\le\frac{g(x)}{x}=1+\frac{2}{3+\ln(1+x+x^2)}\le 2=\delta_2,$$

$$\begin{aligned}|\,g'(x)\,| &= |\,1+2\frac{(x^2+2x+3+(1+x+x^2)\ln(x^2+x+1))}{(\ln(x^2+x+1)+3)^2(x^2+x+1)}\,| \\ &\le 2=\delta,\ g(0)=0.\end{aligned}$$

Obviously, $P,Q,R\in\mathcal{C}^1(I,(0,\infty)),\ I=[t_1,\infty);\ g\in\mathcal{C}^1(\mathbb{R},\mathbb{R});\ \Omega\in\mathcal{C}^2(\mathbb{R},\mathbb{R})$.

(iii)

$$P'(t)=2\frac{(\cos t+\cos t\cosh t-3\sinh t\ln(\sin t+3)-\sin t\sinh t\ln(\sin t+3))}{(\sin t+3)(\cosh 2t+4\cosh t+3)},$$

$$\begin{aligned}\int_{t_1}^{+\infty}|P'(t)|\,dt &\le \int_0^{+\infty}|P'(t)|\,dt=0.624\,35 \\ Q'(t) &= -\frac{e^{-t^2}-2t^2e^{-t^2}}{(te^{-t^2}+1)^2}; \\ \int_{t_1}^{+\infty}|Q'(t)|\,dt &\le \int_0^{+\infty}\left|\left(e^{-t^2}-2t^2e^{-t^2}\right)\right|dt=\sqrt{2}e^{-\frac{1}{2}}\end{aligned}$$

$$R'(t)=-\frac{\sinh t}{(\cosh t+1)^2}; \qquad \int_{t_1}^{+\infty}|R'(t)|\,dt\le\int_0^{+\infty}|R'(t)|\,dt=\frac{1}{2},$$

then

$$\int_{t_1}^{+\infty}(|P'(t)|+|Q'(t)|+|R'(t)|)\,dt\le 0.624\,35+\sqrt{2}e^{-\frac{1}{2}}+\frac{1}{2}<\alpha_1=2.$$

(iv)

$$\begin{aligned}H(t) &= \Phi(x'(t))=\Omega'(x'(t))\,x'(t)+\Omega(x'(t)) \\ &= 2+2\frac{x'(t)}{(x'^2(t)+1)^2},\end{aligned}$$

then

$$\frac{4}{3} = h_0 \leq H(t) \leq h_1 = \frac{8}{3}$$

$$\Phi(y) = 2 + 2\frac{y}{(y^2+1)^2}; \quad \Phi'(y) = 2\frac{1-3y^2}{(y^2+1)^3}$$

and $$\int_{-\infty}^{+\infty} |\Phi'(y)|\, dy = \frac{3}{2}\sqrt{3} < \alpha_2 = 3,$$

$$\begin{aligned}
&|\psi(t,x(t),x(t-\ell),x'(t),x'(t-\ell),x''(t))| \\
= &\left|\frac{2 + e^{-x''^2(t)} - \ln\left(\frac{1+t+t^2}{1+t^2}\right) + \cos(x(t-\ell)) - \sin(x'(t-\ell))}{1+t+t^2+x^2(t)+x'^2(t)}\right| \\
\leq &\left|\frac{5 - \ln\left(\frac{1+t+t^2}{1+t^2}\right)}{1+t+t^2}\right| = |e(t)| \leq e_1 = 5
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_1}^{+\infty} |e(s)|\, ds &= \int_{t_1}^{\infty} \left|\frac{5 - \ln\left(\frac{1+s+s^2}{1+s^2}\right)}{1+s+s^2}\right| ds \\
&\leq \int_0^{\infty} \frac{5}{|s^2+s+1|}\, ds = \frac{10}{9}\sqrt{3}\pi,
\end{aligned}$$

clearly $\psi \in \mathcal{C}(I \times \mathbb{R}^5, \mathbb{R})$. The above constants lead to the following

$$\begin{aligned}
\rho = 0.03 < \min\Bigg\{&1,\ \frac{2kQ_0 - 2R_1h_1\delta - 2kP_1 - k}{Q_1 + k + 2(1+k)R_1\delta\ell + (\ell k + 2h_1)R_1\delta},\ \frac{2kR_0\delta_1}{R_1(\frac{2}{2}+\ell k\delta)}, \\
&\frac{2h_0}{4P_1 + 2k + 2\ell R_1\delta h_0 + h_0(Q_0 + 2R_1 + k)}\left(\frac{P_0}{h_1} - \frac{k(2+h_0)}{2h_0}\right)\Bigg\} \\
= \min\{&0.429\,12; 0.306\,12; 0.03128\,3,\}
\end{aligned}$$

where

$$\begin{cases}
k = 1.5 < \ k < \min\left\{\frac{P_0}{3},\ \frac{Q_0}{6},\ \frac{Q_0^2h_0}{4P_1^2},\ 2\frac{P_0h_0}{h_1(2+h_0)}\right\} \\
\qquad = \min\{2.\,333\,3; 2.\,833\,3; 1.\,505\,2; 2.\,1\} \\
Q_0 = 17 > \max\left\{\frac{4\delta h_1R_1}{k},\ \frac{R_1h_1\delta}{k} + P_1 + \frac{1}{2}\right\} = \max\{14.\,222; 12.\,056\}.
\end{cases}$$

The functions given above for constructing the equation of the example are suitable for the existence of all the constants that satisfy the conditions of the two theorems then all the results in the two theorems hold for the equation of the example.

## 2.2 A Study of a Second Class of Third-Order Neutral Delay Differential Equations

In this section, we extend our qualitative investigation to a second class of third-order neutral delay differential equations, as defined in equation (2.24). Similar to the previous analysis, our objective is to establish sufficient conditions that ensure the boundedness, square integrability, and uniform asymptotic stability of the solutions. This specific model incorporates different functional structures and nonlinear perturbations, requiring a dedicated approach to verify its stability properties. By constructing appropriate Lyapunov functionals and satisfying the required parametric constraints, we demonstrate that the system's trajectories remain controlled and converge toward the equilibrium state over time.

Let the following neutral differential equation

$$\begin{aligned}&\Big[x'(t)+\rho_1 x'(t-r)+h(x)+\rho_1 h\big(x(t-r)\big)\Big]''+P(t)x''(t)+Q(t)x'(t)\\&+R(t)\Big[g\big(x(t)\big)+\rho_2 g\big(x(t-\sigma)\big)\Big]=\psi\big(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)\big)\end{aligned}\tag{2.24}$$

for all $t \geq t_1 = t_0 + \max\{r,\sigma\}$, where, $\max\{\rho_1,\rho_2\} = \rho < 1$, $r, \sigma, \rho_1$ and $\rho_2$ some positive constants, and $P, Q, R \in \mathcal{C}^1(\mathbb{R}^+, (0,\infty))$, $\mathbb{R}^+ = [0,\infty)$; $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$; $h \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$; $\psi \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^5, \mathbb{R})$ and $g(0) = 0$.

By solution of (2.24) we mean a continuous function $x : [t_x,\infty) \to \mathbb{R}$ for $t_x \geq t_1$ which satisfy the equation (2.24) in $[t_x,\infty)$ and such that

$$x(t) + \rho_1 x(t-r) \in \mathcal{C}^3([t_x,\infty),\mathbb{R}).$$

### 2.2.1 Basic Assumptions

This subsection outlines the fundamental hypotheses required to perform a comprehensive qualitative analysis of the second neutral differential equation (2.24). These assumptions define the necessary mathematical constraints on the functional coefficients $P(t), Q(t)$, and $R(t)$, as well as the nonlinear terms and time-varying delays involved in the system. By establishing these prerequisites, we ensure that the conditions for the existence and stability of solutions are rigorously met. These criteria form the essential foundation for the subsequent stability theorems and integral inequalities that characterize the long-term behavior of the trajectories within this specific model.

The following hypotheses on the functions appearing in the equation (2.24) will be useful in next subsequent sections, assume that there are positive constants $p_0, p_1, R_0, R_1, q_0, q_1$, $\xi_1, \xi_2, \xi_3$ and $\gamma$ such that the following conditions are satisfied :

i) $0 < p_0 \leq P(t) \leq p_1,\ \ 0 < q_0 \leq Q(t) \leq q_1,\ \ 0 < R_0 \leq R(t) \leq R_1$

$$Q'(t) \leq 0,\ R'(t) \leq 0 \ \text{ for all } \ t \geq t_1;$$

ii) $g(0) = 0,\ \ \xi_1 \leq \dfrac{g(x)}{x} \leq \xi_2 \ \ (x \neq 0)$ and $\mid g'(x) \mid \leq \xi_3$ for all $x \in \mathbb{R}$;

iii) $\displaystyle\int_{t_1}^{t} \Big( \mid P'(s) \mid - R'(s)\Big) ds \leq \gamma.$

The equation (2.24) is equivalent to the system

$$\begin{cases} x'(t) = y(t) - h(x(t)), \\ y'(t) = z(t), \\ Z'(t) = \ \ -P(t)z(t) + P(t)h'(x)\vartheta(t) - Q(t)\vartheta(t) - (1+\rho_2)R(t)g(x) \\ \qquad\qquad +\rho_2 R(t) \int_{t-\sigma}^{t} \Big(y(s) - h(x(s))\Big) g'(x(s)ds \\ \qquad\qquad +\psi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)), \end{cases} \tag{2.25}$$

where

$$\begin{cases} y(t) - h(x(t)) = \vartheta(t), \\ \\ y(t) = x'(t) + h(x(t)), \\ \\ Z(t) = y'(t) + \rho_1 y'(t-r) = z(t) + \rho_1 z(t-r). \end{cases}$$

### 2.2.2 Boundedness And Square Integrability

In this subsection, we investigate the fundamental properties of boundedness and square integrability for the solutions of the second neutral differential equation (2.24). Our analysis focuses on the general case where the system is subject to nonlinear perturbations, represented by the conditions $h \neq 0$ and $\psi \neq 0$. The primary objective is to establish The following theorem, which identifies a set of sufficient conditions and parametric constraints that guarantee the trajectories and their derivatives remain within finite limits over time. By ensuring these criteria are met, we provide mathematical assurance that the system's energy dissipates effectively, preventing any divergent behavior and confirming the stability of the physical process modeled by the equation.

Let the following theorem where $h \neq 0$ and $\psi \neq 0$.

**Theorem 2.3.** *In addition to assumptions (i)-(iii), assume that there are positive constants $\delta$, $\varphi_1$ and $D_1$ such that the following conditions are satisfied:*

*H1)* $| h'(u) | \leq \delta$ *for all* $u \in \mathbb{R}$ *and* $\delta < \min\left\{\frac{2kR_0\xi_1}{R_1\xi_2^2}, \frac{2p_0 - 3k}{p_1 - k}\right\}$,

*H2)* $| \psi(t,x(t),x(t-r),x'(t),x'(t-r),x''(t)) | \leq \varphi(t) \leq \varphi_1$ *and* $\int_{t_1}^{t} \varphi(s)ds \leq D_1$.

*Then, there exists a finite positive constant $\eta$ such that all the solutions $x(.)$ of (2.24) and their derivatives $x'(.)$ and $x''(.)$ fulfill*

***I)*** $| x(t) | \leq \eta$, $| x'(t) | \leq \eta$ *and* $| x''(t) + \rho_1 x''(t-r) | \leq \eta$, *for all* $t \geq t_1$,

***II)*** $\displaystyle\int_{t_1}^{\infty}\Big(x^2(s)+x'^2(s)+\big(x''(s)+\rho_1 x''(s-r)\big)^2\Big)ds<\infty,$

*provided that*

$$\rho<\min\Bigg\{\frac{2(k+\delta)q_0-R_1(2\xi_3+\delta)-k(1+2p_1)-\delta(p_1-k)}{R_1\xi_3\big[(2+3k)\sigma+2\big]+q_1+k+R_1\delta},\ \frac{2kR_0\xi_1+\delta R_1\xi_2^2}{R_1(\xi_3\sigma k+\xi_2^2)},\ \frac{2p_0-3k-\delta(p_1-k)}{2R_1(\xi_3\sigma+1)+3k+8p_1+q_1}\Bigg\}, \tag{2.26}$$

*where*

$$\begin{cases} k<\min\Big\{\dfrac{p_0}{2},\ \dfrac{q_0^2}{4p_1^2},\ \dfrac{q_0}{4}\Big\},\\[2ex] q_0>\max\Big\{\dfrac{R_1(2\xi_3+\delta)+k(1+2p_1)+\delta(p_1-k)}{2(k+\delta)},\ \dfrac{4R_1\xi_3}{k}\Big\}.\end{cases} \tag{2.27}$$

**Proof.**

Our main tool is the continuously differentiable function $V=V(t;x;y;z)$ defined by

$$V=W\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big), \tag{2.28}$$

in which $\Delta(t)=\mid P'(t)\mid -R'(t)$, the function $W=W(t;x;y;z)$ is given by

$$\begin{aligned} W \;=\; & \frac{1}{2}Z^2+k\vartheta Z+\frac{k}{2}P(t)\vartheta^2+(1+\rho_2)kR(t)G(x)+(1+\rho_2)R(t)\vartheta g(x)+\frac{Q(t)}{2}\vartheta^2\\ & +kxZ+\frac{1}{2}kQ(t)x^2+kP(t)x\vartheta+\mu\int_{t-r}^{t}z^2(s)ds+\lambda\int_{-\sigma}^{0}\int_{t+s}^{t}\vartheta^2(\tau)d\tau ds, \end{aligned}$$

with $\Gamma$,$\mu$ and $\lambda$ are positive constants to be determined later in the proof and $G(x)=\int_0^x g(u)du$. Rewriting $W$ as follow

$$W=W_1+W_2+W_3+W_4+\mu\int_{t-r}^{t}z^2(s)ds+\lambda\int_{-\sigma}^{0}\int_{t+s}^{t}\vartheta^2(\tau)d\tau ds,$$

where

$$\begin{aligned} W_1 &= \frac{1}{4}Z^2 + k\vartheta Z + \frac{1}{2}kP(t)\vartheta^2, \\ W_2 &= (1+\rho_2)kR(t)G(x) + (1+\rho_2)R(t)\vartheta g(x) + \frac{1}{4}Q(t)\vartheta^2, \\ W_3 &= \frac{1}{4}kQ(t)x^2 + kP(t)x\vartheta + \frac{1}{4}Q(t)\vartheta^2, \\ W_4 &= \frac{1}{4}Z^2 + kxZ + \frac{1}{4}kQ(t)x^2. \end{aligned}$$

In view of conditions (i), (ii) and (2.27) we have

$$\begin{aligned} W_1 &= \frac{1}{4}Z^2 + k\vartheta Z + \frac{1}{2}kP(t)\vartheta^2 \\ &= \frac{1}{4}\Big[(Z+2k\vartheta)^2 + 2k\big(P(t)-2k\big)\vartheta^2\Big] \\ &\geq \frac{1}{2}k(p_0-2k)\vartheta^2 \\ &\geq k_1\vartheta^2, \end{aligned}$$

where

$$k_1 = \frac{1}{2}k(p_0-2k).$$

Rearrange $W_2$ we obtain the estimate

$$\begin{aligned} W_2 =&(1+\rho_2)kR(t)G(x) + (1+\rho_2)R(t)\vartheta g(x) + \frac{1}{4}Q(t)\vartheta^2 \\ =&(1+\rho_2)kR(t)G(x) + \frac{Q(t)}{4}\left[\vartheta^2 + \frac{4(1+\rho_2)R(t)g(x)\vartheta}{Q(t)}\right] \\ =&(1+\rho_2)kR(t)\int_0^x g(u)du \\ &+ \frac{Q(t)}{4}\left[\Big(\vartheta + \frac{2(1+\rho_2)R(t)}{Q(t)}g(x)\Big)^2 - 4(1+\rho_2)^2\frac{R^2(t)}{Q^2(t)}g^2(x)\right] \\ \geq&(1+\rho_2)kR(t)\int_0^x g(u)du - (1+\rho_2)^2\frac{R^2(t)}{Q(t)}g^2(x) \\ \geq&(1+\rho_2)kR(t)\left[\int_0^x g(u)du - 2(1+\rho_2)\frac{R(t)}{kQ(t)}\int_0^x g(u)g'(u)du\right] \\ \geq&(1+\rho_2)kR_0\int_0^x\Big(1-\frac{4R_1\xi_3}{kq_0}\Big)g(u)du = (1+\rho_2)kR_0\Big(1-\frac{4R_1\xi_3}{kq_0}\Big)G(x). \end{aligned}$$

Note that by (ii) we have

$$\xi_1^2 \leq \frac{g^2(x)}{x^2},$$

which implies

$$\frac{\xi_1^2}{2\xi_3}x^2 \leq \frac{1}{2\xi_3}g^2(x) = \frac{1}{\xi_3}\int_0^x g(u)g'(u)du \leq G(x),$$

so

$$W_2 \geq k_2 x^2$$

where

$$k_2 = \frac{\xi_1^2}{2\xi_3}(1+\rho_2)kR_0\Big(1 - \frac{4R_1\xi_3}{kq_0}\Big).$$

We have also,

$$\begin{aligned}
W_3 &= \frac{1}{4}Q(t)\vartheta^2 + kP(t)x\vartheta + \frac{1}{4}kQ(t)x^2 \\
&= \frac{1}{4}Q(t)\Big[\big(\vartheta + \frac{2kP(t)}{Q(t)}x\big)^2 - \frac{4k^2P^2(t)}{Q^2(t)}x^2 + kx^2\Big] \\
&= \frac{1}{4}Q(t)\Big[\big(\vartheta + \frac{2kP(t)}{Q(t)}x\big)^2 + k[1 - \frac{4kP^2(t)}{Q^2(t)}]x^2\Big] \\
&\geq \frac{1}{4}q_0k\Big[1 - \frac{4kp_1^2}{q_0^2}\Big]x^2 \\
&\geq k_3x^2,
\end{aligned}$$

where

$$k_3 = \frac{1}{4}q_0k\Big[1 - \frac{4kp_1^2}{q_0^2}\Big].$$

Rearranging $W_4$ we get

$$\begin{aligned} W_4 &= \frac{1}{4}kQ(t)x^2 + kxZ + \frac{1}{4}Z^2 \\ &= \frac{1}{4}kQ(t)\Big[x^2 + \frac{4}{Q(t)}xZ + \frac{1}{kQ(t)}Z^2\Big] \\ &= \frac{1}{4}kQ(t)\Big[\big(x + \frac{2}{Q(t)}Z\big)^2 + \frac{1}{Q(t)}\big(\frac{1}{k} - \frac{4}{Q(t)}\big)Z^2\Big] \\ &\geq \frac{1}{4}k\Big[\frac{1}{k} - \frac{4}{q_0}\Big]Z^2 \\ &\geq k_4Z^2, \end{aligned}$$

where

$$k_4 = k\Big[\frac{1}{k} - \frac{4}{q_0}\Big].$$

Since

$$\mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} \vartheta^2(\tau)d\tau ds > 0,$$

it follows that

$$W \geq k_5\Big(x^2 + \vartheta^2 + Z^2\Big), \tag{2.29}$$

where

$$k_5 = \min\{k_1,k_2,k_3,k_4\}.$$

By (iii) we conclude that

$$1 \geq \exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big) \geq \exp(\frac{-\gamma}{\Gamma}), \tag{2.30}$$

using (2.28) and (2.29) we obtain

$$V \geq k_6\Big(x^2 + \vartheta^2 + Z^2\Big),$$

where

$$k_6 = \exp(\frac{-\gamma}{\Gamma})k_5.$$

For the time derivative of the function $W$ along the trajectories of system (2.25), a straight forward calculation yields

$$\dot{W}_{(2.25)} = W_5 + W_6 + W_7, \tag{2.31}$$

such that

$$\begin{aligned} W_5 &= (k - P(t))z^2 + h'(x)\vartheta\Big[(P(t) - k)Z - (1+\rho_2)R(t)g(x) - Q(t)\vartheta\Big] \\ &\quad +\rho_1 kzz(t-r) - \rho_1 Q(t)\vartheta z(t-r) - \rho_1 P(t)zz(t-r) \\ &\quad -\rho_1(1+\rho_2)R(t)g(x)z(t-r) - kQ(t)\vartheta^2 + (1+\rho_2)R(t)g'(x)\,\vartheta^2 + k\vartheta Z \\ &\quad -k(1+\rho_2)R(t)g(x)x + kP(t)\vartheta^2 + \mu z^2 - \mu z^2(t-r) - \lambda\int_{t-\sigma}^{t}\vartheta^2(s)ds \\ &\quad +\lambda\sigma\vartheta^2 + \Big[k(x+\vartheta) + Z\Big]\psi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)), \end{aligned}$$

$$W_6 = \rho_2 R(t)\Big[z + \rho_1 z(t-r) + k\vartheta + kx\Big]\int_{t-\sigma}^{t} g'(x(s))\vartheta(s)ds$$

and

$$W_7 = \frac{\partial W}{\partial t}.$$

By conditions (i), (ii) and by applying the estimate $2st \le s^2 + t^2$

$$\begin{aligned} W_6 &\le \rho_2 R_1\Big[\frac{\xi_3\sigma}{2}z^2 + \frac{\xi_3}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds + \rho_1\frac{\xi_3\sigma}{2}z^2(t-\sigma) + \rho_1\frac{\xi_3}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds \\ &\quad +\frac{\xi_3\sigma}{2}k\vartheta^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds + \frac{\xi_3\sigma}{2}kx^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds\Big] \\ &\le \rho R_1\Big[\frac{\xi_3\sigma}{2}z^2 + \frac{\xi_3}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds + \rho\frac{\xi_3\sigma}{2}z^2(t-\sigma) + \rho\frac{\xi_3}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds \\ &\quad +\frac{\xi_3\sigma}{2}k\vartheta^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds + (1+\rho)\frac{\xi_3\sigma}{2}kx^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t}\vartheta^2(s)ds\Big]. \end{aligned}$$

From conditions (i), (ii), (H1), (2.26), (2.27) and the estimate $u \leq \mid u \mid \leq u^2+1$

$$\begin{aligned}
W_5 \ &\leq (k-p_0)z^2 + \frac{\delta}{2}\Big[(p_1-k)(Z^2+\vartheta^2) + (1+\rho_2)R_1(g^2(x)+\vartheta^2) - 2q_0\vartheta^2\Big] \\
&+ \frac{\rho_1 k}{2}z^2 + \frac{\rho_1 k}{2}z^2(t-r) + \frac{\rho_1}{2}q_1\vartheta^2 + \frac{\rho_1}{2}q_1 z^2(t-r) + \frac{\rho_1}{2}p_1 z^2 + \frac{\rho_1}{2}p_1 z^2(t-r) \\
&+ \frac{\rho_1}{2}(1+\rho_2)R_1\xi_2^2 x^2 + \frac{\rho_1}{2}(1+\rho_2)R_1 z^2(t-r) - kq_0\vartheta^2 + (1+\rho_2)R_1\xi_3\vartheta^2 \\
&+ \frac{k}{2}\vartheta^2 + \frac{k}{2}z^2 + \frac{\rho_1 k}{2}\vartheta^2 + \frac{\rho_1 k}{2}z^2(t-r) - k(1+\rho_2)R_0\xi_1 x^2 + kp_1\vartheta^2 \\
&+ \mu z^2 - \mu z^2(t-r) + \lambda\sigma\vartheta^2 - \lambda\int_{t-\sigma}^{t}\vartheta^2(s)ds \\
&+ \Big[k(x^2+\vartheta^2+2) + Z^2 + 1\Big] \mid \psi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)) \mid \\
&- 2\rho_1 P(t) \mid zz(t-r) \mid + 2\rho_1 P(t) \mid zz(t-r) \mid + (\rho_1-\rho_1^2)P(t)z^2(t-r), \\
&\leq (k-p_0)z^2 + \frac{\delta}{2}\Big[(p_1-k)(Z^2+\vartheta^2) + (1+\rho_2)R_1(g^2(x)+\vartheta^2) - 2q_0\vartheta^2\Big] \\
&+ \frac{\rho_1 k}{2}z^2 + \frac{\rho_1 k}{2}z^2(t-r) + \frac{\rho_1}{2}q_1\vartheta^2 + \frac{\rho_1}{2}q_1 z^2(t-r) + \frac{\rho_1}{2}p_1 z^2 + \frac{\rho_1}{2}p_1 z^2(t-r) \\
&+ \frac{\rho_1}{2}(1+\rho_2)R_1\xi_2^2 x^2 + \frac{\rho_1}{2}(1+\rho_2)R_1 z^2(t-r) - kq_0\vartheta^2 + (1+\rho_2)R_1\xi_3\vartheta^2 \\
&+ \frac{k}{2}\vartheta^2 + \frac{k}{2}z^2 + \frac{\rho_1 k}{2}\vartheta^2 + \frac{\rho_1 k}{2}z^2(t-r) - k(1+\rho_2)R_0\xi_1 x^2 + kp_1\vartheta^2 \\
&+ \mu z^2 - \mu z^2(t-r) + \lambda\sigma\vartheta^2 - \lambda\int_{t-\sigma}^{t}\vartheta^2(s)ds \\
&+ \Big[k(x^2+\vartheta^2+2) + Z^2 + 1\Big] \mid \psi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)) \mid \\
&+ \rho_1 P(t)z^2 + 2\rho_1 P(t)z^2(t-r) - 2\rho_1 P(t) \mid zz(t-r) \mid - \rho_1^2 P(t)z^2(t-r).
\end{aligned}$$

Therefore,

$$\begin{aligned}
W_5 + W_6 \ \ \leq \ \ & -\Big[(k+\delta)q_0 - \frac{\rho}{2}(R_1\xi_3\sigma k + q_1 + k) - (1+\rho)R_1(\xi_3 + \frac{\delta}{2}) \\
& - \frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k) - \lambda\sigma\Big]\vartheta^2 \\
& -(1+\rho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\rho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2 \\
& -\Big[p_0 - \frac{\rho}{2}(R_1\xi_3\sigma + k + 5p_1) - \frac{3}{2}k - \mu\Big]z^2 + \frac{\delta}{2}(p_1-k)Z^2 \\
& +\Big[\frac{\rho}{2}\Big(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 3p_1\Big) - \mu\Big]z^2(t-r) \\
& +\Big[\rho R_1\xi_3(1+k) - \lambda\Big]\int_{t-\sigma}^{t}\vartheta^2(s)ds \\
& +d\Big(x^2+\vartheta^2+Z^2\Big)\varphi(t) + 3d\varphi(t) - 2\rho_1 p_0|zz(t-r)| - \rho_1^2 p_0 z^2(t-r).
\end{aligned}$$

By taking

$$\begin{aligned}\lambda &= \rho R_1\xi_3(1+k)\\ \mu &= \frac{\rho}{2}\Big(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 5p_1\Big)\\ d &= \max\{k,1\}\end{aligned}$$

and by (2.29)

$$\begin{aligned}W_5+W_6 &\le -\Big[(k+\delta)q_0 - \frac{\rho}{2}\Big(R_1\xi_3\big[(2+3k)\sigma+2\big] + q_1 + k + R_1\delta\Big) - R_1(\xi_3+\frac{\delta}{2})\\ &\quad -\frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k)\Big]\vartheta^2\\ &\quad -(1+\rho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\rho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2 + \frac{\delta}{2}(p_1-k)Z^2\\ &\quad -\Big[p_0 - \frac{\rho}{2}\Big(2R_1(\xi_3\sigma+1) + 3k + 8p_1 + q_1\Big) - \frac{3k}{2}\Big]\Big[z^2 + 2\rho_1|zz(t-r)|\\ &\quad +\rho_1^2 z^2(t-r)\Big] + d\Big(x^2+\vartheta^2+Z^2\Big)\varphi(t) + 3d\varphi(t),\\[1em] &\le -\Big[(k+\delta)q_0 - \frac{\rho}{2}\Big(R_1\xi_3\big[(2+3k)\sigma+2\big] + q_1 + k + R_1\delta\Big) - R_1(\xi_3+\frac{\delta}{2})\\ &\quad -\frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k)\Big]\vartheta^2\\ &\quad -(1+\rho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\rho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2\\ &\quad -\Big[p_0 - \frac{\rho}{2}\Big(2R_1(\xi_3\sigma+1) + 3k + 8p_1 + q_1\Big) - \frac{3k}{2} - \frac{\delta}{2}(p_1-k)\Big]Z^2\\ &\quad +d\Big(x^2+\vartheta^2+Z^2\Big)\varphi(t) + 3d\varphi(t),\end{aligned}$$

provided that

$$\rho < \min\left\{\frac{2(k+\delta)q_0 - R_1(2\xi_3+\delta) - k(1+2p_1) - \delta(p_1-k)}{R_1\xi_3\big[(2+3k)\sigma+2\big] + q_1 + k + R_1\delta},\right.$$
$$\left.\frac{2kR_0\xi_1 + \delta R_1\xi_2^2}{R_1(\xi_3\sigma k + \xi_2^2)}, \frac{2p_0 - 3k - \delta(p_1-k)}{2R_1(\xi_3\sigma+1) + 3k + 8p_1 + q_1}\right\},$$

Hence, there exists a positive constant $S$ such that,

$$\begin{aligned}W_5+W_6 &\le -S\Big[x^2+\vartheta^2+Z^2\Big] + d\Big(x^2+\vartheta^2+Z^2\Big)\varphi(t) + 3d\varphi(t)\\ &\le (d\varphi_1 - S)\Big(x^2+\vartheta^2+Z^2\Big) + 3d\varphi(t),\end{aligned}\tag{2.32}$$

where $S > d\varphi_1$ and

$$S=\min\Big\{(k+\delta)q_0-\frac{\rho}{2}\Big(R_1\xi_3\big[(2+3k)\sigma+2\big]+q_1+k+R_1\delta\Big)-R_1(\xi_3+\frac{\delta}{2})$$
$$-\frac{k}{2}(1+2p_1)-\frac{\delta}{2}(p_1-k),\ \ (1+\rho_2)\Big[kR_0\xi_1-\frac{\delta}{2}R_1\xi_2^2-\frac{\rho R_1}{2}(\xi_3\sigma k+\xi_2^2)\Big],$$
$$p_0-\frac{\rho}{2}\Big(2R_1(\xi_3\sigma+1)+3k+8p_1+q_1\Big)-\frac{3k}{2}-\frac{\delta}{2}(p_1-k)\Big\},$$

also

$$\begin{aligned}
W_7 &= \frac{1}{2}kP'(t)\vartheta^2+(1+\rho_2)R'(t)g(x)\vartheta+k(1+\rho_2)R'(t)G(x)+\frac{1}{2}Q'(t)\vartheta^2\\
&\quad+\frac{1}{2}kQ'(t)x^2+kP'(t)x\vartheta\\
&= \frac{1}{2}kP'(t)\vartheta^2+(1+\rho_2)R'(t)\Big[g(x)\vartheta+kG(x)\Big]+\frac{1}{2}Q'(t)(\vartheta^2+kx^2)+kP'(t)x\vartheta\\
&\le \frac{1}{2}k\mid P'(t)\mid\vartheta^2+(1+\rho)\mid R'(t)\mid\Big[\frac{1}{2}g^2(x)+\frac{1}{2}\vartheta^2\Big]+\frac{1}{2}k\mid P'(t)\mid(x^2+\vartheta^2)\\
&\le \frac{1}{2}k\Delta(t)(x^2+2\vartheta^2)+\frac{1}{2}(1+\rho)\Delta(t)\Big[\xi_2^2x^2+\vartheta^2\Big]\\
&\le \omega\Delta(t)(x^2+\vartheta^2+Z^2), \qquad (2.33)
\end{aligned}$$

where $\omega=\dfrac{2k+(1+\rho)(\xi_2^2+1)}{2}$.

By (2.29), (2.32), (2.33) and expression (2.31) can be rewritten as

$$\dot{W}_{(2.25)}\le -M\big(x^2+\vartheta^2+Z^2\big)+\frac{\omega}{k_5}\Delta(t)W+3d\varphi(t),$$

where $M=S-d\varphi_1$

The derivative of the functional $V$ along the trajectories of system ( 2.25) is given by

$$\begin{aligned}
\dot{V}_{(2.25)} &=\Big[\dot{W}_{(2.25)}-\frac{1}{\Gamma}\Delta(t)W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big)\\
&\le\Big[-M\big(x^2+\vartheta^2+Z^2\big)+\frac{\omega}{k_5}\Delta(t)W+3d\varphi(t)-\frac{\Delta(t)}{\Gamma}W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big).
\end{aligned}$$

Let $\Gamma^{-1}=\dfrac{\omega}{k_5}$, hence

$$\dot{V}_{(2.51)}\le\Big[-M\big(x^2+\vartheta^2+Z^2\big)+3d\varphi(t)\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big).$$

By inequality (2.57)

$$\dot{V}_{(2.51)} \leq -N\Big(x^2 + \vartheta^2 + Z^2\Big) + 3d\varphi(t), \tag{2.34}$$

where $N = M\exp(\frac{-\gamma}{\Gamma})$, by integrating (2.34) from $t_1$ to $t$ where $t \geq t_1$,

$$\begin{aligned} V(t) &\leq V(t_1) + 3d\int_{t_1}^{t} \varphi(s)ds \\ &\leq D_2, \end{aligned} \tag{2.35}$$

where $D_2 = V(t_1) + 3dD_1$. By (2.28)

$$W = V\exp(\frac{1}{\Gamma}\int_{t_1}^{t} \Delta(s)ds)$$

and from (2.57) and (2.35)

$$W \leq D_2 \exp(\frac{\gamma}{\Gamma}).$$

Due to the boundedness of $W$, there exists a positive constant $\eta$ such that

$$\mid x(t) \mid \leq \eta, \quad \mid \vartheta(t) \mid \leq \eta \ \ and \quad \mid Z(t) \mid \leq \eta, \tag{2.36}$$

we have

$$\mid x'(t) \mid = \mid y(t) - h(x(t)) \mid = \mid \vartheta(t) \mid \leq \eta,$$

thanks to the boundedness of $x(t)$ and $(H1)$,

$$\begin{aligned} \mid x''(t) + \rho_1 x''(t-r) \mid &= \mid Z(t) - h'(x(t))x'(t) - \rho_1 h'(x(t-r))x'(t-r) \mid \\ &\leq \mid Z(t) \mid + \mid h'(x(t)) \mid\mid x'(t) \mid + \rho_1 \mid h'(x(t-r)) \mid\mid x'(t-r) \mid \\ &\leq \eta_1 \end{aligned}$$

where $\eta_1 = \eta\Big(1 + (1+\rho_1)\delta\Big)$.

Finally $x$, $x'$ and $x''(t) + \rho_1 x''(t-r)$ are bounded.

We will show that the solutions and their derivatives belong to $L^2$.

We define the function

$$E(t) = V(t) + \alpha \int_{t_1}^{t} \Big( x^2(s) + \vartheta^2(s) + Z^2(s) \Big) ds, \quad \forall t \geq t_1, \ \alpha > 0, \tag{2.37}$$

according to (2.34)

$$\begin{aligned} \dot{E}(t) &\leq \dot{V}(t) + \alpha \Big( x^2(t) + \vartheta^2(t) + Z^2(t) \Big) \\ &\leq (\alpha - N) \Big( x^2(t) + \vartheta^2(t) + Z^2(t) \Big) + 3d\varphi(t), \end{aligned}$$

if we take $\alpha < N$ then

$$\dot{E}(t) \quad \leq 3d\varphi(t). \tag{2.38}$$

Integrating (2.38) from $t_1$ to $t$, we obtain

$$\begin{aligned} E(t) &\leq E(t_1) + 3d \int_{t_1}^{t} \varphi(s) ds \\ &\leq E(t_1) + 3dD_1, \end{aligned} \tag{2.39}$$

we use (2.37) and (2.39)

$$\begin{aligned} \alpha \int_{t_1}^{t} \Big( x^2(t) + \vartheta^2(t) + Z^2(t) \Big) ds &\leq V(t) + \alpha \int_{t_1}^{t} \Big( x^2(t) + \vartheta^2(t) + Z^2(t) \Big) ds \\ &\leq E(t_1) + 3dD_1, \end{aligned}$$

while $E(t_1) = V(t_1)$ it follows that

$$\int_{t_1}^{t} \Big( x^2(t) + \vartheta^2(t) + Z^2(t) \Big) ds \leq \frac{V(t_1) + 3dD_1}{\alpha} = D_2.$$

Therefore,

$$\int_{t_1}^{t} x^2(s) ds \leq D_2, \quad \int_{t_1}^{t} \vartheta^2(s) ds \leq D_2 \ \ \textit{and} \ \ \int_{t_1}^{t} Z^2(s) ds \leq D_2, \tag{2.40}$$

we have

$$\int_{t_1}^{t} x'^2(s) ds = \int_{t_1}^{t} \vartheta^2(s) ds \leq D_2.$$

By (2.40) and (*H1*) we have

$$\begin{aligned}
&\int_{t_1}^{t}\Big[x''(s)+\rho_1 x''(s-r)\Big]^2 ds\\
&=\int_{t_1}^{t}\Big[Z(s)-h'(x(s))x'(s)-\rho_1 h'(x(s-r))x'(s-r)\Big]^2 ds\\
&=\int_{t_1}^{t}\Bigg[Z^2(s)+h'^2(x(s))x'^2(s)+\rho_1^2 h'^2(x(s-r))x'^2(s-r)\\
&\qquad -2Zh'(x(s))x'(s)-2\rho_1 Zh'(x(s-r))x'(s-r)\\
&\qquad +2\rho_1 h'(x(s))h'(x(s-r))x'(s)x'(s-r)\Bigg]ds\\
&\le \Big(1+\delta(1+\rho_1)\Big)\int_{t_1}^{t}Z^2(s)ds+\Big(\delta^2(1+\rho_1)+\delta\Big)\int_{t_1}^{t}x'^2(s)ds\\
&\qquad +\rho_1\delta\Big(1+\delta(1+\rho_1)\Big)\int_{t_1}^{t}x'^2(s-r)ds
\end{aligned}$$

and from (2.40)

$$\int_{t_1}^{t}x'^2(s-r)ds=\int_{t_1-r}^{t-r}x'^2(u)du\le\int_{t_1-r}^{t_1}x'^2(u)du+D_2\le\eta^2 r+D_2.$$

Finally

$$\begin{aligned}
\int_{t_1-r}^{t}\Big[x''(s)+\rho_1 x''(s-r)\Big]^2 ds \;\le\; &\Big(1+\delta(1+\rho_1)\Big)D_2+\Big(\delta^2(1+\rho_1)+\delta\Big)D_2\\
&+\rho_1\delta\Big(1+\delta(1+\rho_1)\Big)(\eta^2 r+D_2).
\end{aligned}$$

The proof of Theorem 2.3 is completed. □

### 2.2.3 Uniform Asymptotic Stability

In this subsection, we establish the criteria for the uniform asymptotic stability of the zero solution for the second neutral differential equation (2.24). This analysis is conducted for the unperturbed case, where $h\equiv 0$ and $\psi\equiv 0$. The primary goal is to present Theorem 2.4, which provides a set of sufficient conditions involving the interaction of system parameters and time delays. By ensuring that the derived inequality involving the minimum values of the system's coefficients is satisfied, we prove that all solutions starting near the equilibrium will not only remain bounded but will also converge to the equilibrium state independently of the initial time. These results reinforce the structural stability of the second model under consideration.

For the case $h \equiv 0$ and $\psi \equiv 0$. we have the following theorem

**Theorem 2.4.** *Suppose that assumptions (i)-(iii) hold. Then every solution of (2.24) is uniformly asymptotically stable, provided that*

$$\rho < \min\left\{\frac{2kq_0 - 2R_1\xi_3 - k(1+2p_1)}{R_1\xi_3\big[(2+3k)\sigma+2\big]+q_1+k}, \frac{2kR_0\xi_1}{R_1(\xi_3\sigma k+\xi_2^2)}, \frac{2p_0-3k}{2R_1(\xi_3\sigma+1)+3k+8p_1+q_1}\right\}, \tag{2.41}$$

*where*

$$\begin{cases} k < \min\left\{\dfrac{p_0}{2}, \dfrac{q_0^2}{4p_1^2}, \dfrac{q_0}{4}\right\}, \\ q_0 > \max\left\{\dfrac{2R_1\xi_3+k(1+2p_1)}{2k}, \dfrac{4R_1\xi_3}{k}\right\}. \end{cases} \tag{2.42}$$

**Proof.** Proof of Theorem 2.4. In this case, equation (2.24) becomes,

$$[x'(t)+\rho_1 x'(t-r)]'' + P(t)x''(t) + Q(t)x'(t) + R(t)[g(x(t)) + \rho_2 g(x(t-\sigma))] = 0. \tag{2.43}$$

The equation (2.43) is equivalent to the system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ Z'(t) = -P(t)z - Q(t)y - (1+\rho_2)R(t)g(x) + \rho_2 R(t)\int_{t-\sigma}^{t} y(s)g'(x(s))ds, \end{cases} \tag{2.44}$$

where

$$Z(t) = y'(t) + \rho_1 y'(t-r) = z(t) + \rho_1 z(t-r).$$

The proof depends on some fundamental properties of a continuously differentiable functional $V = V(t,x(t),y(t),z(t))$ defined by

$$V = W\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big),$$

in which $\Delta(t) =\mid P'(t) \mid -R'(t)$, the function $W = W(t;x;y;z)$ is defined by

$$\begin{aligned} W &= \frac{1}{2}Z^2 + kyZ + \frac{1}{2}kP(t)y^2 + (1+\rho_2)kR(t)G(x) + (1+\rho_2)R(t)yg(x) + \frac{Q(t)}{2}y^2 \\ &\quad + kxZ + \frac{1}{2}kQ(t)x^2 + kP(t)xy + \mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)d\tau ds, \end{aligned}$$

with $\Gamma$,$\mu$ and $\lambda$ are positive constants to be determined later in the proof. Rewriting $W$ as follows

$$W = W_1 + W_2 + W_3 + W_4 + \mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)d\tau ds,$$

where

$$\begin{aligned} W_1 &= \frac{1}{4}Z^2 + kyZ + \frac{1}{2}kP(t)y^2, \\ W_2 &= (1+\rho_2)kR(t)G(x) + (1+\rho_2)R(t)yg(x) + \frac{1}{4}Q(t)y^2, \\ W_3 &= \frac{1}{4}kQ(t)x^2 + kP(t)xy + \frac{1}{4}Q(t)y^2, \\ W_4 &= \frac{1}{4}Z^2 + kxZ + \frac{1}{4}kQ(t)x^2, \end{aligned}$$

with similar steps to the previous proof we obtain the next results.

Firstly let's show that $W$ is positive definite.

$$W_1 \geq k_1y^2, \quad W_2 \geq k_2x^2, \quad W_3 \geq k_3x^2, \quad W_4 \geq k_4Z^2,$$

where

$$\begin{aligned} k_1 &= \frac{1}{2}k(p_0 - 2k) \\ k_2 &= \frac{\xi_1^2}{2\xi_3}(1+\rho_2)kR_0\Big(1 - \frac{2R_1\xi_3}{kq_0}\Big) \\ k_3 &= \frac{1}{4}q_0k\Big[1 - \frac{4kp_1^2}{q_0^2}\Big] \\ k_4 &= k\Big[\frac{1}{k} - \frac{4}{q_0}\Big]. \end{aligned}$$

Since

$$\mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)d\tau ds > 0,$$

it follows that

$$W \geq k_5\Big(x^2 + y^2 + Z^2\Big), \tag{2.45}$$

where

$$k_5 = \min\{k_1, k_2, k_3, k_4\},$$

by (iii) we conclude that

$$\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big) \geq \exp(\frac{-\gamma}{\Gamma}), \tag{2.46}$$

we use (2.45) and (2.46) we obtain

$$V \geq k_6\Big(x^2 + y^2 + Z^2\Big),$$

where

$$k_6 = \exp(\frac{-\gamma}{\Gamma})k_5.$$

Also, it is esay to see that there is a positive constant $\delta_1$ such that

$$V \leq \delta_1(x^2 + y^2 + Z^2) + \mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)d\tau ds, \tag{2.47}$$

for all $x$,$y$ and $Z$, and all $t \geq t_1$.

For the time derivative of the function $W$ along the trajectories of system (2.44), a straight forward calculation yields

$$\dot{W}_{(2.44)} = W_5 + W_6 + W_7,$$

such that

$$\begin{aligned} W_5 \;=\;& (k - P(t))z^2 + \rho_1 kzz(t-r) - \rho_1 Q(t)yz(t-r) - \rho_1 P(t)zz(t-r) \\ & -\rho_1(1+\rho_2)R(t)g(x)z(t-r) - kQ(t)y^2 + (1+\rho_2)R(t)g'(x)\,y^2 + kyZ \\ & -k(1+\rho_2)R(t)g(x)x + kP(t)y^2 + \mu z^2 - \mu z^2(t-r) + \lambda\sigma y^2 - \lambda\int_{t-\sigma}^{t} y^2(s)ds, \end{aligned}$$

$$W_6 = \rho_2 R(t)\Big[z + \rho z(t-r) + ky + kx\Big]\int_{t-\sigma}^{t} y(s)g'(x(s))ds$$

and

$$W_7 = \frac{\partial W}{\partial t}.$$

By conditions (i), (ii) , (iii), (2.41), (2.42) and by applying the estimate $2uv \le u^2 + v^2$ we obtain

$$\begin{aligned} W_5 + W_6 \;\; \le \;\; & -\Big[kq_0 - \frac{\rho}{2}(R_1\xi_3\sigma k + q_1 + k) - (1+\rho)R_1\xi_3 - \frac{k}{2}(1+2p_1) - \lambda\sigma\Big]y^2 \\ & -(1+\rho_2)\Big[kR_0\xi_1 - \frac{\rho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2 \\ & -\Big[p_0 - \frac{\rho}{2}(R_1\xi_3\sigma + k + 3p_1) - \frac{3}{2}k - \mu\Big]z^2 \\ & +\Big[\frac{\rho}{2}(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 5p_1) - \mu\Big]z^2(t-r) \\ & +\Big[\rho R_1\xi_3(1+k) - \lambda\Big]\int_{t-\sigma}^{t} y^2(s)ds - 2\rho_1 p_0 z z(t-r) - \rho_1^2 p_0 z^2(t-r). \end{aligned}$$

Let

$$\begin{aligned} \lambda &= \rho R_1\xi_3(1+k) \\ \mu &= \frac{\rho}{2}(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 5p_1), \end{aligned}$$

$$\rho < \min\left\{\frac{2kq_0 - 2R_1\xi_3 - k(1+2p_1)}{R_1\xi_3\big[(2+3k)\sigma + 2\big] + q_1 + k},\right.$$

$$\left.\frac{2kR_0\xi_1}{R_1(\xi_3\sigma k + \xi_2^2)},\;\; \frac{2p_0 - 3k}{2R_1(\xi_3\sigma + 1) + 3k + 8p_1 + q_1}\right\},$$

The last inequality becomes

$$W_5 + W_6 \le -S\Big(x^2 + y^2 + Z^2\Big),$$

where

$$S = \min\left\{kq_0 - \frac{\rho}{2}\Big(R_1\xi_3\big[(2+3k)\sigma + 2\big] + q_1 + k\Big) - R_1\xi_3 - \frac{k}{2}(1+2p_1),\; p_0,\right.$$

$$\left.(1+\rho_2)\Big[kR_0\xi_1 - \frac{\rho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big],\; p_0 - \frac{\rho}{2}\Big(2R_1(\xi_3\sigma + 1) + 3k + 8p_1 + q_1\Big) - \frac{3k}{2}\right\},$$

$$\begin{aligned} W_7 &= \frac{1}{2}kP'(t)y^2 + (1+\rho_2)R'(t)g(x)y + (1+\rho_2)kR'(t)G(x) \\ &\quad + \frac{1}{2}Q'(t)y^2 + \frac{1}{2}kQ'(t)x^2 + kP'(t)xy \\ &\leq \omega\Delta(t)(x^2+y^2+Z^2) \end{aligned}$$

and $\omega = \dfrac{2k + (1+\rho)(\xi_2^2+1)}{2}$.
The above estimates lead to

$$\dot{W}_{(2.44)} \leq -S\big(x^2+y^2+Z^2\big) + \omega\Delta(t)(x^2+y^2+Z^2).$$

Finally by (2.45) the derivative of the functional $V$ along the trajectories of system (2.44) is given by

$$\begin{aligned} \dot{V}_{(2.44)} &= \Big[\dot{W}_{(2.44)} - \frac{1}{\Gamma}\Delta(t)W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big) \\ &\leq \Big[-S\big(x^2+y^2+Z^2\big) + \frac{\omega}{k_5}\Delta(t)W - \frac{1}{\Gamma}\Delta(t)W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big). \end{aligned}$$

Let $\Gamma^{-1} = \dfrac{\omega}{k_5}$,
so

$$\dot{V}_{(2.44)} \leq -S\big(x^2+y^2+Z^2\big)\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big), \tag{2.48}$$

from (2.46) and (2.48) we have

$$\dot{V}_{(2.44)} \leq -\delta_2\big(x^2+y^2+Z^2\big). \tag{2.49}$$

where $\delta_2 = S\exp\big(\frac{-\gamma}{\Gamma}\big)$.

We have established that the zero solution of (2.44) is uniformly asymptotically stable. This fact completes the proof of Theorem 2.4. □

### 2.2.4 Example

We consider the following third order non-autonomous delay neutral diffential equation

$$\begin{aligned}
& \Big[ \quad x'(t)+0.02x'(t-0{,}5)+\frac{1}{5}\sin(x(t))+\frac{0.02}{5}\sin(x(t-0.5))\Big]'' \\
& \quad +\Big(5+\frac{2}{\pi}\arctan(t)\Big)x''(t)+\Big(17+\frac{1}{2+t^2}\Big)x'(t) \\
& \quad +\Big(4+\frac{1}{\pi}\arctan(-t)\Big)\Big[\Big(\frac{3}{5}x(t)+\frac{x(t)}{1+x^2(t)}\Big) \\
& \quad +0.03\Big(\frac{3}{5}x(t-0{,}2)+\frac{x(t-0{,}2)}{1+x^2(t-0{,}2)}\Big)\Big] \\
= & \quad \frac{\cos(t)\sin(x'(t-0.2))}{1+t^2+x^2(t)+x^2(t-0{,}2)}.
\end{aligned}$$

For all $t \geq t_1 = t_0 + 0.5$. It is esay to see that:

(i)

$$\begin{aligned}
& 4 = p_0 \leq P(t) = 5+\frac{2}{\pi}\arctan(t) \leq 6 = p_1 \\
& \frac{7}{2} = R_0 \leq R(t) = 4+\frac{1}{\pi}\arctan(-t) \leq \frac{9}{2} = R_1, \\
& 17 = q_0 \leq Q(t) = 17+\frac{1}{2+t^2} \leq \frac{35}{2} = q_1, \\
& Q'(t) = \frac{-2t}{(2+t^2)^2} \leq 0, \\
& R'(t) = \frac{-1}{\pi(1+t^2)} \leq 0.
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{3}{5} &= \xi_1 \leq \frac{g(x)}{x} = \frac{3}{5}+\frac{1}{1+x^2} \leq \frac{8}{5} = \xi_2, \\
\mid g'(x) \mid &= \mid \frac{3}{5}+\frac{1-x^2}{(1+x^2)^2} \mid \leq \frac{8}{5} = \xi_3,\ g(0)=0.
\end{aligned}$$

(iii)

$$\begin{aligned}\int_{t_1}^{t}\Big(\mid P'(s)\mid -R'(s)\Big)ds &= \int_{t_1}^{t}\Big(\frac{2}{\pi(1+s^2)}+\frac{1}{\pi(1+s^2)}\Big)ds\\ &= \int_{t_1}^{t}\Big(\frac{3}{\pi(1+s^2)}\Big)ds\\ &\leq \int_{0}^{+\infty}\Big(\frac{3}{\pi(1+s^2)}\Big)ds \leq \frac{3}{2} < \infty.\end{aligned}$$

(H1)

$$\mid \quad h'(x)\mid = \frac{1}{5}\mid \cos(x(t))\mid \leq \frac{1}{5} = \delta \quad \text{for all} \quad x\in\mathbb{R}$$

and

$$\frac{1}{5} = \delta < \min\Big\{\frac{2kR_0\xi_1}{R_1\xi_2^2};\frac{2p_0-3k}{p_1-k}\Big\} = \min\Big\{0.57;0.78\Big\}.$$

(H2)

$$\begin{aligned}&\mid \psi(t,x(t),x(t-0.2),x'(t),x'(t-0.2),x''(t))\mid\\ =&\mid \frac{\cos(t)\sin(x'(t-0.2))}{1+t^2+x^2(t)+x^2(t-0.2)}\mid\\ \leq&\ \frac{\mid\cos(t)\mid}{1+t^2} = \varphi(t)\\ \leq&\ \varphi_1 = 1\end{aligned}$$

and

$$\int_{t_1}^{t}\varphi(s)ds = \int_{t_1}^{t}\frac{\mid\cos(s)\mid}{1+s^2}ds \leq \int_{t_1}^{+\infty}\frac{1}{1+s^2}ds \leq \int_{0}^{+\infty}\frac{1}{1+s^2}ds = \frac{\pi}{2} = D_1.$$

For $d = 1.5$ we obtain

$$\begin{aligned}\rho = 0.03 < \min\Bigg\{&\frac{2(k+\delta)q_0 - R_1(2\xi_3+\delta) - k(1+2p_1) - \delta(p_1-k)}{R_1\xi_3\big[(2+3k)\sigma+2\big]+q_1+k+R_1\delta},\\ &\frac{2kR_0\xi_1-\delta R_1\xi_2^2}{R_1(\xi_3\sigma k+\xi_2^2)},\ \frac{2p_0-3k-\delta(p_1-k)}{2R_1(\xi_3\sigma+1)+3k+8p_1+q_1}\Bigg\}\\ = \min\Big\{&0.5;0.29;0.0317\Big\},\end{aligned}$$

$$\text{where}\begin{cases} k = 1.5 < \min\{\frac{p_0}{2};\ \frac{q_0^2}{4p_1^2};\ \frac{q_0}{4}\} = \min\{2; 2601; 4.25\} \\ q_0 = 17 > \max\left\{\frac{R_1(2\xi_3+\delta)+k(1+2p_1)+\delta(p_1-k)}{2(k+\delta)};\ \frac{2R_1\xi_3}{k}\right\} \\ \qquad = \max\{15.41; 16\} \end{cases}$$

and

$$\begin{aligned} S = 1.8873 = \min\Bigg\{ & (k+\delta)q_0 - \frac{\rho}{2}\Big(R_1\xi_3\big[(2+3k)\sigma+2\big]+q_1+k+R_1\delta\Big) - R_1(\xi_3+\frac{\delta}{2}) \\ & -\frac{k}{2}(1+2p_1)-\frac{\delta}{2}(p_1-k),\ (1+\rho_2)\Big[kR_0\xi_1-\frac{\delta}{2}R_1\xi_2^2-\frac{\rho R_1}{2}(\xi_3\sigma k+\xi_2^2)\Big], \\ & p_0-\frac{\rho}{2}\Big(2R_1(\xi_3\sigma+1)+3k+6p_1+q_1\Big)-\frac{3k}{2}-\frac{\delta}{2}(p_1-k)\Bigg\} \\ & = \min\big\{32.810,\ \ 1.8873,\ \ 59.122\big\} \\ & > 1.5 = d\varphi_1. \end{aligned}$$

The given constants in the example guarantee the existence of the parameters $\rho$,$k$,$q_0$ and $S$ which satisfy all the conditions of theorem 2.5 and the theorem 2.4 then all the solutions are bounded and square integrable and if $h \equiv 0$ and $\psi \equiv 0$. they are uniformly asymptotically stable.

## 2.3 A Study of the Exponential Stability of Solutions of a Class of Third-Order Neutral Equations

The primary objective of this section is to establish two new stability results for a specific class of third-order neutral differential equations with delay, as formulated in equation (2.50). By employing the Lyapunov second method, we aim to derive a set of sufficient conditions that guarantee the boundedness and square integrability of the solutions. This analysis is crucial for ensuring that the system's trajectories remain within finite limits and that its energy dissipates effectively over time. By constructing a suitable

Lyapunov functional and satisfying the required parametric constraints, we provide a rigorous theoretical framework to confirm the stability of the system, even in the presence of complex delay terms and nonlinear components.

Let a third order neutral differential equation with delay of the form

$$\begin{aligned}\Big[x'(t)+\varrho_1 x'(t-r)+f(x)+\varrho_1 f\big(x(t-r)\big)\Big]''+\Big[V\big(x(t)\big)x'(t)\Big]' \\ +W\big(x(t)\big)x'(t)+\varphi\big(x(t)\big)+\varrho_2\varphi\big(x(t-\sigma)\big)=\Phi\big(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)\big)\end{aligned} \tag{2.50}$$

for all $t \geq t_1 = t_0+\max\{r,\sigma\}$ where $\max\{\varrho_1,\varrho_2\} = \varrho < 1$, $r,\sigma,\varrho_1$ and $\varrho_2$ are positive constants and $V,\ W \in \mathcal{C}^1(\mathbb{R},(0,\infty))$; $\varphi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$; $f \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$; $\Phi \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^5,\mathbb{R})$, $\mathbb{R}^+ = [0,\infty)$ and $\varphi(0) = 0$.

While the second part is devoted to prove a new interesting result about the exponential stability of solutions for the above equation in the case $f \equiv 0$, $\Phi \equiv 0$ and $\varrho_1 = \varrho_2 = 0$.

The solution of equation (2.50) is by definition a continuous function $x : [t_x, +\infty) \longrightarrow \mathbb{R}$ for which the given equation is satisfied in $[t_x,\infty)$ for $t_x \geq t_1$ and such that

$$x(t)+\varrho_1 x(t-r) \in \mathbf{C}^3([t_x,\infty),\mathbb{R}).$$

### 2.3.1 Basic Assumptions

In this subsection, we establish the essential mathematical framework for analyzing the third-order neutral delay differential equation (2.50). To ensure the validity of the stability and boundedness results presented in this section, it is necessary to impose specific hypotheses on the functions $V(x(.)), W(x(.))$, and $\phi(x(.))$. These assumptions define the required positive constants and functional bounds, such as the constraints on the nonlinear term $\varphi(x)$ and the integrability of the derivatives $V'$ and $W'$. By satisfying these foundational criteria, we provide a consistent basis for the subsequent proofs and the construction of Lyapunov functionals, ensuring that the system's qualitative behavior remains predictable and stable.

The given functions appearing in the equation (2.50) have to satisfy some conditions that are necessary in the proof of the theorems of this section, assume that there are positive constants $v_0, v_1, w_0, w_1, A, B$, and $C$ such that the following conditions are satisfied :

i) $v_0 \leq V(x(t)) \leq v_1, \quad w_0 \leq W(x(t)) \leq w_1;$

ii) $\varphi(0) = 0, \quad A \leq \dfrac{\varphi(x)}{x} \leq B$ ( for all $x \neq 0$) and $\mid \varphi'(x) \mid \leq C$ for all $x \in \mathbb{R}$;

iii) $\int_{-\infty}^{+\infty} \Big( \mid V'(u) \mid + \mid W'(u) \mid \Big) du \leq \gamma.$

For brevity, we put

$$\varepsilon_1(t) = V'(x(t))x'(t),$$
$$\varepsilon_2(t) = W'(x(t))x'(t).$$

The equation (2.50) is equivalent to the system

$$\begin{cases} x'(t) = y(t) - f(x(t)), \\ y'(t) = z(t), \\ Z'(t) = \; -V(x(t))z(t) + V(x(t))f'(x)U(t) - \varepsilon_1(t)U(t) - W(x(t))U(t) \\ \qquad\qquad -(1+\varrho_2)\varphi(x(t)) + \varrho_2 \int_{t-\sigma}^{t} \Big(y(s) - f(x(s))\Big)\varphi'(x(s))ds \\ \qquad\qquad +\Phi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)), \end{cases} \tag{2.51}$$

where

$$\begin{cases} y(t) - f(x(t)) = U(t), \\ \\ Z(t) = y'(t) + \varrho_1 y'(t-r) = z(t) + \varrho_1 z(t-r). \end{cases}$$

### 2.3.2 Boundedness And Square Integrability

In this subsection, we focus on establishing the properties of boundedness and square integrability for the solutions of the third-order neutral differential equation (2.50). This investigation considers the general perturbed case where the functions $f \neq 0$ and $\Phi \neq 0$. The central result is presented in the following theorem, which derives sufficient conditions and specific parametric constraints including a requirement on the smallness of the constant $\delta$. These conditions ensure that the trajectories of the system, along with their first and second derivatives, remain within finite bounds and that their integral energy over time is finite. This analysis confirms that the system maintains its qualitative stability even when subjected to nonlinear external influences.

The following theorem is for $f \neq 0$ and $\Phi \neq 0$.

**Theorem 2.5.** *In addition to assumptions (i)-(iii), assume that there are positive constants $k$, $\delta$, $\phi_1$, and $D_1$ such that the following conditions are satisfied:*

*H1)* $\mid f'(u) \mid \leq \delta$ *for all* $u \in \mathbb{R}$ *and* $\delta < \min\left\{\dfrac{2kA}{B^2}, \dfrac{2v_0 - 3k}{v_1 - k}\right\}$,

*H2)* $\mid \Phi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t)) \mid \leq \phi(t) \leq \phi_1$ *and* $\displaystyle\int_{t_1}^{t} \phi(s)ds \leq D_1$.

*Then, there exists a finite positive constant $\eta$ such that every solution $x(.)$ of (2.50) and their derivatives $x'(.)$ and $x''(.)$ satisfy*

1. $\mid x(t) \mid \leq \eta$, $\mid x'(t) \mid \leq \eta$ *and* $\mid x''(t) + \varrho_1 x''(t-r) \mid \leq \eta$, *for all* $t \geq t_1$,

2. $\displaystyle\int_{t_1}^{\infty} \Big(x^2(s) + x'^2(s) + \big(x''(s) + \varrho_1 x''(s-r)\big)^2\Big)ds < \infty$,

*provided that*

$$\varrho < \min\left\{\frac{2(k+\delta)w_0 - (2C+\delta) - k(1+2v_1) - \delta(v_1-k)}{C\big[(2+3k)\sigma + 2\big] + w_1 + k + \delta}, \frac{2kA + \delta B^2}{C\sigma k + B^2}, \frac{2v_0 - 3k - \delta(v_1-k)}{2(C\sigma+1) + 3k + 6v_1 + w_1}\right\} \tag{2.52}$$

$$\text{where} \quad \begin{cases} k < \min\left\{\dfrac{v_0}{2}, \dfrac{w_0^2}{4v_1^2}, \dfrac{w_0}{4}\right\}, \\[2ex] w_0 > \max\left\{\dfrac{2C + \delta + k(1+2v_1) + \delta(p_1 - k)}{2(k+\delta)}, \dfrac{4C}{k}\right\}. \end{cases} \tag{2.53}$$

**Proof.** Let's define the differentiable function
$\Psi = \Psi(t,x,y,z)$ defined by

$$\Psi = \Omega \exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t} \Delta(s)ds\Big), \tag{2.54}$$

where

$$\begin{aligned}
\int_{t_1}^{t} \Delta(s)ds &= \int_{t_1}^{t} \Big[ \mid \varepsilon_1(s) \mid + \mid \varepsilon_2(s) \mid \Big] ds \\
&= \int_{t_1}^{t} \Big[ \mid V'(x(s))\, x'(s) \mid + \mid W'(x(s))\, x'(s) \mid \Big] ds \\
&= \int_{\beta_1(t)}^{\beta_2(t)} \Big[ \mid V'(u) \mid + \mid W'(u) \mid \Big] du \\
&\leq \int_{-\infty}^{+\infty} \Big[ \mid V'(u) \mid + \mid W'(u) \mid \Big] du < +\infty, \qquad (2.55)
\end{aligned}$$

for $\Delta(t) = \mid \varepsilon_1(t) \mid + \mid \varepsilon_2(t) \mid$ , $\beta_1(t) = \min\{x(t_1), x(t)\}$ , $\beta_2(t) = \max\{x(t_1), x(t)\}$ and the function $\Omega = \Omega(t, x, y, z)$ is given by

$$\begin{aligned}
\Omega = &\ \frac{1}{2}Z^2 + kUZ + \frac{1}{2}kV(x(t))U^2 + (1+\varrho_2)kG(x) + (1+\varrho_2)U\varphi(x) + \frac{1}{2}W(x(t))U^2 \\
&+ kxZ + \frac{1}{2}kW(x(t))x^2 + kV(x(t))xU + \mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} U^2(\tau)d\tau ds,
\end{aligned}$$

such that $G(x) = \int_0^x \varphi(u)du$, with $\Gamma$, $\mu$ and $\lambda$ are positive constants to be determined later in the proof.

Rewriting $\Omega$ as follow

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} U^2(\tau)d\tau ds, \qquad (2.56)$$

where

$$\begin{aligned}
\Omega_1 &= \frac{1}{4}Z^2 + kUZ + \frac{1}{2}kV(x)U^2, \\
\Omega_2 &= (1+\varrho_2)kG(x) + (1+\varrho_2)U\varphi(x) + \frac{1}{4}W(x)U^2, \\
\Omega_3 &= \frac{1}{4}kW(x)x^2 + kV(x)xU + \frac{1}{4}W(x)U^2, \\
\Omega_4 &= \frac{1}{4}Z^2 + kxZ + \frac{1}{4}kW(x)x^2.
\end{aligned}$$

In view of conditions $(i)$, $(ii)$ and (2.53) we have

$$\begin{aligned}
\Omega_1 &= \frac{1}{4}\Big[(Z+2kU)^2 + 2k\big(V(x) - 2k\big)U^2\Big] \\
&\geq \frac{1}{2}k(v_0 - 2k)U^2 \\
&\geq k_1U^2,
\end{aligned}$$

where

$$k_1 = \frac{1}{2}k(v_0 - 2k).$$

Since $\varphi(0) = 0$ and $\varrho_2 < 1$ it follows that

$$\varphi^2(x) = 2\int_0^x \varphi(s)\varphi'(s)ds,$$

$$\begin{aligned}
\therefore \Omega_2 &= \frac{W(x)}{4}\Big[\Big(U + \frac{2(1+\varrho_2)}{W(x)}\varphi(x)\Big)^2 - 4\frac{(1+\varrho_2)^2}{W^2(x)}\varphi^2(x)\Big] + (1+\varrho_2)k\int_0^x \varphi(s)ds \\
&\geq (1+\varrho_2)k\int_0^x \Big[1 - \frac{2(1+\varrho_2)}{kW(x)}\varphi'(s)\Big]\varphi(s)ds \\
&\geq (1+\varrho_2)k\int_0^x \Big(1 - \frac{4C}{kw_0}\Big)\varphi(s)ds.
\end{aligned}$$

Note that by $(ii)$ we have

$$A^2 \leq \frac{\varphi^2(x)}{x^2},$$

$$\therefore \frac{A^2}{2C}x^2 \leq \frac{1}{2C}\varphi^2(x) = \frac{1}{C}\int_0^x \varphi(s)\varphi'(s)ds \leq \int_0^x \varphi(s)ds,$$

so

$$\Omega_2 \geq k_2 x^2,$$

where

$$k_2 = \frac{A^2}{2C}(1+\varrho_2)k\Big(1 - \frac{4C}{kw_0}\Big)$$

and

$$\begin{aligned}
\Omega_3 &= \frac{1}{4}W(x)\Big[\big(U + \frac{2kV(x)}{W(x)}x\big)^2 - \frac{4k^2V^2(x)}{W^2(x)}x^2 + kx^2\Big] \\
&= \frac{1}{4}W(x)\Big[\big(U + \frac{2kV(x)}{W(x)}x\big)^2 + k[1 - \frac{4kV^2(x)}{W^2(x)}]x^2\Big] \\
&\geq \frac{1}{4}w_0 k\Big[1 - \frac{4kv_1^2}{w_0^2}\Big]x^2 \\
&\geq k_3 x^2,
\end{aligned}$$

where

$$k_3 = \frac{1}{4}w_0 k\Big[1 - \frac{4kv_1^2}{w_0^2}\Big].$$

Moreover

$$\begin{aligned}\Omega_4 &= \frac{1}{4}kW(x)\Big[x^2 + \frac{4}{W(x)}xZ + \frac{1}{kW(x)}Z^2\Big]\\ &= \frac{1}{4}kW(x)\Big[\big(x + \frac{2}{W(x)}Z\big)^2 + \frac{1}{W(x)}[\frac{1}{k} - \frac{4}{W(x)}]Z^2\Big]\\ &\geq k\Big[\frac{1}{k} - \frac{4}{w_0}\Big]Z^2\\ &\geq k_4 Z^2,\end{aligned}$$

where

$$k_4 = k\Big[\frac{1}{k} - \frac{4}{w_0}\Big].$$

Since

$$\mu\int_{t-r}^{t} z^2(s)ds + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} U^2(\tau)d\tau ds > 0,$$

it follows that

$$\Omega \geq k_5\Big(x^2 + U^2 + Z^2\Big), \tag{2.57}$$

such that

$$\begin{aligned}k_5 &= \min_{i\in\{1,2,3,4\}}\{k_i\}\\ &= \min\{\frac{1}{2}k(v_0 - 2k),\ \frac{A^2}{2C}(1+\varrho_2)k\Big(1 - \frac{4C}{kw_0}\Big),\ \frac{1}{4}w_0 k\Big[1 - \frac{4kv_1^2}{w_0^2}\Big],\ k\Big[\frac{1}{k} - \frac{4}{w_0}\Big]\}.\end{aligned}$$

By (2.53) we get

$$k_5 > 0,$$

by (2.55) we obtain

$$1 \geq \exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big) \geq \exp(\frac{-\gamma}{\Gamma}). \tag{2.58}$$

From (2.57) and (2.58) it results

$$\Psi \geq k_6\Big(x^2 + U^2 + Z^2\Big), \tag{2.59}$$

where
$$k_6 = k_5 \exp(\frac{-\gamma}{\Gamma}).$$

For the time derivative of the function $\Omega$ along the trajectories of the system (2.51), a straight forward calculation yields
$$\dot{\Omega}_{(2.51)} = \Omega_5 + \Omega_6 + \Omega_7, \tag{2.60}$$

such that
$$\begin{aligned}
\Omega_5 \;=\;& (k - V(x))z^2 + f'(x)U\Big[(V(x) - k)Z - (1+\varrho_2)\varphi(x) - W(x)U\Big] \\
& +\varrho_1 kzz(t-r) - \varrho_1 W(x)Uz(t-r) - \varrho_1 V(x)zz(t-r) - \varrho_1(1+\varrho_2)\varphi(x)z(t-r) \\
& -kW(x)U^2 + (1+\varrho_2)\varphi'(x)\,U^2 + kUZ - k(1+\varrho_2)\varphi(x)x \\
& +kV(x)U^2 + \mu z^2 - \mu z^2(t-r) + \lambda\sigma U^2 - \lambda\int_{t-\sigma}^{t} U^2(s)ds \\
& +\Big[k(x+U) + Z\Big]\Phi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t))
\end{aligned}$$
$$\Omega_6 = \varrho_2\Big[z + \varrho_1 z(t-r) + kU + kx\Big]\int_{t-\sigma}^{t} U(s)\varphi'(x(s))ds$$

and
$$\Omega_7 = -\frac{1}{2}k\varepsilon_1(t)U^2 + \frac{1}{2}\varepsilon_2(t)U^2 + \frac{1}{2}k\varepsilon_2(t)x^2 - \varepsilon_1(t)UZ.$$

By conditions $(i)$, $(ii)$ and by applying the estimate $2st \leq s^2 + t^2$ we obtain
$$\begin{aligned}
\Omega_6 \;\leq\;& \varrho_2\Big[\frac{C\sigma}{2}z^2 + \frac{C}{2}\int_{t-\sigma}^{t} U^2(s)ds + \varrho_1\frac{C\sigma}{2}z^2(t-r) + \varrho_1\frac{C}{2}\int_{t-\sigma}^{t} U^2(s)ds \\
& +\frac{C\sigma}{2}kU^2 + \frac{Ck}{2}\int_{t-\sigma}^{t} U^2(s)ds + \frac{C\sigma}{2}kx^2 + \frac{Ck}{2}\int_{t-\sigma}^{t} U^2(s)ds\Big] \\
\leq\;& \varrho\Big[\frac{C\sigma}{2}z^2 + \frac{C}{2}\int_{t-\sigma}^{t} U^2(s)ds + \varrho\frac{C\sigma}{2}z^2(t-r) + \varrho\frac{C}{2}\int_{t-\sigma}^{t} U^2(s)ds \\
& +\frac{C\sigma}{2}kU^2 + \frac{Ck}{2}\int_{t-\sigma}^{t} U^2(s)ds + (1+\varrho)\frac{C\sigma}{2}kx^2 + \frac{Ck}{2}\int_{t-\sigma}^{t} U^2(s)ds\Big].
\end{aligned}$$

From conditions $(i)$, $(ii)$, $(H1)$,(2.52),(2.53) and the estimate $\tau \leq\mid \tau \mid\leq \tau^2+1$

$$\begin{aligned}\Omega_5 \leq{} & (k-v_0)z^2+\frac{\delta}{2}\Big[(v_1-k)(Z^2+U^2)+(1+\varrho_2)(\varphi^2(x)+U^2)-2w_0U^2\Big]\\ & +\frac{\varrho_1 k}{2}z^2+\frac{\varrho_1 k}{2}z^2(t-r)+\frac{\varrho_1}{2}w_1U^2+\frac{\varrho_1}{2}w_1z^2(t-r)+\frac{\varrho_1}{2}v_1z^2+\frac{\varrho_1}{2}v_1z^2(t-r)\\ & +\frac{\varrho_1}{2}(1+\varrho_2)B^2x^2+\frac{\varrho_1}{2}(1+\varrho_2)z^2(t-r)-kw_0U^2+(1+\varrho_2)CU^2\\ & +\frac{k}{2}U^2+\frac{k}{2}z^2+\frac{\varrho_1 k}{2}U^2+\frac{\varrho_1 k}{2}z^2(t-r)-k(1+\varrho_2)Ax^2+kv_1U^2\\ & +\mu z^2-\mu z^2(t-r)+\lambda\sigma U^2-\lambda\int_{t-\sigma}^{t}U^2(s)ds\\ & +\Big[k(x^2+U^2+2)+Z^2+1\Big]\mid \Phi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t))\mid\\ & -2\varrho_1 V(x)\mid zz(t-r)\mid +2\varrho_1 V(x)\mid zz(t-r)\mid +(\varrho_1-\varrho_1^2)V(x)z^2(t-r),\end{aligned}$$

it follows

$$\begin{aligned}\Omega_5 \leq{} & (k-v_0)z^2+\frac{\delta}{2}\Big[(v_1-k)(Z^2+U^2)+(1+\varrho_2)(\varphi^2(x)+U^2)-2w_0U^2\Big]\\ & +\frac{\varrho_1 k}{2}z^2+\frac{\varrho_1 k}{2}z^2(t-r)+\frac{\varrho_1}{2}w_1U^2+\frac{\varrho_1}{2}w_1z^2(t-r)+\frac{\varrho_1}{2}v_1z^2+\frac{\varrho_1}{2}v_1z^2(t-r)\\ & +\frac{\varrho_1}{2}(1+\varrho_2)B^2x^2+\frac{\varrho_1}{2}(1+\varrho_2)z^2(t-r)-kw_0U^2+(1+\varrho_2)CU^2\\ & +\frac{k}{2}U^2+\frac{k}{2}z^2+\frac{\varrho_1 k}{2}U^2+\frac{\varrho_1 k}{2}z^2(t-r)-k(1+\varrho_2)Ax^2+kv_1U^2\\ & +\mu z^2-\mu z^2(t-r)+\lambda\sigma U^2-\lambda\int_{t-\sigma}^{t}U^2(s)ds\\ & +\Big[k(x^2+U^2+2)+Z^2+1\Big]\mid \Phi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t))\mid\\ & +\varrho_1 V(x)z^2+2\varrho_1 V(x)z^2(t-r)-2\varrho_1 V(x)\mid zz(t-r)\mid -\varrho_1^2V(x)z^2(t-r).\end{aligned}$$

Therefore,

$$\begin{aligned}\Omega_5+\Omega_6 &\le -\Big[(k+\delta)w_0-\frac{\varrho}{2}(C\sigma k+w_1+k)-(1+\varrho)(C+\frac{\delta}{2})\\&-\frac{k}{2}(1+2v_1)-\frac{\delta}{2}(v_1-k)-\lambda\sigma\Big]U^2\\&-(1+\varrho_2)\Big[kA-\frac{\delta}{2}B^2-\frac{\varrho}{2}(C\sigma k+B^2)\Big]x^2\\&-\Big[v_0-\frac{\varrho}{2}(C\sigma+k+3v_1)-\frac{3}{2}k-\mu\Big]z^2+\frac{\delta}{2}(v_1-k)Z^2\\&+\Big[\frac{\varrho}{2}\Big(C\sigma+w_1+2+2k+3v_1\Big)-\mu\Big]z^2(t-r)\\&+\Big[\varrho C(1+k)-\lambda\Big]\int_{t-\sigma}^{t}U^2(s)ds\\&+d\Big(x^2+U^2+Z^2\Big)\phi(t)+3d\phi(t)-2\varrho_1v_0zz(t-r)-\varrho_1^2v_0z^2(t-r).\end{aligned}$$

By taking

$$\begin{aligned}\lambda &= \varrho C(1+k)\\ \mu &= \frac{\varrho}{2}\Big(C\sigma+w_1+2+2k+3v_1\Big)\\ d &= \max\{k,1\}\end{aligned}$$

and by (2.57) we obtain

$$\begin{aligned}\Omega_5+\Omega_6 &\le -\Big[(k+\delta)w_0-\frac{\varrho}{2}\Big(C\big[(2+3k)\sigma+2\big]+w_1+k+\delta\Big)-(C+\frac{\delta}{2})\\&-\frac{k}{2}(1+2v_1)-\frac{\delta}{2}(v_1-k)\Big]U^2\\&-(1+\varrho_2)\Big[kA-\frac{\delta}{2}B^2-\frac{\varrho}{2}(C\sigma k+B^2)\Big]x^2+\frac{\delta}{2}(v_1-k)Z^2\\&-\Big[v_0-\frac{\varrho}{2}\Big(2(C\sigma+1)+3k+6v_1+w_1\Big)-\frac{3k}{2}\Big]\Big[z^2+2\varrho_1zz(t-r)\\&+\varrho_1^2z^2(t-r)\Big]+d\Big(x^2+U^2+Z^2\Big)\phi(t)+3d\phi(t),\end{aligned}$$

$$\begin{aligned}\Omega_5+\Omega_6 &\le -\Big[(k+\delta)w_0-\frac{\varrho}{2}\Big(C\big[(2+3k)\sigma+2\big]+w_1+k+\delta\Big)-(C+\frac{\delta}{2})\\&-\frac{k}{2}(1+2v_1)-\frac{\delta}{2}(v_1-k)\Big]U^2\\&-(1+\varrho_2)\Big[kA-\frac{\delta}{2}B^2-\frac{\varrho}{2}(C\sigma k+B^2)\Big]x^2\\&-\Big[v_0-\frac{\varrho}{2}\Big(2(C\sigma+1)+3k+6v_1+w_1\Big)-\frac{3k}{2}-\frac{\delta}{2}(v_1-k)\Big]Z^2\\&+d\Big(x^2+U^2+Z^2\Big)\phi(t)+3d\phi(t),\end{aligned}$$

provided that

$$\varrho < \min\left\{\frac{2(k+\delta)w_0-(2C+\delta)-k(1+2v_1)-\delta(v_1-k)}{C\big[(2+3k)\sigma+2\big]+w_1+k+\delta},\right.$$
$$\left.\frac{2kA+\delta B^2}{C\sigma k+B^2},\ \frac{2v_0-3k-\delta(v_1-k)}{2(C\sigma+1)+3k+6v_1+w_1}\right\}$$

Hence, there exists a positive constant $S$ such that,

$$\begin{aligned}\Omega_5+\Omega_6 &\le -S\Big[x^2+U^2+Z^2\Big]+d\Big(x^2+U^2+Z^2\Big)\phi(t)+3d\phi(t)\\ &\le (d\phi_1-S)\Big(x^2+U^2+Z^2\Big)+3d\phi(t),\end{aligned} \tag{2.61}$$

where $S>d\phi_1$ and

$$\begin{aligned}S=\min\Big\{&(k+\delta)w_0-\frac{\varrho}{2}\Big(C\big[(2+3k)\sigma+2\big]+w_1+k+\delta\Big)-(C+\frac{\delta}{2})\\ &-\frac{k}{2}(1+2v_1)-\frac{\delta}{2}(v_1-k),\ (1+\varrho_2)\Big[kA-\frac{\delta}{2}B^2-\frac{\varrho}{2}(C\sigma k+B^2)\Big],\\ &v_0-\frac{\varrho}{2}\Big(2(C\sigma+1)+3k+6v_1+w_1\Big)-\frac{3k}{2}-\frac{\delta}{2}(v_1-k)\Big\}\end{aligned}$$

We have also

$$\begin{aligned}\Omega_7 &= -\frac{1}{2}\varepsilon_1(t)\Big[kU^2+2UZ\Big]+\frac{1}{2}\varepsilon_2(t)\Big[U^2+kx^2\Big]\\ &\le \frac{1}{2}\mid\varepsilon_1(t)\mid\Big[(k+1)\,U^2+Z^2\Big]+\frac{1}{2}\mid\varepsilon_2(t)\mid\Big[U^2+kx^2\Big]\\ &\le \omega\Delta(t)(x^2+U^2+Z^2),\end{aligned} \tag{2.62}$$

where $\omega=\dfrac{1+k}{2}$.

By (2.57) , (2.61), (2.62) and expression (2.60) becomes

$$\dot{\Omega}_{(2.51)} \le -M\Big(x^2+U^2+Z^2\Big)+\frac{\omega}{k_5}\Delta(t)\Omega+3d\phi(t),$$

where $M=S-d\phi_1$.

The derivative of the functional $\Psi$ along the trajectories of system (2.51) is given by

$$\begin{aligned}\dot{\Psi}_{(2.51)} &= \Big[\dot{\Omega}_{(2.51)} - \frac{1}{\mathit{\Gamma}}\Delta(t)\Omega\Big] \exp\Big(\frac{-1}{\mathit{\Gamma}}\int_{t_1}^{t}\Delta(s)ds\Big)\\ &\leq \Big[-M\big(x^2+U^2+Z^2\big)+\frac{\omega}{k_5}\Delta(t)\Omega+3d\phi(t)-\frac{1}{\mathit{\Gamma}}\Delta(t)\Omega\Big]\exp\Big(\frac{-1}{\mathit{\Gamma}}\int_{t_1}^{t}\Delta(s)ds\Big).\end{aligned}$$

Let $\mathit{\Gamma}^{-1} = \dfrac{\omega}{k_5}$, hence

$$\dot{\Psi}_{(2.51)} \leq \Big[-M\big(x^2+U^2+Z^2\big)+3d\phi(t)\Big]\exp\Big(\frac{-1}{\mathit{\Gamma}}\int_{t_1}^{t}\Delta(s)ds\Big).$$

By (2.58) we obtain

$$\dot{\Psi}_{(2.51)} \leq -N\big(x^2+U^2+Z^2\big)+3d\phi(t), \tag{2.63}$$

where $N = M\exp(\frac{-\gamma}{\mathit{\Gamma}})$.

We integrate (2.63) from $t_1$ to $t$, $t \geq t_1 = t_0 + r$ and using $(H_2)$ we get

$$\begin{aligned}\Psi(t) &\leq \Psi(t_1) + 3d\int_{t_1}^{t}\phi(s)ds\\ &\leq D_2,\end{aligned} \tag{2.64}$$

where $D_2 = \Psi(t_1) + 3dD_1$.

Due to the boundedness of $\Psi$ and by (2.59) there exists a positive constant $\eta$ such that

$$\mid x(t) \mid\leq \eta, \quad \mid U(t) \mid\leq \eta \;\; and \quad \mid Z(t) \mid\leq \eta, \tag{2.65}$$

we have

$$\mid x'(t) \mid=\mid y(t) - f(x(t)) \mid=\mid U(t) \mid\leq \eta,$$

from the boundedness of $x(t)$ and by $(H_1)$,and

$$\begin{cases} Z(t) = y'(t) + \varrho_1 y'(t-r), \\ \\ y(t) = x'(t) + f(x(t)), \end{cases} \tag{2.66}$$

we have

$$\begin{aligned} \mid x''(t)+\varrho_1 x''(t-r)\mid &= \mid Z(t)-f'(x(t))x'(t)-\varrho_1 f'(x(t-r))x'(t-r)\mid \\ &\leq \mid Z(t)\mid + \mid f'(x(t))\mid\mid x'(t)\mid + \varrho_1 \mid f'(x(t-r))\mid\mid x'(t-r)\mid \\ &\leq \eta_1, \end{aligned}$$

where $\eta_1=\eta\Big(1+(1+\varrho_1)\delta\Big)$.

We deduce that $x$, $x'$, $x''(t)+\varrho_1 x''(t-r)$ are bounded.

We will show that the solutions and their derivatives are square integrable.

We define the function

$$E(t)=\Psi(t)+\alpha\int_{t_1}^{t}\Big(x^2(s)+U^2(s)+Z^2(s)\Big)ds, \quad \forall t\geq t_1, \alpha>0, \tag{2.67}$$

according to (2.63)

$$\begin{aligned} \dot{E}(t) &\leq \dot{\Psi}_{(2.51)}(t)+\alpha\Big(x^2(t)+U^2(t)+Z^2(t)\Big) \\ &\leq (\alpha-N)\Big(x^2(t)+U^2(t)+Z^2(t)\Big)+3d\phi(t), \end{aligned}$$

if we take $\alpha<N$ then

$$\dot{E}(t) \leq 3d\phi(t). \tag{2.68}$$

Integrating (2.68) from $t_1$ to $t$ and using $(H_2)$ we obtain

$$\begin{aligned} E(t) &\leq E(t_1)+3d\int_{t_1}^{t}\phi(s)ds \\ &\leq E(t_1)+3dD_1, \end{aligned} \tag{2.69}$$

we use (2.67) and (2.69)

$$\begin{aligned} \alpha\int_{t_1}^{t}\Big(x^2(s)+U^2(s)+Z^2(s)\Big)ds &\leq E(t) \\ &\leq E(t_1)+3dD_1, \end{aligned}$$

while $E(t_1)=\Psi(t_1)$, so

$$\int_{t_1}^{t}\Big(x^2(s)+U^2(s)+Z^2(s)\Big)ds\leq\frac{\Psi(t_1)+3dD_1}{\alpha}=L,$$

$$\therefore \int_{t_1}^{t} x^2(s)ds \le L, \quad \int_{t_1}^{t} U^2(s)ds \le L \quad and \quad \int_{t_1}^{t} Z^2(s)ds \le L,$$

we have

$$\int_{t_1}^{t} x'^2(s)ds = \int_{t_1}^{t} U^2(s)ds \le L. \tag{2.70}$$

By (2.66) we get

$$\begin{aligned}
\int_{t_1}^{t} \Big[x''(s) + \varrho_1 x''(s-r)\Big]^2 ds &= \int_{t_1}^{t} \Big[Z(s) - f'(x(s))x'(s) - \varrho_1 f'(x(s-r))x'(s-r)\Big]^2 ds \\
&= \int_{t_1}^{t} \Big[Z^2(s) + f'^2(x(s))x'^2(s) + \varrho_1^2 f'^2(x(s-r))x'^2(s-r) \\
&\qquad - 2Zf'(x(s))x'(s) - 2\varrho_1 Zf'(x(s-r))x'(s-r) \\
&\qquad + 2\varrho_1 f'(x(s))f'(x(s-r))x'(s)x'(s-r)\Big] ds \\
&\le (1+\delta + \varrho_1\delta) \int_{t_1}^{t} Z^2(s)ds \\
&\qquad + \Big(\delta^2(1+\varrho_1) + \delta\Big) \int_{t_1}^{t} x'^2(s)ds \\
&\qquad + \varrho_1\delta\Big(1+\delta+\varrho_1\delta)\Big) \int_{t_1}^{t} x'^2(s-r)ds,
\end{aligned}$$

using (2.70) and by putting $\kappa_2 = \int_{t_1-r}^{t_1} x'^2(u)du$ we obtain

$$\int_{t_1}^{t} x'^2(s-r)ds = \int_{t_1-r}^{t-r} x'^2(u)du \le \int_{t_1-r}^{t_1} x'^2(u)du + \int_{t_1}^{t} x'^2(u)du \le \kappa_2 + L,$$

so

$$\begin{aligned}
\int_{t_1}^{t} \Big[x''(s) + \varrho_1 x''(s-r)\Big]^2 ds \quad &\le \quad \Big(1+\delta+\varrho_1\delta\Big)L + \Big(\delta^2(1+\varrho_1)+\delta\Big)L \\
&+ \quad \varrho_1\delta\Big(1+\delta+\varrho_1\delta\Big)(\kappa_2 + L).
\end{aligned}$$

The proof of Theorem 2.5 is completed. □

### 2.3.3 The Stability Exponential

In this subsection, we investigate the exponential stability of the zero solution for the third-order neutral differential equation (2.50). While previous sections established uniform asymptotic stability, this part focuses on ensuring that solutions converge to the equilibrium state at an exponential rate. The analysis is conducted for the unperturbed case where $f \equiv 0, \Phi \equiv 0$, and $\varrho_1 = \varrho_2 = 0$. We present the following theorem, which provides sufficient conditions and specific threshold values for the parameters $k$ and $w_0$. By satisfying these mathematical constraints, we demonstrate that the system exhibits a strong form of stability, ensuring a rapid return to equilibrium, which is a critical requirement for many high-performance dynamical systems.

The following theorem is for $f \equiv 0,\ \Phi \equiv 0$ and $\varrho_1 = \varrho_2 = 0$.

**Theorem 2.6.** *Suppose that assumptions (i)-(iii) hold. Then every solution of (2.50) is exponentially stable, provided that*

$$\begin{cases} k < \min\left\{\dfrac{v_0}{2}, \dfrac{w_0^2}{4v_1^2}, \dfrac{w_0}{4}\right\}, \\ \\ w_0 > \max\left\{\dfrac{2C + k(1+2v_1)}{2k}, \dfrac{4C}{k}\right\}. \end{cases} \tag{2.71}$$

**Proof.** In this case, equation (2.50) becomes

$$x''' + \Big[V(x(t))x'(t)\Big]' + W(x(t))x'(t) + \varphi(x(t)) = 0. \tag{2.72}$$

The equation (2.72) is equivalent to the system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ z'(t) = \ -V(x(t))z\,(t) - \varepsilon_1\,(t)\,y\,(t) - W(x(t))y\,(t) - \varphi(x\,(t)) \end{cases} \tag{2.73}$$

The proof is similar to the proof of the previous theorem 2.5 using new differentiable function $\Xi = \Xi(t,x(t),y(t),z(t))$ defined by

$$\Xi = \Theta \exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t} \Delta(s)ds\Big), \tag{2.74}$$

where $\Delta(t) =\mid \varepsilon_1(t) \mid + \mid \varepsilon_2(t) \mid$, the function $\Theta = \Theta(t, x(t), y(t), z(t))$ is defined by

$$\begin{aligned} \Theta \ = \ & \frac{1}{2}z^2 + kyz + \frac{1}{2}kV(x(t))y^2 + kG(x) + y\varphi(x) + \frac{1}{2}W(x(t))y^2 \\ & + kxz + \frac{1}{2}kW(x(t))x^2 + kV(x(t))xy \end{aligned}$$

with $\Gamma$,$\mu$ and $\lambda$ is positive constants to be determined later in the proof and $G(x) = \int_0^x \varphi(u)du$. Rewriting $\Theta$ as follows

$$\Theta = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4$$

where

$$\begin{aligned}
\Theta_1 &= \frac{1}{4}z^2 + kyz + \frac{1}{2}kV(x(t))y^2, \\
\Theta_2 &= kG(x) + y\varphi(x) + \frac{1}{4}W(x(t))y^2, \\
\Theta_3 &= \frac{1}{4}kW(x(t))x^2 + kV(x(t))xy + \frac{1}{4}W(x(t))y^2, \\
\Theta_4 &= \frac{1}{4}z^2 + kxz + \frac{1}{4}kW(x(t))x^2,
\end{aligned}$$

with similar steps to the previous proof we obtain the next results.

We will first show that $\Theta$ is positive definite,

$$\begin{aligned}
\Theta_1 &\geq k_1 y^2, \\
\Theta_2 &\geq k_2 x^2, \\
\Theta_3 &\geq k_3 x^2, \\
\Theta_4 &\geq k_4 z^2,
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \frac{1}{2}k(v_0 - 2k), \\
k_2 &= \frac{A^2}{2C}k\Big(1 - \frac{4C}{kw_0}\Big), \\
k_3 &= \frac{1}{4}w_0 k\Big[1 - \frac{4kv_1^2}{w_0^2}\Big], \\
k_4 &= k\Big[\frac{1}{k} - \frac{4}{w_0}\Big].
\end{aligned}$$

Then

$$\Theta \geq k_5\Big(x^2 + y^2 + z^2\Big), \tag{2.75}$$

where

$$k_5 = \min_{i \in \{1,2,3,4\}} \{k_i\}.$$

By (2.71) we get

$$k_5 > 0,$$

by (2.55) we conclude that

$$1 \geq \exp\Big(\frac{-1}{\mathit{\Gamma}}\int_{t_1}^{t}\Delta(s)ds\Big) \geq \exp(\frac{-\gamma}{\mathit{\Gamma}}), \tag{2.76}$$

we use (2.75) and (2.76) we obtain

$$\Xi \geq k_6\Big(x^2 + y^2 + z^2\Big), \tag{2.77}$$

where

$$k_6 = \exp(\frac{-\gamma}{\mathit{\Gamma}})k_5.$$

Since $\varphi(0) = 0$ and $|\varphi'(x)| \leq C$, we see that $|\varphi(x)| \leq C|x|$.

From i), ii) and inequality $|\varphi(x)| \leq C|x|$ and the fact that $2\alpha\beta \leq \alpha^2 + \beta^2$ we get

$$\Theta \leq \delta_1(x^2 + y^2 + z^2), \tag{2.78}$$

for all $x$,$y$ and $z$ where

$$\delta_1 = \frac{1}{2}\max\Big\{\ kB + B^2 + k + kw_1 + kv_1,\ k + kv_1 + 1 + w_1,\ 2k\Big\}.$$

By (2.78) and the estimate $\exp\Big(\frac{-1}{\mathit{\Gamma}}\int_{t_1}^{t}\Delta(s)ds\Big) \leq 1$ we get

$$\Xi \leq \delta_1(x^2 + y^2 + z^2), \tag{2.79}$$

for all $x$,$y$ and $z$.

Therefore, we can find positive definite functions $V_1(\|X\|)$ and $V_2(\|X\|)$ such that $V_1(\|X\|) \leq \Xi \leq V_2(\|X\|)$.

For the time derivative of the function $\Theta$ along the trajectories of system (2.73), a straight for ward calculation yields

$$\dot{\Theta}_{(2.73)} = \Theta_5 + \Theta_6, \tag{2.80}$$

$$\Theta_5 \quad = \quad (k - V(x(t)))z^2 - kW(x(t))y^2 + \varphi'(x)\,y^2 + kyz - k\varphi(x)x + kV(x(t))y^2,$$

and

$$\Theta_6 = -\frac{1}{2}k\varepsilon_1(t)y^2 - \varepsilon_1(t)yz + \frac{1}{2}\varepsilon_2(t)y^2 + \frac{1}{2}k\varepsilon_2(t)x^2.$$

By conditions (i), (ii) , (iii), (2.71) and by applying the estimate $2\alpha\beta \leq \alpha^2 + \beta^2$ we obtain

$$\begin{aligned}\Theta_5 &\leq -\Big[kw_0 - C - \frac{k}{2}(1 + 2v_1)\Big]y^2 - kAx^2 - \Big[v_0 - \frac{3}{2}k\Big]z^2 \\ &\leq -S\Big(x^2 + y^2 + z^2\Big),\end{aligned}$$

where

$$S = \min\left\{kw_0 - C - \frac{k}{2}(1 + 2v_1),\ kA,\ v_0 - \frac{3k}{2}\right\},$$

$$\begin{aligned}\Theta_6 &= -\frac{1}{2}k\varepsilon_1(t)\Big[ky^2 + 2yz\Big] + \frac{1}{2}\varepsilon_2(t)\Big[y^2 + kx^2\Big] \\ &\leq \omega\Delta(t)(x^2 + y^2 + z^2),\end{aligned}$$

with $\omega = \dfrac{k+1}{2}$, then

$$\dot{\Theta}_{(2.73)} \leq -S\big(x^2 + y^2 + z^2\big) + \omega\Delta(t)(x^2 + y^2 + z^2).$$

Finally by (2.75) the derivative of the functional $\Xi$ along the trajectories of system (2.73) is given by

$$\begin{aligned}\dot{\Xi}_{(2.73)} &= \Big[\dot{\Theta}_{(2.51)} - \frac{1}{\Gamma}\Delta(t)\Theta\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big) \\ &\leq \Big[-S\big(x^2 + y^2 + z^2\big) + \frac{\omega}{k_5}\Delta(t)\Theta - \frac{1}{\Gamma}\Delta(t)\Theta\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big).\end{aligned}$$

Let $\Gamma^{-1} = \dfrac{\omega}{k_5}$, so

$$\dot{\Xi}_{(2.73)} \leq -S\big(x^2 + y^2 + z^2\big)\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)ds\Big), \tag{2.81}$$

from (2.79) and (2.81) we have

$$\dot{\Xi}_{(2.73)} \leq -\frac{S}{\delta_1}\Xi\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\Big)ds. \tag{2.82}$$

By (iii), the inequality (2.82) becomes

$$\dot{\Xi}_{(2.73)} \leq -\frac{S}{\delta_1}\exp(\frac{-\gamma}{\Gamma})\Xi.$$

We have established that the zero solution of (2.73) is exponentially stable. This fact completes the proof of Theorem 2.6. □

### 2.3.4 Example

This section is devoted to the construction of an example which illustrate and authenticate the obtained results and guarantee that these results may occur in applications, for that reason consider the third order non-autonomous delay neutral differential equation of the forme :

$$\begin{aligned} & [x'(t) + \frac{3}{100}x'(t-0.5) + \frac{1}{2}e^{-x^2(t)}\sin(x(t)) + \frac{3}{200}e^{-x^2(t-0.5)}\sin(x(t-0.5))]'' \\ & +\Big(5 + \frac{\sin^2 x(t)}{1+x^2(t)}\Big)x''(t) + \Big(16 + \frac{1}{x^2(t)+3}\Big)x'(t) + x(t) + \frac{x(t)}{1+2x^2(t)} \\ & +\frac{5}{100}\Big(x(t-0.02) + \frac{x(t-0.02)}{1+2x^2(t-0.02)}\Big) \\ = \ & \frac{2\arctan(x(t-0.02)) + \pi}{8\pi\,(1+t^2+x^2(t)+x^2(t-0.02)+x'^2(t))}, \end{aligned}$$

for all $t \geq t_1 = t_0 + 0.5$. It is esay to see that :

i) $v_0 = 5 \leq V(x(t)) = 5 + \dfrac{\sin^2 x(t)}{1+x^2(t)} \leq 6 = v_1$, $w_0 = 16 \leq W(x(t)) = 16 + \frac{1}{x^2(t)+2} \leq \frac{33}{2} = w_1$.

ii) $A = 1 \leq \dfrac{\varphi(x)}{x} = 1 + \dfrac{1}{1+2x^2} \leq 2 = B$ and $\mid \varphi'(x) \mid = \mid \frac{2}{(2x^2+1)^2}(2x^4+x^2+1) \mid \leq 2 = C$, $\varphi(0) = 0$.

$$\begin{aligned} iii)\ \int_{-\infty}^{+\infty} & \mid V'(u) \mid + \mid W'(u) \mid du \\ & \leq \int_{-\infty}^{+\infty} \left| 2(\cos u)\frac{\sin u}{u^2+1} - 2u\frac{\sin^2 u}{(u^2+1)^2} \right| du + \int_{-\infty}^{+\infty} \mid \frac{-2u}{(u^2+2)^2} \mid du \\ & \leq \int_{-\infty}^{+\infty} \frac{2}{u^2+1}du + \int_{-\infty}^{+\infty} \frac{2|u|}{(u^2+1)^2}du + \int_{-\infty}^{+\infty} \mid \frac{-2u}{(u^2+2)^2} \mid du \\ & \leq 2\pi + 3. \end{aligned}$$

H1) $\mid f'(x) \mid = \mid \dfrac{1}{2}e^{-x^2}(\cos x - 2x\sin x) \mid \leq \dfrac{1}{2} = \delta$ for all $x \in \mathbb{R}$, and

$$\delta = \frac{1}{2} < \min\left\{\frac{2kA}{B^2}, \frac{2v_0 - 3k}{v_1 - k}\right\} = \min\{0.75,\ 1.2222\}.$$

$$\begin{aligned} H_2) \ \mid \Phi(t,x(t),x(t-r),x'(t),x'(t-r),x''(t)) \mid &= \mid \frac{2\arctan(x(t-0.5)) + \pi}{8\pi\left(1 + t^2 + x^2(t) + x^2(t-0.5) + x'^2(t)\right)} \mid, \\ &\leq \frac{1}{4(1+t^2)} = \phi(t) \\ &\leq \frac{1}{4} = \phi_1 \end{aligned}$$

and

$$\begin{aligned} \int_{t_1}^{t} \phi(s)ds = \int_{t_1}^{t} \frac{1}{4(1+s^2)}ds \quad &\leq \int_0^{+\infty} \frac{1}{4(1+s^2)}ds \\ &\leq \frac{\pi}{8} = D_1. \end{aligned}$$

For $d = \frac{3}{2}$ we get

$$\begin{aligned} \varrho = 0.05 < \min\Bigg\{ &\frac{2(k+\delta)w_0 - (2C+\delta) - k(1+2v_1) - \delta(v_1 - k)}{C\big[(2+3k)\sigma + 2\big] + w_1 + k + \delta}, \\ &\frac{2kA + \delta B^2}{C\sigma k + B^2}, \frac{2v_0 - 3k - \delta(v_1 - k)}{2(C\sigma + 1) + 3k + 6v_1 + w_1}\Bigg\} \\ &< \min\big\{1.818\,4, 1.231\,5, 0.055\,01\big\}, \end{aligned}$$

where

$$\begin{cases} k = \\ \frac{3}{2} < \min\left\{\frac{v_0}{2}, \frac{w_0^2}{4v_1^2}, \frac{w_0}{4}\right\} = \min\{2.5, 1.7778, 4\} \\ \\ w_0 = 16 > \max\left\{\dfrac{2C + \delta + k(1+2v_1) + \delta(v_1 - k)}{2(k+\delta)}, \dfrac{4C}{k}\right\} = \max\{6.562, .333\ 3\} \end{cases}$$

and

$$\begin{aligned} S = \min \Bigg\{ & (k+\delta)w_0 - \frac{\varrho}{2}\Big(C\big[(2+3k)\sigma + 2\big] + w_1 + k + \delta\Big) - (C + \frac{\delta}{2}) \\ & - \frac{k}{2}(1+2v_1) - \frac{\delta}{2}(v_1 - k), 1 + \varrho_2)\Big[kA - \frac{\delta}{2}B^2 - \frac{\varrho}{2}(C\sigma k + B^2)\Big], \\ & v_0 - \frac{\varrho}{2}\Big(2(C\sigma + 1) + 3k + 6v_1 + w_1\Big) - \frac{3k}{2} - \frac{\delta}{2}(v_1 - k)\Bigg\} \\ & = \min\{9.\ 369, 0.473\,5, 58.\ 573\}, \\ & > \frac{3}{8} = d\phi_1. \end{aligned}$$

The equation of the example is based on constants that confirm the existence of $\varrho, k, w_0$ and $S$ which appear in the theorems 2.5 and 2.6 then all the solutions are bounded and square integrable and if $f \equiv 0$, $\Phi \equiv 0$ and $\varrho_1 = \varrho_2 = 0$ they are exponentially stable.

Chapter 3

# Fourth-Order Differential Equations With Multiple Delays

In this chapter, we study fourth-order differential equations with multiple delays, which arise in various applied problems. The presence of delays increases the complexity of the analysis, since the evolution of the system depends not only on its current state but also on its past states. One of the main challenges in this context is the construction of suitable Lyapunov functions, which are essential tools for studying stability but are often difficult to obtain for such systems. Therefore, we aim to develop appropriate techniques and establish sufficient conditions to analyze the asymptotic stability, boundedness and square integrability of the solutions.

We investigate some asymptotic properties of solutions for the fourth order nonlinear neutral delay differential equation of the form

$$\begin{aligned}\Big(x\,(t)+\rho\sum_{i=1}^{n}x\,(t-r_i)\Big)^{''''}+\Gamma\,(t)\,\varphi\big(x''(t)\big)x'''+\Omega\,(t)\sum_{i=1}^{n}f_i\big(x'(t-r_i)\ ,\ x''(t-r_i)\big)\\ +\Phi\,(t)\sum_{i=1}^{n}h_i\big(x'(t-r_i)\big)+\Psi\,(t)\sum_{i=1}^{n}g_i\big(x(t-r_i)\big)=P(t,x(t),x'(t),x''(t);x'''(t)),\end{aligned}\tag{3.1}$$

where $r_i$ for $i\in\{1,...,n\}$ are a positive constants to be determined later and let $r\geq\max\limits_{1\leq i\leq n}\{r_i\}$

Equation (3.1) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = z, \\ z' = w, \\ W' = -\Gamma(t)\varphi(z)\,w - \Omega(t)\sum_{i=1}^{n} f_i(y,z) - \Phi(t)\sum_{i=1}^{n} h_i(y) - \Psi(t)\sum_{i=1}^{n} g_i(x) \\ \qquad +\Omega(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} f_{iy}\big(y(s),z(s)\big)z(s)ds + \Omega(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} f_{iz}\big(y(s),z(s)\big)w(s)ds \\ \qquad +\Phi(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} h_i'(y(s))\,z(s)ds + \Psi(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} g_i'(x(s))\,y(s)ds + P(t,x,y,z,w), \end{cases} \tag{3.2}$$

where

$$W(t) = \Big(x'''(t) + \rho\sum_{i=1}^{n} x'''(t-r_i)\Big).$$

It is esay to see from (3.2) that

$$\begin{cases} X(t) = \Big(x(t) + \rho\sum_{i=1}^{n} x(t-r_i)\Big) = x(t) + \rho x_t \\ X'(t) = \Big(x'(t) + \rho\sum_{i=1}^{n} x'(t-r_i)\Big) = \Big(y(t) + \rho\sum_{i=1}^{n} y(t-r_i)\Big) = y(t) + \rho y_t = Y(t) \\ X''(t) = \Big(x''(t) + \rho\sum_{i=1}^{n} x''(t-r_i)\Big) = \Big(z(t) + \rho\sum_{i=1}^{n} z(t-r_i)\Big) = z(t) + \rho z_t = Z(t) \\ X'''(t) = \Big(x'''(t) + \rho\sum_{i=1}^{n} x'''(t-r_i)\Big) = \Big(w(t) + \rho\sum_{i=1}^{n} w(t-r_i)\Big) = w(t) + \rho w_t = W(t). \end{cases}$$

The functions $\Gamma,\Omega,\Phi,\Psi$ are continuously differentiable functions in $I = [0,\infty)$. The functions $h_i(x), \frac{dh_i}{dx} = h_i'(x)$, $g_i(x), \frac{dg_i}{dx} = g_i'(x), \varphi(z), \frac{d\varphi}{dz} = \varphi'(z)$, are continuous in $\mathbb{R}$ and $P(t,x,y,z,w)$ is continuous in $I \times \mathbb{R}^4$ and $f_i(y,z)$, $\frac{\partial f_i}{\partial y}(y,z) = f_{iy}(y,z)$, $\frac{\partial f_i}{\partial z}(y,z) = f_{iz}(y,z)$, exist and are continuous in $\mathbb{R}^2$.

By solution of (3.1) we mean a continuous function $x : [t_x,\infty) \to \mathbb{R}$ which satisfy the equation (3.1) in $[t_x,\infty)$ and such that

$$x(t) + \rho\sum_{i=1}^{n} x(t-r_i) \in \mathcal{C}^4([t_x,\infty),\mathbb{R}).$$

## 3.1 Basic Assumptions

In this section, we present the basic assumptions and notations that will be used throughout the analysis. These assumptions provide a structured framework for studying the considered fourth-order differential equations with multiple delays. By imposing suitable conditions on the involved functions and parameters, we ensure the well-posedness of the problem for facilitating the use of the Lyapunov function methods. These hypotheses play a fundamental role in establishing the main results related to stability, boundedness, and integrability of the solutions.

Hereafter we use the following notations:

$$E(z)=\begin{cases}\dfrac{1}{z}\displaystyle\int_0^z \varphi(\eta)\,d\eta, & z\neq 0\\ \varphi(0), & z=0\end{cases} \quad \text{and} \quad H(y)=\begin{cases}\displaystyle\sum_{i=1}^{n}\frac{h_i(y)}{y}, & y\neq 0\\ \displaystyle\sum_{i=1}^{n}h_i(0), & y=0\end{cases}$$

Suppose that there are positive constants $\Gamma_0$,$\Omega_0$,$\Phi_0$,$\Psi_0$,$E_0$,$H_0$,$k$,$L$,$\varphi_0$,$f_0$,$\Gamma_1$,$\Omega_1$, $\Phi_1$,$\Psi_1$,$E_1$, $H_1$,$\varphi_1$,$f_1$,
$g_0$,$m$,$M$,$\widetilde{\delta}$,$\delta_0$,$\epsilon$ and $\eta_1$ such that the following conditions are satisfied :

i) $0<\Gamma_0\leq\Gamma(t)\leq\Gamma_1;\ 0<\Omega_0\leq\Omega(t)\leq\Omega_1;\quad 0<\Phi_0\leq\Phi(t)\leq\Phi_1;$
$0<\Psi_0\leq\Psi(t)\leq\Psi_1,$ and $\Psi'(t)\leq 0$ for $t\geq 0$.

ii) $0<H_0\leq H(y)\leq H_1;\ \varphi_0\leq\varphi(z)\leq\varphi_1;$ for all $y,z\in\mathbb{R}$;
$0\leq \displaystyle\sum_{i=1}^{n}\frac{f_i(y,z)}{z}-f_0\leq\frac{\epsilon\Phi_0^3m^3}{2\Omega_1\Psi_1^2g_0^2}$ (for $z\neq 0$); $f_i(y,0)=0$ for all $(y,z)\in\mathbb{R}^2$;
$f_1=\dfrac{\epsilon\Phi_0^3m^3}{2\Omega_1\Psi_1^2g_0^2}+f_0;$ and $0<m<\min\{H_0,\varphi_0,1\},\ M>\max\{H_1,,\varphi_1,1\}$.

iii) $\dfrac{g_i(x)}{x}\geq\widetilde{\delta}>0$ ( for $x\neq 0$); $g_i(0)=0.$ for all $i\in\{1,...,n\}$

iv) $g_0-\dfrac{\Gamma_0\delta_0}{\Psi_1}\leq\displaystyle\sum_{i=1}^{n}g_i'(x)\leq\frac{g_0}{2};\ \ |\sum_{i=1}^{n}h_i'(y)|\leq\theta$ for all $x,y\in\mathbb{R}$;
$-k\leq\displaystyle\sum_{i=1}^{n}f_{iy}(y,z)\leq 0;\ \ |\sum_{i=1}^{n}f_{iz}(y,z)|\leq L$ for all $(y,z)\in\mathbb{R}^2$.

v) $\delta_1=2\dfrac{\Gamma_1g_0\Psi_1M}{\Phi_0m^3}+2\dfrac{M^2(\theta\Phi_1+\delta_0+\Phi_1)}{\Gamma_0m^2}+\Gamma_0\Gamma_1+\dfrac{\Phi_0\Phi_1M}{\Psi_1g_0}<\Omega_0f_0.$

vi) $\int_0^{+\infty} \left(\left|\Gamma'(t)\right| + \left|\Omega'(t)\right| + \left|\Phi'(t)\right| - \Psi'(t)\right) dt < \eta_1 < +\infty.$

Now we dispose of the following lemma which will be required in the proof of next theorem.

**Lemma 3.1.** *[42] Let* $g(0) = 0,\ xg(x) > 0\ (x \neq 0)$ *and* $\delta(t) - g'(x) \geq 0\ (\delta(t) > 0)$. *Then*

$$2\delta(t)G(x) \geq g^2(x) \qquad \textit{where} \quad G(x) = \int_0^x g(s)ds.$$

## 3.2 Asymptotic Stability

This section is dedicated to investigating the asymptotic stability of solutions for the fourth-order nonlinear neutral delay differential equation introduced previously. Establishing stability is a fundamental objective in the qualitative theory of differential equations, as it ensures that the system's trajectories converge toward an equilibrium state over time. The analysis presented here relies on the application of a generalized Lyapunov functional and the derivation of specific criteria involving the system's parameters. We begin by introducing a preliminary lemma ([42]) that provides fundamontal integral inequalities required for the subsequent proofs. The core of this section is a theorem, which establishes the sufficient conditions under which every solution $x(t)$ along with its first, second, and third derivatives is asymptotically stable. These conditions are formulated through a set of complex inequalities involving constants such as $\rho, \epsilon$, and various system functions $\Psi, \Gamma$ and $\Omega$. By satisfying these constraints, we ensure that the energy of the system dissipates effectively, leading to the long-term stability of the system under consideration.

We have the following theorem where $P(t,x,y,z,w) = 0$.

**Theorem 3.1.** *Suppose that assumptions i) $\sim$ vi) hold. Then every solution $x(t)$ of (3.1)* *and their derivatives $x'(t)$, $x''(t)$ and $x'''(t)$ are asymptotically stable, provided that*

$$\rho < \min\left\{1,\ \frac{2\epsilon}{\alpha g_0},\ \frac{\epsilon\Phi_0 m}{\alpha\Phi_1 M+\alpha\Psi_1\lambda_0},\ \frac{2\epsilon\Gamma_0 m+2(M-1)}{\alpha\Gamma_1 M+2+\mu_2'},\frac{2\Omega_0 f_0-2\delta_1-2\epsilon\left(\Gamma_1 M+\Phi_1\theta\right)}{\alpha\Omega_1 f_1+\beta+\mu_1'}\right\}.$$

$$\textit{where}\quad \begin{cases} \lambda=\Psi_1\lambda_0(\alpha+\alpha\rho+\beta+1), & \lambda_0=\max\left\{\dfrac{g_0}{2},\ \left|g_0-\dfrac{\Gamma_0\delta_0}{\Psi_1}\right|\right\}, \\ \delta=\left(\Omega_1 k+\Phi_1\theta\right)(\alpha+\alpha\rho+\beta+1)\ , & \sigma=\Omega_1 L(\alpha+\alpha\rho+\beta+1) \end{cases} \tag{3.3}$$

*and*

$$\epsilon<\min\left\{\frac{M}{\Gamma_0 m}\ ,\ \frac{\Psi_1 g_0}{\Phi_0 m}\ ,\ \frac{\Omega_0 f_0-\delta_1}{\Gamma_1 M+\Phi_1 N}\right\},\ N=\max\left\{M,\ \theta\right\}.$$

**Proof.** The proof depend on some fundamental properties of a continuously differentiable Lyapunov functional $V=V(t,x_t,y_t,z_t,w_t)$ defined by

$$V=e^{-\frac{1}{\eta}\int_0^t\gamma(s)\,ds}Q, \tag{3.4}$$

where

$$\gamma(t)=\left|\Gamma'(t)\right|+\left|\Omega'(t)\right|+\left|\Phi'(t)\right|-\Psi'(t),$$

$$\begin{aligned} 2Q &= 2Q_0(t,x_t,y_t,z_t,w_t)+\mu_1\sum_{i=1}^{n}\int_{t-r_i}^{t}z^2(s)\,ds+\mu_2\sum_{i=1}^{n}\int_{t-r_i}^{t}w^2(s)\,ds \\ &\quad +\lambda\sum_{i=1}^{n}\int_{-r_i}^{0}\int_{t+s}^{t}y^2(\theta)\,d\theta ds+\delta\sum_{i=1}^{n}\int_{-r_i}^{0}\int_{t+s}^{t}z^2(\theta)\,d\theta ds+\sigma\sum_{i=1}^{n}\int_{-r_i}^{0}\int_{t+s}^{t}W^2(\theta)\,d\theta ds, \end{aligned}$$

and

$$\begin{aligned} 2Q_0 &= 2\beta\Psi(t)\int_0^x\sum_{i=1}^{n}g_i(\eta)\,d\eta+2\Phi(t)\int_0^y\sum_{i=1}^{n}h_i(\eta)\,d\eta+2\alpha\Omega(t)\int_0^z\sum_{i=1}^{n}f_i(y,\eta)\,d\eta \\ &\quad +2\Gamma(t)\int_0^z\varphi(\eta)\,\eta d\eta+2\beta\Gamma(t)\,y\int_0^z\varphi(\eta)\,d\eta+\Big(\beta f_0\Omega(t)-\alpha g_0\Psi(t)\Big)y^2 \\ &\quad -\beta z^2+\alpha W^2+2\Psi(t)\,y\sum_{i=1}^{n}g_i(x)+2\alpha\Psi(t)Z\sum_{i=1}^{n}g_i(x)+\alpha\rho\Psi(t)\,z_t^2 \\ &\quad +2\alpha\Phi(t)\,z\sum_{i=1}^{n}h_i(y)+2\beta yW+2zW, \end{aligned}$$

with $\alpha = \dfrac{M}{\Gamma_0 m} + \epsilon$ , $\beta = \dfrac{g_0}{\Phi_0 m} + \epsilon$ and $\eta$ is positive constant to be determined later in the proof. $2Q_0$ can be rearranged as the following

$$\begin{aligned} 2Q_0 &= \Gamma(t)\,E(z)\Big(\frac{W}{\Gamma(t)\,E(z)} + z + \beta y\Big)^2 + \Phi(t)\,H(y)\left[\frac{\Psi(t)\sum_{i=1}^{n} g_i(x)}{\Phi(t)\,H(y)} + y + \alpha z\right]^2 \\ &+ \frac{\Psi^2(t)\left(\sum_{i=1}^{n} g_i(x)\right)^2}{\Phi(t)\,H(y)} + Q_1 + Q_2 + Q_3 + Q_4, \end{aligned}$$

where

$$\begin{aligned} Q_1 &= 2\Psi(t)\int_0^x \sum_{i=1}^{n} g_i(s)\Big(\frac{\Psi_1 g_0}{\Phi_0 m} - 2\frac{\Psi(t)}{\Phi(t)\,H(y)}\sum_{i=1}^{n} g_i'(s)\Big)ds, \\ Q_2 &= 2\alpha\Omega(t)\int_0^z \Big(\sum_{i=1}^{n} f_i(y,\eta) - f_0\eta\Big)d\eta + 2\Gamma(t)\int_0^z \varphi(\eta)\,\eta d\eta \\ &\quad + \Big(\alpha f_0\Omega(t) - \beta - \alpha^2\Phi(t)\,H(y) - \Gamma(t)\,E(z)\Big)z^2, \\ Q_3 &= \Big(\alpha - \frac{1}{\Gamma(t)\,E(z)}\Big)W^2 + 2\Phi(t)\int_0^y H(\eta)\,\eta d\eta \\ &\quad + \Big(\beta f_0\Omega(t) - \alpha g_0\Psi(t) - \beta^2\Gamma(t)\,E(z) - \Phi(t)\,H(y)\Big)y^2 \end{aligned}$$

and

$$Q_4 = 2\epsilon\Psi(t)\,G(x) + 2\alpha\rho\Psi(t)\,g(x)\,z_t + \alpha\rho\Psi(t)\,z_t^2,$$

where $G(x) = \displaystyle\int_0^x g(s)ds$ and $g(x) = \displaystyle\sum_{i=1}^{n} g_i(x)$.

We prove that $Q$ is positive definite that is we show that $Q_1$, $Q_2$, $Q_3$ and $Q_4$ are positive.

Using conditions i) - v), and inequality (3.1), we obtain

$$\begin{aligned} Q_1 &\geq 2\Psi(t)\int_0^x 2\sum_{i=1}^{n} g_i(s)\frac{\Psi_1}{\Phi_0 m}\Big(\frac{g_0}{2} - \sum_{i=1}^{n} g_i'(s)\Big)ds \\ &\geq 4\Psi_0\frac{\Psi_1}{\Phi_0 m}\int_0^x \sum_{i=1}^{n} g_i(s)\left(\frac{g_0}{2} - \sum_{i=1}^{n} g_i'(s)\right)ds \geq 0. \end{aligned}$$

Condition (ii) imply that

$$0 < \varphi_0 \leq E(z) \leq \varphi_1, \tag{3.5}$$

and since (3.1) we get

$$\frac{M}{\Gamma_0 m} < \alpha < 2\frac{M}{\Gamma_0 m}, \quad \frac{\Psi_1 g_0}{\Phi_0 m} < \beta < 2\frac{\Psi_1 g_0}{\Phi_0 m}. \tag{3.6}$$

Using (3.5), (3.6) and rearranging $Q_2$ we obtain

$$\begin{aligned}
Q_2 &= 2\alpha\Omega(t)\int_0^z \Big(\sum_{i=1}^{n}\frac{f_i(y,\eta)}{\eta} - f_0\Big)\eta d\eta + 2\Gamma(t)\frac{1}{f(x)}\int_0^z \varphi(\eta)\eta d\eta \\
&\quad + \alpha\Big(f_0\Omega(t) - \beta\Gamma(t) - \alpha\Phi(t)H(y) - \frac{\Gamma(t)}{\alpha}E(z)\Big)z^2 + \beta\big(\alpha\Gamma(t) - 1\big)z^2 \\
&\geq \alpha\Big(\Omega_0 f_0 - (\frac{\Psi_1 g_0}{\Phi_0 m} + \epsilon)\Gamma_1 - (\frac{M}{\Gamma_0 m} + \epsilon)\Phi_1 M - \Gamma_0\Gamma_1 m\Big)z^2 + \beta\big(\alpha\Gamma_0 - 1\big)z^2 \\
&\geq \alpha\Big(\Omega_0 f_0 - \frac{\Psi_1 g_0 \Gamma_1}{\Phi_0 m^2} - \frac{\Phi_1 M^2}{\Gamma_0 m} - \Gamma_1\Gamma_0 m - \epsilon\big(\Gamma_1 + \Phi_1 M\big)\Big)z^2,
\end{aligned}$$

From condition (v) we get

$$Q_2 \geq \alpha\Big(\Omega_0 f_0 - \delta_1 - \epsilon\left(\Gamma_1 + \Phi_1 M\right)\Big)z^2 \geq 0.$$

Also

$$\begin{aligned}
Q_3 &\geq \beta\Big(\Omega_0 f_0 - \frac{\alpha}{\beta}g_0\Psi_1 - \Gamma_1\beta M - \frac{\Phi_1 M}{\beta}\Big)y^2 + \Big(\frac{M-1}{\Gamma_0 m}\Big)W^2 \\
&\geq \beta\Big(\Omega_0 f_0 m - 2\frac{M}{\Gamma_0}\Phi_0 - 2\Gamma_1\frac{\Psi_1 g_0 M}{\Phi_0 m} - \frac{\Phi_0\Phi_1 m M}{\Psi_1 g_0}\Big)y^2 + \Big(\frac{M-1}{\Gamma_0 m}\Big)W^2 \\
&\geq \beta m\big(\Omega_0 f_0 - \delta_1\big)y^2 + \Big(\frac{M-1}{\Gamma_0 m}\Big)W^2.
\end{aligned}$$

Choosing $\rho < \frac{2\epsilon}{\alpha g_0}$ we have

$$\begin{aligned}
Q_4 &= 2\epsilon\Psi(t)\int_0^x g(\xi)d\xi + \alpha\rho\Psi(t)\left[(z_t + g(x))^2 - g^2(x)\right] \\
&\geq 2\Psi(t)\int_0^x \Big(\epsilon - \frac{\alpha\rho g_0}{2}\Big)g(\xi)d\xi \\
&\geq 2\Psi_0\Big(\epsilon - \frac{\alpha\rho g_0}{2}\Big)G(x).
\end{aligned}$$

Using the fact that the integral $\int_{-r_i}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ , $\int_{-r_i}^{0}\int_{t+s}^{t} z^2(\theta)\,d\theta ds$ $\int_{-r_i}^{0}\int_{t+s}^{t} w^2(\theta)\,d\theta ds$, $\int_{t-r_i}^{t} z^2(\theta)\,d\theta$ and $\int_{t-r_i}^{t} w^2(\theta)\,d\theta$ for $i \in \{1,...,n\}$ are positives, we deduce that there exists a positive number $B_0$ such that

$$2Q \geq B_0\left(y^2 + z^2 + W^2 + G(x)\right). \tag{3.7}$$

By Lemma 3.1 and conditions iii) and iv) we conclude that there exists a positive number $B_1$ such that

$$2Q \geq B_1\left(x^2 + y^2 + z^2 + W^2\right); \tag{3.8}$$

thus $Q$ is positive definite which implies that $V$ is also positive definite.
By inequalities (3.4), and ( 3.8 ), we have

$$V \geq B_2(x^2 + y^2 + z^2 + W^2), \text{ where } \ B_2 = \frac{B_1}{2}e^{-\frac{\eta_1}{\eta}}. \tag{3.9}$$

Therefore, we can find a positive definite functions $W_1(\|X\|)$ and $W_2(\|X\|)$ such that

$$W_1(\|X\|) \leq V \leq W_2(\|X\|).$$

Now we prove that $\dot{Q}$ is a negative definite functional.
The derivative of $Q$ along any solution $(x(t),y(t),z(t),W(t))$ of system (3.2), we have

$$2\dot{Q}_{(3.2)} = Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + 2\frac{\partial Q_0}{\partial t},$$

where

$$\begin{aligned}
Q_5 &= -2\left(\frac{\Psi_1 g_0}{\Phi_0 m}\Phi\left(t\right)H(y) - \Psi\left(t\right)\sum_{i=1}^{n} g_i'(x)\right)y^2 - 2\alpha\Psi\left(t\right)\left(g_0 - \sum_{i=1}^{n} g_i'(x)\right)yz, \\
Q_6 &= -2\left(\Omega\left(t\right)f_0 - \alpha\Phi\left(t\right)\sum_{i=1}^{n} h_i'(y) - \beta\Gamma\left(t\right)E(z)\right)z^2 \\
&\quad -2\Omega(t)\Big(\sum_{i=1}^{n}\frac{f_i\left(y,z\right)}{z} - f_0\Big)\Big(z + \frac{\beta}{2}y\Big)^2, \\
Q_7 &= -2\epsilon\Phi\left(t\right)H(y)y^2 + \frac{\beta^2}{2}\Omega(t)\Big(\sum_{i=1}^{n}\frac{f_i(y,z)}{z} - f_0\Big)y^2 - 2\alpha\rho\Gamma(t)\varphi\left(z\right)w_t w \\
&\quad -2\alpha\rho\Omega(t)\sum_{i=1}^{n} f_i(y,z)zw_t - 2\alpha\rho\Phi\left(t\right)\sum_{i=1}^{n} h_i(y)yw_t + 2\alpha\rho\Psi\left(t\right)\sum_{i=1}^{n} g_i'(x)yz_t \\
&\quad +\mu_1 z^2 + \mu_2 w^2 - \mu_1 z_t^2 - \mu_2 w_t^2 + 2\alpha\rho\Psi\left(t\right)z_t w_t + 2\rho w w_t + 2\beta\rho z w_t, \\
Q_8 &= -2\left(\alpha\Gamma\left(t\right)\varphi(z) - 1\right)w^2
\end{aligned}$$

and

$$\begin{aligned} Q_9 &= 2\alpha\Omega(t)\, z\sum_{i=1}^{n}\int_0^z f_{iy}(y,\eta)\, d\eta + \lambda\sum_{i=1}^{n} r_i y^2(t) - \lambda\sum_{i=1}^{n}\int_{t-r_i}^{t} y^2(u)\, du + \delta\sum_{i=1}^{n} r_i z^2(t) \\ &\quad -\delta\sum_{i=1}^{n}\int_{t-r_i}^{t} z^2(u)\, du + \sigma\sum_{i=1}^{n} r_i w^2(t) - \sigma\sum_{i=1}^{n}\int_{t-r_i}^{t} w^2(u)\, du \\ &\quad +2(\beta y + z + \alpha W)\Omega(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} (f_i)_y\big(y(s),z(s)\big) z(s) ds \\ &\quad +2(\beta y + z + \alpha W)\Omega(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} (f_i)_z\big(y(s),z(s)\big) w(s) ds \\ &\quad +2(\beta y + z + \alpha W)\Phi(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} h_i'(y(s))\, z(s) ds \\ &\quad +2(\beta y + z + \alpha W)\Psi(t)\sum_{i=1}^{n}\int_{t-r_i}^{t} g_i'(x(s))\, y(s) ds, \end{aligned}$$

By conditions i), ii), iv), v) and inequalities ( 3.3), ( 3.1) and ( 3.6),

$$\begin{aligned} Q_5 &\leq -2\Psi(t)\Big(g_0 - \sum_{i=1}^{n} g_i'(x)\Big) y^2 - 2\alpha\Psi(t)\Big(g_0 - \sum_{i=1}^{n} g_i'(x)\Big) yz \\ &\leq -2\Psi(t)\Big(g_0 - \sum_{i=1}^{n} g_i'(x)\Big)\Big[\Big(y + \frac{\alpha}{2}z\Big)^2 - \Big(\frac{\alpha}{2}z\Big)^2\Big] \\ &\leq \frac{\alpha^2}{2}\Psi(t)\Big(g_0 - \sum_{i=1}^{n} g_i'(x)\Big) z^2. \end{aligned}$$

Hence,

$$\begin{aligned} Q_5 + Q_6 &\leq -2\Big[\Omega(t) f_0 - \alpha\Phi(t)\Big|\sum_{i=1}^{n} h_i'(y)\Big| - \beta\Gamma(t) E(z) - \frac{\alpha^2}{4}\Psi(t)\Big(g_0 - \sum_{i=1}^{n} g_i'(x)\Big)\Big] z^2 \\ &\leq -2\Big[\Omega_0 f_0 - \big(\frac{M}{\Gamma_0 m} + \epsilon\big)\Phi_1\theta - \big(\frac{\Psi_1 g_0}{\Phi_0 m} + \epsilon\big)\Gamma_1 M - \frac{\alpha^2}{4}\big(\Gamma_0\delta_0\big)\Big] z^2 \\ &\leq -2\Big[\Omega_0 f_0 - \frac{M\theta}{\Gamma_0 m^2}\Phi_1 - \frac{\Psi_1 g_0\Gamma_1 M}{\Phi_0 m} - \frac{M^2\delta_0}{\Gamma_0 m^2} - \epsilon\left(\Gamma_1 M + \Phi_1\theta\right)\Big] z^2 \\ &\leq -2\Big(\Omega_0 f_0 - \delta_1 - \epsilon\left(\Gamma_1 M + \Phi_1\theta\right)\Big) z^2 \leq 0. \end{aligned}$$

We have also,

$$\begin{aligned}
Q_7 \leq & \left[-2\epsilon\Phi_0 m + 2\frac{\Psi_1^2 g_0^2}{\Phi_0^2 m^2}\Omega_1\Big(\frac{\epsilon\Phi_0^3 m^3}{2\Omega_1\Psi_1^2 g_0^2}\Big) + \alpha\rho\Phi_1 M + \alpha\rho\Psi_1\lambda_0\right] y^2 \\
& + \left[\alpha\rho\Omega_1 f_1 + \mu_1 + \beta\rho\right] z^2 + \left[\alpha\rho\Psi_1\lambda_0 - \mu_1 + \alpha\rho\Psi_1\right] z_t^2 \\
& + \left[\alpha\rho\Gamma_1 M + \mu_2 + 2\rho\right] w^2 - 2\rho|ww_t| \\
& + \left[\alpha\rho\Gamma_1 M + \alpha\rho\Omega_1 f_1 + \alpha\rho\Phi_1 M - \mu_2 + \alpha\rho\Psi_1 + 2\rho + \beta\rho + (\rho - \rho^2)\right] w_t^2 \\
\leq & -(\epsilon\Phi_0 m - \alpha\rho\Phi_1 M - \alpha\rho\Psi_1\lambda_0)\, y^2 + (\alpha\rho\Omega_1 f_1 + \beta\rho + \mu_1)\, z^2 + (\alpha\rho\Gamma_1 M + 2\rho + \mu_2)\, w^2 \\
& + \Big(\alpha\rho\Gamma_1 M + \alpha\rho\Omega_1 (f_1 + nrL + nrk) + \beta\rho\Phi_1 M + \alpha\rho nr\Phi_1\theta + \alpha\rho\Psi_1 (1 + nr\lambda_0) + \beta\rho \\
& + 3\rho - \mu_2\Big) w_t^2 + (\alpha\rho\Psi_1\lambda_0 + \alpha\rho\Psi_1 - \mu_1)\, z_t^2 - \rho^2 w_t^2 - 2\rho|ww_t| \\
& - \alpha\rho nr\,(\Omega_1 L + \Omega_1 k + \Phi_1\theta + \Psi_1\lambda_0)\, w_t^2,
\end{aligned}$$

By taking

$$\begin{cases}
\mu_1 = \rho\mu_1', \quad \text{where} \quad \mu_1' = \alpha\Psi_1\lambda_0 + \alpha\Psi_1, \\
\mu_2 = \rho\mu_2', \quad \text{where} \\
\mu_2' = \alpha\Gamma_1 M + \alpha\Omega_1 (f_1 + nrL + nrk) + \beta\Phi_1 M + \alpha\Phi_1\theta nr + \alpha\Psi_1 (1 + nr\lambda_0) + \beta + 3,
\end{cases}$$

we obtain

$$\begin{aligned}
Q_7 \leq & -(\epsilon\Phi_0 m - \alpha\rho\Phi_1 M - \alpha\rho\Psi_1\lambda_0)\, y^2 + (\alpha\rho\Omega_1 f_1 + \beta\rho + \mu_1)\, z^2 + (\alpha\rho\Gamma_1 M + 2\rho + \mu_2)\, w^2 \\
& - \rho^2 w_t^2 - 2\rho|ww_t| - \alpha\rho nr\,(\Omega_1 L + \Omega_1 k + \Phi_1\theta + \Psi_1\lambda_0)\, w_t^2
\end{aligned}$$

and

$$Q_8 \leq -2\,(\alpha\Gamma_0 m - 1)\, w^2 = -2\,(\epsilon\Gamma_0 m + M - 1)\, w^2 \leq 0.$$

By Cauchy-Schwartz inequality we have

$$\begin{aligned}\sum_{i=5}^{9} Q_i \leq &- (\epsilon\Phi_0 m - \alpha\rho\Phi_1 M - \alpha\rho\Psi_1\lambda_0)\, y^2 - \left(2\epsilon\Gamma_0 m + 2(M-1) - \rho\left(\alpha\Gamma_1 M + 2 + \mu_2^{'}\right)\right) w^2 \\ &- \left(2\Omega_0 f_0 - 2\delta_1 - 2\epsilon\left(\Gamma_1 M + \Phi_1\theta\right) - \rho\left(\alpha\Omega_1 f_1 + \beta + \mu_1^{'}\right)\right) z^2 \\ &- \rho^2 w_t^2 - 2\rho|ww_t| + \Omega_1 Lnr\left(\beta y^2 + z^2 + \alpha w^2\right) + \sigma nrw^2 \\ &+ \left(\Omega_1 L(\alpha + \alpha\rho + \beta + 1) - \sigma\right)\sum_{i=1}^{n}\int_{t-r_i}^{t} w^2(s)\,ds + (\Omega_1 k + \Phi_1\theta)\, nr\left(\beta y^2 + z^2 + \alpha w^2\right) \\ &+ \delta nrz^2 + ((\Omega_1 k + \Phi_1\theta)(\alpha + \alpha\rho + \beta + 1) - \delta)\sum_{i=1}^{n}\int_{t-r_i}^{t} z^2(s)\,ds + \lambda nry^2 \\ &+ \Psi_1\lambda_0 nr\left(\beta y^2 + z^2 + \alpha w^2\right) + \left(\Psi_1\lambda_0(\alpha + \alpha\rho + \beta + 1) - \lambda\right)\sum_{i=1}^{n}\int_{t-r_i}^{t} y^2(s)\,ds\end{aligned}$$

provided that

$$\rho < \min\left\{1,\, \frac{2\epsilon}{\alpha g_0},\, \frac{\epsilon\Phi_0 m}{\alpha\Phi_1 M + \alpha\Psi_1\lambda_0},\, \frac{2\epsilon\Gamma_0 m + 2(M-1)}{\alpha\Gamma_1 M + 2 + \mu_2^{'}},\, \frac{2\Omega_0 f_0 - 2\delta_1 - 2\epsilon\big(\Gamma_1 M + \Phi_1\theta\big)}{\alpha\Omega_1 f_1 + \beta + \mu_1^{'}}\right\}.$$

Hence, there exists a positive constant $\lambda_1$ such that,

$$\begin{aligned}\sum_{i=5}^{9} Q_i \leq &- 2\lambda_1\left(y^2 + z^2 + w^2\right) + \Omega_1 L(\alpha + \beta + 1)nr\left(y^2 + z^2 + w^2\right) + \sigma nrw^2 \\ &+ (\Omega_1 k + \Phi_1\theta)(\alpha + \beta + 1)nr\left(y^2 + z^2 + w^2\right) + \delta nrz^2 \\ &+ \Psi_1\lambda_0(\alpha + \beta + 1)nr\left(y^2 + z^2 + w^2\right) + \lambda nry^2 \\ &- \rho^2 w_t^2 - 2\rho|ww_t| \\ \leq &- 2B_3\left(y^2 + z^2 + w^2\right) - \rho^2 w_t^2 - 2\rho|ww_t|. \end{aligned} \tag{3.10}$$

Where $B_3 = \lambda_1 - (\delta + \lambda + \sigma)r$. It can be seen that if

$$r < \frac{\lambda_1}{n}\min\left\{\frac{1}{\delta + \lambda + \sigma}\ ,\ \frac{1}{(\Omega_1 L + \Omega_1 k + \Phi_1\theta + \Psi_1\lambda_0)(\alpha + \beta + 1)}\right\},$$

then $B_3 > 0$. Putting $B_4 = \min\left\{\frac{1}{2},\, B_3\right\}$ we have

$$\begin{aligned}\sum_{i=5}^{9} Q_i &\leq - 2B_4\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho|ww_t|\right) \\ &\leq - 2B_4\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho ww_t\right) \\ &= - 2B_4\left(y^2 + z^2 + W^2\right). \end{aligned} \tag{3.11}$$

Now by the inequalities (ii), (3.11) and the Cauchy-Schwartz inequality together with condition (iv) and the Lemma 3.1 we get,

$$\begin{aligned} 2\frac{\partial Q_0}{\partial t} &= \Psi'(t)\left[2\beta\int_0^x\sum_{i=1}^n g_i(\eta)\,d\eta - \alpha g_0 y^2 + 2y\sum_{i=1}^n g_i(x) + 2\alpha z\sum_{i=1}^n g_i(x)\right] \\ &+\Phi'(t)\left[2\int_0^y H(\eta)\,\eta d\eta + 2\alpha yzH(y)\right] \\ &+\Omega'(t)\left[2\alpha\int_0^z\Big(\sum_{i=1}^n\frac{f_i(y,z)}{z} - f_0\Big)\eta d\eta + \alpha f_0 z^2 + \beta f_0 y^2\right] \\ &+\Gamma'(t)\left[2\int_0^z\varphi(\eta)\,\eta d\eta + 2\beta y\int_0^z\varphi(\eta)\,d\eta\right] \\ &+\alpha\rho\Psi'(t)\left[z_t + \sum_{i=1}^n g_i(x)\right]^2 - \alpha\rho\Psi'(t)\left(\sum_{i=1}^n g_i(x)\right)^2. \end{aligned}$$

There exist a positive constant $\lambda_2$ such that

$$\begin{aligned} 2\left|\frac{\partial Q_0}{\partial t}\right| &\leq -\Psi'(t)\Bigg[2\beta G(x) + \alpha g_0 y^2 + \left(y^2 + \left(\sum_{i=1}^n g_i(x)\right)^2\right) + \alpha\rho\left(\sum_{i=1}^n g_i(x)\right)^2 \\ &+\alpha\left(z^2 + \left(\sum_{i=1}^n g_i(x)\right)^2\right)\Bigg] + |\Phi'(t)|M\Big(y^2 + \alpha\left(y^2+z^2\right)\Big) \\ &+|\Omega'(t)|\left(\alpha\left(\frac{\epsilon\Phi_0^3 m^3}{2\Omega_1\Psi_1^2 g_0^2} + f_0\right)z^2 + \beta f_0 y^2\right) \\ &+|\Gamma'(t)|M\Big(z^2 + \beta\left(y^2+z^2\right)\Big) \\ &\leq \lambda_2\Big(|\Gamma'(t)| + |\Omega'(t)| + |\Phi'(t)| - \Psi'(t)\Big)\left(y^2+z^2+W^2+G(x)\right) \\ &\leq 2\frac{\lambda_2}{B_0}\Big(|\Gamma'(t)| + |\Omega'(t)| + |\Phi'(t)| - \Psi'(t)\Big)V, \end{aligned}$$

Thus for $\frac{1}{\eta} = \frac{\lambda_2}{B_0}$ we have

$$\dot{Q}_{(3.2)} \leq -B_4(y^2+z^2+W^2) + \frac{1}{\eta}\Big(|\Gamma'(t)| + |\Omega'(t)| + |\Phi'(t)| - \Psi'(t)\Big)V. \tag{3.12}$$

By condition vi) and inequality (3.12 ) we have

$$\begin{aligned} \dot{Q}_{(3.2)} &= \left(\dot{Q}_{(3.2)} - \frac{1}{\eta}\gamma(t)\,Q\right)e^{-\frac{1}{\eta}\int_0^t\gamma(s)\,ds} \\ &\leq -B_4\left(y^2+z^2+W^2\right)e^{-\frac{1}{\eta}\int_0^t\gamma(s)\,ds} \\ &\leq -B_5\left(y^2+z^2+W^2\right), \end{aligned}$$

where $B_5 = B_4 e^{-\frac{\eta_1}{\eta}}$. Now take $W_3(\|X\|) = B_5(x^2 + y^2 + z^2)$. From (3.2) it is esay to see that $W_3$ is positive definite function. Thus, we conclude that the solution of system (3.2) are asymptotically stable, which complete the proof of Theorem 3.1. □

## 3.3 Boundedness And Square Integrability

While the previous section established the conditions for asymptotic stability, this section extends the qualitative analysis to explore the boundedness and square integrability of the solutions for the perturbed system (where $P(t, x, y, z, w) \neq 0$). In many practical applications, it is not enough to know that a system returns to equilibrium; it is equally critical to guarantee that the state variables do not exceed certain physical or mathematical limits, even in the presence of external perturbations. Furthermore, ensuring that the solutions are square integrable over an infinite interval provides deep insights into the energy dissipation and the $L^2$-stability of the system. The primary focus here is the theorem, which provides the sufficient conditions expressed through the convergence of a certain function $e(t)$ and the parameters inherited from the previous stability criteria to ensure that the solution $x(t)$ and its higher-order derivatives remain bounded for all $t \geq 0$. Additionally, we establish that the integral of the sum of squares of these solutions remains finite, confirming their square-integrable nature.

For the case where $P(t,x,y,z,w) \neq 0$ we state the following theorem:

**Theorem 3.2.** *Let all the conditions of Theorem 3.1 and the assumption*

$$|P(t,x,y,z,w)| \leq e(t) \quad \text{and} \quad \int_0^{+\infty} e\,(s)\,ds < \eta_2 < +\infty \tag{3.13}$$

*where $\eta_2$ is positive constant. Then, there exists a finite positive constant $K$ such that every solution $x(.)$ of (3.1) and their derivatives $x'(.)$, $x''(.)$, $x'''(.)$ and $X'''(.)$ satisfy*

1. $|x(t)| \leq \sqrt{K},\ |x'(t)| \leq \sqrt{K},\ |x''(t)| \leq \sqrt{K},\ |X'''(t)| \leq \sqrt{K},$ *for all* $t \geq 0$;

2. $\displaystyle\int_0^{\infty} \big(x'^2(s) + x''^2(s) + x'''^2(s)\big)ds < \infty.$

**Proof.** From (3.9), we have

$$V \geq B_2(x^2+y^2+z^2+W^2), \text{ where } B_2 = \frac{B_1}{2}e^{-\frac{\eta_1}{\eta}}. \tag{3.14}$$

Taking the time derivative of $Q$ with respect to t along the trajectory of system (3.2), we obtain

$$\begin{aligned}\dot{Q}_{(3.2)} &\leq -B_4(y^2+z^2+W^2) + \frac{1}{\eta}\Big( |\Gamma'(t)| + |\Omega'(t)| + |\Phi'(t)| - \Psi'(t) \Big)Q \\ &\quad + \big(\beta y + z + \alpha W\big)P(t,x,y,z,w). \end{aligned} \tag{3.15}$$

By condition iv) and inequalities ( 3.13 ), ( 3.14 ), ( 3.15 ) and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
\dot{V}_{(3.2)} &= \left(\dot{Q}_{(3.2)} - \frac{1}{\eta}\gamma(t)\,Q\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\
&\leq \Big( -B_4\left(y^2+z^2+W^2\right) + \big(\beta y + z + \alpha W\big)P(t,x,y,z,w)\Big)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\
&\leq (\beta|y| + |z| + \alpha|W|)\,|P(t,x,y,z,w)| \\
&\leq B_5\,(|y|+|z|+|W|)\,e(t) \\
&\leq B_5\left(3+y^2+z^2+W^2\right)e(t) \\
&\leq B_5\left(3+\frac{1}{B_2}V\right)e(t) \\
&\leq 3B_5e(t) + \frac{B_5}{B_2}Ve(t),
\end{aligned} \tag{3.16}$$

where $B_5 = \max\{\alpha,\beta,1\}$.

Integrating (3.16) from 0 to $t$, using the inequality ( 3.13 ) and the Gronwall inequality, we have

$$\begin{aligned}
V(t,x,y,z,w) &\leq V\big(0,x(0),y(0),z(0),w(0)\big) + 3B_5\eta_2 \\
&\quad + \frac{B_5}{B_2}\int_0^t V\big(s,x(s),y(s),z(s),w(s)\big)e(s)ds \\
&\leq \Big(V\big(0,x(0),y(0),z(0),w(0)\big) + 3B_5\eta_2\Big)e^{\frac{B_5}{B_2}\int_0^t e(s)ds} \\
&\leq \Big(V\big(0,x(0),y(0),z(0),w(0)\big) + 3B_5\eta_2\Big)e^{\frac{B_5}{B_2}\eta_2} = K_1 < \infty.
\end{aligned} \tag{3.17}$$

In view of inequalities (3.14) and (3.17),

$$(x^2+y^2+z^2+W^2) \le \frac{1}{B_2}V \le K$$

where $K=\frac{K_1}{B_2}$. Aforementioned inequality implies that

$$|x(t)| \le \sqrt{K},\ |y(t)| \le \sqrt{K},\ |z(t)| \le \sqrt{K},\ |W(t)| \le \sqrt{K} \quad \text{for all} \quad t \ge 0.$$

Hence,

$$|x(t)| \le \sqrt{K},\ |x'(t)| \le \sqrt{K},\ |x''(t)| \le \sqrt{K},\ |X'''(t)| \le \sqrt{K} \quad \text{for all} \quad t \ge 0. \tag{3.18}$$

First from (3.10 ) we obtain

$$Q_5+Q_6+Q_7+Q_8+Q_9 \le -2B_3\left(y^2+z^2+w^2\right)$$

and

$$\dot{Q}_{(3.2)} \le -B_3(y^2+z^2+w^2)+\frac{1}{\eta}\gamma\,(t)\,Q+\big(\beta y+z+\alpha W\big)P(t,x,y,z,w). \tag{3.19}$$

From (vi), ( 3.4), ( 3.19 ) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\dot{V}_{(3.2)} &= \left(\dot{Q}_{(3.2)}-\frac{1}{\eta}\gamma\,(t)\,Q\right)e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds} \\ &\le \Big(-B_3\left(y^2+z^2+w^2\right)+\big(\beta y+z+\alpha W\big)P(t,x,y,z,w)\Big)e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds} \qquad (3.20)\end{aligned}$$

Now, we define $U_t=U(t,x(t),y(t),z(t),w(t))$ by

$$U_t=V+\sigma\int_0^t \big(y^2(s)+z^2(s)+w^2(s)\big)\,ds,$$

where $\sigma>0$. It is esay to see that $U_t$ is positive definite, since $W=W(t,x,y,z,w)$ is already positive definite. Using the estimate $e^{-\frac{\eta_1}{\eta}} \le e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds} \le 1$, (3.20) implies

$$\begin{aligned}\dot{U}_{t(3.2)} &\le -B_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1}{\eta}}+B_4\Big(|y(t)|+|z(t)|+|W(t)|\Big)|P(t,x,y,z,w)| \\ &\quad +\sigma\Big(y^2(t)+z^2(t)+w^2(t)\Big) \\ &\le -B_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1}{\eta}}+B_4\Big(|y(t)|+|z(t)|+|W(t)|\Big)e(t) \\ &\quad +\sigma\Big(y^2(t)+z^2(t)+w^2(t)\Big). \qquad (3.21)\end{aligned}$$

where $B_4$ is positive constant. By choosing $\sigma = B_3 e^{-\frac{\eta_1}{\eta}}$, we obtain

$$\begin{aligned}\dot{U}_{t(3.2)} &\leq B_4\Big(3 + y^2(t) + z^2(t) + W^2(t)\Big)e(t)\\ &\leq B_4\Big(3 + \frac{1}{B_2}V\Big)e(t)\\ &\leq 3B_4e(t) + \frac{B_4}{B_2}U_te(t). \qquad (3.22)\end{aligned}$$

Integrating the last inequality (3.22) from 0 to $t$, and again using the Gronwall inequality and the condition (3.13), we get

$$\begin{aligned}U_t &\leq U_0 + 3B_4\eta_2 + \frac{B_4}{B_2}\int_0^t U_s e(s)ds\\ &\leq \Big(U_0 + 3B_4\eta_2\Big)e^{\frac{B_4}{B_2}\int_0^t e(s)ds}\\ &\leq \Big(U_0 + 3B_4\eta_2\Big)e^{\frac{B_4}{B_2}\eta_2} = K_2 < \infty. \qquad (3.23)\end{aligned}$$

Therefore,

$$\int_0^\infty y^2(s)ds < K_2 \ \ , \ \ \int_0^\infty z^2(s)ds < K_2 \text{ and } \int_0^\infty w^2(s)ds < K_2,$$

which implies that

$$\int_0^\infty x'^2(s)ds \leq K_2 \ \ , \ \ \int_0^\infty x''^2(s)ds \leq K_2 \ \ , \ \ \int_0^\infty x'''^2(s)ds \leq K_2. \qquad (3.24)$$

The proof of Theorem 3.2 is completed. □

Chapter 4

# Third and Fourth Order Integro-Differential Equations of Neutral Type

This chapter is devoted to the study of third and fourth order nonlinear integro-differential equations of neutral type. Such equations play a fundamental role in describing systems where both present and past states, as well as their derivatives, influence the evolution of the system. The presence of delay and neutral terms makes the analysis more complex and rich in behavior. The main objective of this chapter is to investigate important qualitative properties of solutions, including stability, boundedness, and square integrability. We aim to establish sufficient conditions under which the solutions remain well-behaved over time. The results obtained contribute to a deeper understanding of the influence of nonlinearities and delays on the long-term dynamics of these systems.

## 4.1 Third Order Nonlinear Integro-Differential Equations of Neutral Type

In this section, we study third-order nonlinear integro-differential equations of neutral type with delay. These equations are important in modeling systems where both the current state and past states, together with their derivatives, influence the system's behavior. The presence of delay and neutral terms adds significant complexity to the analysis. Our main goal is to investigate the qualitative properties of solutions, particularly stability, boundedness, and square integrability. We establish sufficient conditions under which the solutions remain well-defined and stable over time. This study helps to better understand the effect of nonlinearities and delays on the behavior of such systems. We propose the following delay integro-differential equation

$$\Big(x(t)+\rho_1 x(t-r)\Big)''' + a(t)x'' + b(t)x' + c(t)\Big[f(x(t))+\rho_2 f(x(t-\sigma))\Big] \tag{4.1}$$
$$= \varphi(t,x(t),x'(t),x''(t),x'''(t)) + \xi\int_0^t \Omega(t,s)x'(s)ds$$

for all $t \geq t_1 = t_0 + r$ where $r$, $\sigma$ and $\xi$ are positive constants and $\rho = \max\{\rho_1,\rho_2\} < 1$, , the primes in (3.1) denote differentiation with respect to t; the functions $a, b, c \ : [0,\infty) \to (0,\infty)$ and $f \ : \mathbb{R} \to \mathbb{R}$ are continuous differentiable and $\Omega(t,s)$ is continuous for $0 \leq s \leq t < \infty$ and $f(0) = 0$.

By solution of (4.1) we mean a continuous function $x : [t_x,\infty) \to \mathbb{R}$ for $t_x \geq t_1$ which satisfy the equation (4.1) in $[t_x,\infty)$ and such that

$$x(t) + \rho_1 x(t-r) \in \mathcal{C}^3([t_x,\infty),\mathbb{R}).$$

### 4.1.1 Basic Assumptions

Suppose that there are positive constants $a_0,a_1,a_2,c_0,b_1,\gamma,\delta,A,m,\eta_2$, and $\eta_3$, such that the following conditions are satisfied

i) $0 < a_0 \leq a(t) \leq a_1$, $\mid a'(t) \mid \leq a_2$, $0 < c_0 \leq c(t) \leq b(t) \leq b_1$, and $b'(t) \leq c'(t) \leq 0$ for $t \geq t_1$;

ii) $\gamma \geq \dfrac{f(x)}{x} \geq \delta > 0$ ( for $x \neq 0$) , $f'(x) \leq A$ *and* $f(0) = 0$;

iii)

$$\begin{cases} A(1+\rho)^2 \leq m \leq \min\{a_0, \dfrac{2a_0}{A\sigma}\}, \\ \\ c_0 \geq \dfrac{1}{2}a_2 + \dfrac{2Ab_1}{m} + \dfrac{A(4+m)}{2m}\sigma + \dfrac{A\sigma}{2} \end{cases}$$

We will study the equation (4.1) in two cases:

### 4.1.2 Asymptotic Stability

First case is for $\varphi(t,x(t),x'(t),x''(t)) = 0$

**Theorem 4.1.** *In addition to conditions (i) $\sim$ (iii) being satisfied, suppose that there are positives constants $m,\eta_2$ such that the following conditions hold:*

*H1)* $\displaystyle\int_0^t |\Omega(t,s)|ds \leq \eta_2$

*and*

$$\rho < \min\left\{ 1, \ \frac{\alpha}{b_1\Big(1+2A+\eta_2(2m+4)\Big)}, \ \frac{2\beta}{a_0+a_1+2b_1+3A\sigma+8\eta_2+6-2m} \right\}.$$

*where*

$$\alpha = c_0 m - \frac{1}{2}ma_2 - 2Ab_1 - \frac{A(4+m)}{2}\sigma - \frac{m\sigma A}{2}$$
$$\beta = a_0 - m\frac{A\sigma}{2}$$

*Then every solution x(.) of (4.1) is asymptotically stable.*

**Proof.** The equation (4.1) is equivalent to the differential system

$$\begin{cases} x' = y \\ y' = z \\ Z' = \xi\displaystyle\int_0^t \Omega(t,s)y(s)ds - a(t)z - b\,(t)\,y - (1+\rho_2)c\,(t)\,f\,(x) \\ \qquad\qquad +\rho_2 c(t)\displaystyle\int_{t-\sigma}^t f'(x(s))y(s)ds, \end{cases} \tag{4.2}$$

It is esay to see from (4.2) that

$$\begin{cases} X'(t) = x'(t) + \rho_1 x'(t-r) = y(t) + \rho_1 y(t-r) = Y(t) \\ \\ X''(t) = x''(t) + \rho_1 x''(t-r) = z(t) + \rho_1 z(t-r) = Z(t). \end{cases}$$

The proof of this theorem depends on properties of the continuously differentiable function $V = V(t,x,y,Z)$ defined as

$$\begin{aligned} V &= mc(t)F(x) + (1+\rho_2)c(t)Yf(x) + \frac{b(t)}{2}Y^2 + \frac{(1+\rho_2)}{2}Z^2 + myZ + \frac{1}{2}ma(t)y^2 \\ &\quad + \mu_1 \int_{t-r}^{t} y^2(s)\,ds + \mu_2 \int_{t-r}^{t} z^2(s)\,ds + \lambda \int_{-\sigma}^{0} \int_{t+s}^{t} y^2(\tau)d\tau ds \\ &\quad + \xi\mu_3 \int_0^t \int_t^{+\infty} |\Omega(\tau,s)| y^2(s) d\tau ds, \end{aligned}$$

Where $F(x) = \int_0^x f(u)du$ and $m > 0$

We rewrite $V$ as

$$\begin{aligned} V = \; & V_1 + V_2 + \mu_1 \int_{t-r}^{t} y^2(s)\,ds + \mu_2 \int_{t-r}^{t} z^2(s)\,ds + \lambda \int_{-\sigma}^{0} \int_{t+s}^{t} y^2(\tau)d\tau ds \\ & + \xi\mu_3 \int_0^t \int_t^{+\infty} |\Omega(\tau,s)| y^2(s) d\tau ds, \end{aligned}$$

where

$$\begin{aligned} V_1 &= mc(t)F(x) + (1+\rho_2)c(t)Yf(x) + \frac{b(t)}{2}Y^2, \\ V_2 &= \frac{(1+\rho_2)}{2}Z^2 + myZ + \frac{1}{2}ma(t)y^2. \end{aligned}$$

Using conditions (i) $\sim$ (iii) we get

$$\begin{aligned} V_1 &= mc(t)F(x) + (1+\rho_2)c(t)Yf(x) + \frac{b(t)}{2}Y^2 \\ &= \frac{b(t)}{2}\Big(Y + (1+\rho_2)\frac{c(t)f(x)}{b(t)}\Big)^2 - (1+\rho_2)^2\frac{c^2(t)f^2(x)}{2b(t)} + mc(t)F(x) \\ &\geq mc(t)F(x) - (1+\rho)^2\frac{c^2(t)f^2(x)}{2b(t)}, \end{aligned}$$

by (ii) we have

$$\frac{1}{2}f^2(x) \quad = \quad \int_0^x f(u)du \leq AF(x) \tag{4.3}$$

so

$$V_1 \quad \geq \quad c_0\Big(m - A(1+\rho)^2\Big)F(x) \tag{4.4}$$

$$\begin{aligned}
V_2 &= \frac{(1+\rho_2)}{2}Z^2 + myZ + \frac{1}{2}ma(t)y^2 \\
&= \frac{(1+\rho_2)}{2}\Big(Z + \frac{m}{1+\rho}y\Big)^2 - \frac{m^2}{2(1+\rho_2)}y^2 + \frac{1}{2}ma(t)y^2 \\
&\geq \frac{m}{2}\Big(a(t) - m\Big)y^2
\end{aligned}$$

also

$$\begin{aligned}
V_2 &= \frac{1}{2}ma(t)\Big(y + \frac{1}{a(t)}Z\Big)^2 + \frac{1}{2}\Big(1 + \rho_2 - \frac{m}{a(t)}\Big)Z^2 \\
&\geq \frac{1}{2}\Big(1 + \rho_2 - \frac{m}{a_0}\Big)Z^2
\end{aligned}$$

so

$$V_2 \geq \frac{m}{4}\Big(a_0 - m\Big)y^2 + \frac{1}{4}\Big(1 + \rho_2 - \frac{m}{a_0}\Big)Z^2$$

$$\begin{aligned}
V_1 + V_2 &\geq c_0\Big(m - A(1+\rho)^2\Big)F(x) + \frac{m}{4}\Big(a_0 - m\Big)y^2 + \frac{1}{4}\Big(1 + \rho_2 - \frac{m}{a_0}\Big)Z^2 \\
&\geq \Gamma_{0_1}F(x) + \Gamma_{0_2}y^2 + \Gamma_{0_3}Z^2
\end{aligned}$$

Thus, there exists positive number $\Gamma_0$ such that

$$V \geq \Gamma_0\left(F(x) + y^2 + Z^2\right), \quad \Gamma_0 = \min\{\Gamma_{0_1},\Gamma_{0_2},\Gamma_{0_3}\}. \tag{4.5}$$

By condition (ii) and (4.3)

$$\frac{1}{2}f^2(x) \geq \frac{A^2}{2}x^2$$

which implies that

$$F(x) \geq \frac{A}{2}x^2$$

it follows that there is a positive constant $\Gamma_1$ such that

$$V \geq \Gamma_1 \left( x^2 + y^2 + Z^2 \right). \tag{4.6}$$

Thus $V$ is positive definite.

The derivative of the function $V$, along any solution $(x(t), y(t), Z(t))$ of system (4.2), with respect to $t$ after simplifying is given by

$$\dot{V}_{(4.2)} = V_3 + V_4$$

where

$$\begin{aligned} V_3 =& \rho_1(1+\rho_2)c(t)f'(x)yy(t-r) + \rho_1 b(t)y(t-r)z + b(t)\rho_1^2 y(t-r)z(t-r) - mb(t)y^2 \\ &+ (1+\rho_2)c(t)f'(x)y^2 + (m-a(t))z^2 + \rho_1(m-a(t))zz(t-r) + \mu_1 y^2 - \mu_1 y^2(t-r) \\ &+ \mu_2 z^2 - \mu_2 z^2(t-r) + \lambda\sigma y^2 + mc'(t)F(x) + (1+\rho_2)c'(t)Yf(x) + \frac{1}{2}b'(t)Y^2 \\ &+ \frac{1}{2}ma'(t)y^2 \end{aligned}$$

$$\begin{aligned} V_4 =& \Bigg( (1+\rho_2)\big(z+\rho_1 z(t-r)\big) + my \Bigg) \int_{t-\sigma}^{t} f'(x(s))y(s)ds - \lambda \int_{t-\sigma}^{t} y^2(s)ds \\ &+ \xi \Bigg( (1+\rho_2)\big(z+\rho_1 z(t-r)\big) + my \Bigg) \int_0^t \Omega(t,s)y(s)ds + \xi\mu_3 y^2(t) \int_t^{+\infty} |\Omega(\tau,t)| d\tau \\ &- \xi\mu_3 \int_0^t |\Omega(t,s)| y^2(s)ds \end{aligned}$$

by (i),(ii), (iii) we have

$$\begin{aligned} mc'(t)F(x) &+ (1+\rho_2)c'(t)Yf(x) + \frac{1}{2}b'(t)Y^2 \\ &\leq c'(t)\left[ mF(x) + (1+\rho_2)Yf(x) + \frac{1}{2}Y^2 \right] \\ &\leq c'(t)\left[ \frac{1}{2}\Big(Y + (1+\rho_2)f(x)\Big)^2 - \frac{(1+\rho_2)^2\big(f(x)\big)^2}{2} + mF(x) \right] \\ &\leq c'(t)\Big(m - (1+\rho)^2 A\Big)F(x) \\ &\leq 0 \end{aligned}$$

By conditions (i), (ii), (H1) we get

$$
\begin{aligned}
V_3 \leq & \frac{\rho_1 b_1}{2}(1+\rho_2)Ay^2 + \frac{\rho_1 b_1}{2}(1+\rho_2)Ay^2(t-r) + \frac{\rho_1 b_1}{2}y^2(t-r) + \frac{\rho_1 b_1}{2}z^2 \\
& + \frac{\rho_1^2 b_1}{2}y^2(t-r) + \frac{\rho_1^2 b_1}{2}z^2(t-r) - mc_0 y^2 + (1+\rho_2)b_1 Ay^2 + (m-a_0)z^2 \\
& + \frac{\rho_1}{2}(m-a_0)z^2 + \frac{\rho_1}{2}(m-a_0)z^2(t-r) + \mu_1 y^2 - \mu_1 y^2(t-r) + \mu_2 z^2 - \mu_2 z^2(t-r) \\
& + \lambda\sigma y^2 + \frac{1}{2}m \mid a'(t) \mid y^2 + 2\rho \mid zz(t-r) \mid -2\rho_1 \mid zz(t-r) \mid + (\rho_1 - \rho_1^2)z^2(t-r)
\end{aligned}
$$

$$
\begin{aligned}
V_4 \leq & \frac{A(1+\rho_2)}{2}\left(\sigma z^2 + \int_{t-\sigma}^{t} y^2(s)ds + \rho_1\sigma z^2(t-r) + \rho_1\int_{t-\sigma}^{t} y^2(s)ds\right) \\
& + \frac{m\sigma A}{2}y^2 + \frac{mA}{2}\int_{t-\sigma}^{t} y^2(s)ds - \lambda\int_{t-\sigma}^{t} y^2(s)ds \\
& + \xi\left((1+\rho_2)\Big(z^2 + \rho_1 z^2(t-r)\Big) + my^2\right)\int_0^t |\Omega(t,s)|ds \\
& + \xi\left((1+\rho_1)^2 + m - \mu_3\right)\int_0^t |\Omega(t,s)|y^2(s)ds + \xi\mu_3 y^2(t)\int_t^{+\infty} |\Omega(\tau,t)|d\tau \\
\leq & \frac{A(1+\rho)\sigma}{2}\big(z^2 + \rho z^2(t-r)\big) + \frac{m\sigma A}{2}y^2 + \left(\frac{A(1+\rho)^2 + mA}{2} - \lambda\right)\int_{t-\sigma}^{t} y^2(s)ds \\
& + \xi\left((1+\rho)\Big(z^2 + \rho z^2(t-r)\Big) + my^2\right)\eta_2 \\
& + \xi\Big((1+\rho)^2 + m - \mu_3\Big)\int_0^t |\Omega(t,s)|y^2(s)ds + \xi\mu_3\eta_2 y^2
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&V_3+V_4\\
&\le \left[\frac{1}{2}ma_2-c_0\left(m-A(1+\frac{\rho}{2})\frac{b_1(1+\rho)}{c_0}\right)+\mu_1+\lambda\sigma+\frac{m\sigma A}{2}+\xi\eta_2(m+\mu_3)\right]y^2\\
&+\left[\mu_2-\frac{(2-\rho)(a_0-m)-\rho b_1}{2}+\frac{A(1+\rho)\sigma}{2}+\xi(1+\rho)\eta_2+\rho\right]z^2\\
&+\left[\frac{\rho b_1}{2}\Big(1+\rho+A(1+\rho)\Big)-\mu_1\right]y^2(t-r)\\
&+\left[\frac{\rho}{2}\left(a_1-m+b_1\rho+A(1+\rho)\sigma+2\xi(1+\rho)\eta_2+4\right)-\mu_2\right]z^2(t-r),\\
&+\left(\frac{A(1+\rho)^2+mA}{2}-\lambda\right)\int_{t-\sigma}^{t}y^2(s)ds\\
&+\xi\Big((1+\rho)^2+m-\mu_3\Big)\int_0^t|\Omega(t,s)|y^2(s)ds-\rho^2z^2(t-r)-2\rho zz(t-\sigma).
\end{aligned}$$

By taking

$$\begin{cases}
\mu_1=\dfrac{b_1\rho}{2}\Big(1+\rho+A(1+\rho)\Big),\\[2ex]
\mu_2=\dfrac{\rho}{2}\left(a_1-m+b_1\rho+A(1+\rho)\sigma+2\xi(1+\rho)\eta_2+4\right),\\
\mu_3=(1+\rho)^2+m\\[1ex]
\lambda=\dfrac{A(1+\rho)^2+mA}{2}
\end{cases}$$

since $\zeta \leq \rho < 1$ we obtain

$$
\begin{aligned}
& V_3 + V_4 \\
& \leq \left[\frac{1}{2}ma_2 - c_0\left(m - \frac{2Ab_1}{c_0}\right) + b_1\rho\Big(1 + 2A + \eta_2(2m+4)\Big) + \frac{A(1+\rho)^2 + mA}{2}\sigma + \frac{m\sigma A}{2}\right]y^2 \\
& + \left[\frac{\rho}{2}\left(a_0 + a_1 + 2b_1 + 3A\sigma + 8\eta_2 + 6 - 2m\right) - a_0 + m + \frac{A\sigma}{2}\right]z^2 \\
& - \rho^2 z^2(t-r) - 2\rho z z(t-r). \\
& \leq -\left[c_0 m - \frac{1}{2}ma_2 - 2Ab_1 - b_1\rho\Big(1 + 2A + \eta_2(2m+4)\Big) - \frac{A(4+m)}{2}\sigma - \frac{m\sigma A}{2}\right]y^2 \\
& - \left[a_0 - \frac{\rho}{2}\left(a_0 + a_1 + 2b_1 + 3A\sigma + 8\eta_2 + 6 - 2m\right) - m\frac{A\sigma}{2}\right]z^2 - \rho^2 z^2(t-r) - 2\rho z z(t-r).
\end{aligned}
$$

provided that

$$
\rho < \min\left\{ 1, \; \frac{\alpha}{b_1\Big(1 + 2A + \eta_2(2m+4)\Big)}, \; \frac{2\beta}{a_0 + a_1 + 2b_1 + 3A\sigma + 8\eta_2 + 6 - 2m} \right\}.
$$

Where

$$
\begin{aligned}
\alpha &= c_0 m - \frac{1}{2}ma_2 - 2Ab_1 - \frac{A(4+m)}{2}\sigma - \frac{m\sigma A}{2} \\
\beta &= a_0 - m\frac{A\sigma}{2}
\end{aligned}
$$

Hence, there exists a positive constant $\Gamma_2$ such that,

$$
\dot{V}_{(4.2)} = V_3 + V_4 \leq -\,\Gamma_2\left(y^2 + Z^2\right) \tag{4.7}
$$

Where

$$
\begin{aligned}
\Gamma_2 = \min\Bigg\{ & 1, \; c_0 m - \frac{1}{2}ma_2 - 2Ab_1 - b_1\rho\Big(1 + 2A + \eta_2(2m+4)\Big) - \frac{A(4+m)}{2}\sigma - \frac{m\sigma A}{2}, \\
& a_0 - \frac{\rho}{2}\Big(a_0 + a_1 + 2b_1 + 3A\sigma + 8\eta_2 + 6 - 2m\Big) - m\frac{A\sigma}{2}\Bigg\}.
\end{aligned}
$$

Now take $V_3(\| \,(y,Z)\, \|) = \Gamma_2\Big(y^2 + Z^2\Big)$, From (4.2) it is esay to see that the function $V_3$ is positive definite. Thus, we conclude that the solution of equation (4.2) is asymptotically stable and completes the proof of the theorem 4.1 . □

### 4.1.3 Boundedness and Square Integrability

Second case is for $\varphi(t,x(t),x'(t),x''(t)) \neq 0$ we have the next theorem

**Theorem 4.2.** *Let all the conditions of Theorem 4.1 hold and assume that the following conditions are satisfied*

$$\begin{aligned} &1) \quad \mid \varphi(t,x(t),x'(t),x''(t)) \mid \leq J(t) \\ &2) \quad \int_{t_1}^{t} J(s)ds \leq \eta_3 \\ &3) \quad \int_{t_1}^{t} \int_{0}^{s} \mid \Omega(s,\tau) \mid d\tau ds \leq \eta_4 \end{aligned}$$

*Then, there exists a finite positive constant $K$ such that every solution $x(.)$ of (4.1) and their derivatives $x'(.)$ and $X''(.)$ are satisfied*

$$1. \quad |x(t)| \leq \sqrt{K}, \ \mid x'(t)| \leq \sqrt{K}, \ |X''(t)| \leq \sqrt{K}, \qquad \text{for all} \ \ t \geq t_1.$$

$$2. \int_{t_1}^{\infty} \left( x'^2(s) + \Big( x''(s) + \rho_1 x''(s-r) \Big)^2 \right) ds < \infty.$$

*where $\eta_3$,$\eta_4$ are positives constants.*

**Proof.** If $\varphi(t,x(t),x'(t),x''(t)) \neq 0$ we have the differential system

$$\begin{cases} x' = y \\ y' = z \\ Z' = \varphi(t,x(t),x'(t),x''(t)) + \xi \int_{0}^{t} \Omega(t,s)y(s)ds - a(t)z - b\,(t)\,y - (1+\rho_2)c\,(t)\,f\,(x) \\ \qquad\qquad +\rho_2 c(t) \int_{t-\sigma}^{t} f'(x(s))y(s)ds, \end{cases} \tag{4.8}$$

From (4.7) and the estimate $u \leq\mid u \mid\leq u^2+1$, the time derivative of $V$ with respect to t along the trajectory of system (4.8), we obtain

$$\begin{aligned}
\dot{V}_{(4.8)} =&\dot{V}_{(4.2)} + \Big((1+\rho_2)\big(z+\rho_1 z(t-r)\big)+my\Big)\varphi(t,x(t),x'(t),x''(t)) \\
&\leq -\Gamma_2\left(y^2+Z^2\right) + \Big((1+\rho_2)\big(z+\rho_1 z(t-r)\big)+my\Big) \mid \varphi(t,x(t),x'(t),x''(t)) \mid \\
&\leq \Big((1+\rho)|Z|+m|y|\Big)J(t) \\
&\leq \Gamma_3\left(3+x^2+y^2+Z^2\right)J(t) \\
&\leq \Gamma_3\left(3+\frac{1}{\Gamma_1}V\right)J(t) \\
&\leq 3\Gamma_3 J(t) + \frac{\Gamma_3}{\Gamma_1}V|J(t)|, \qquad (4.9)
\end{aligned}$$

where $\Gamma_3 = \max\{1+\rho, m, 1\}$. Integrating (4.9) from 0 to $t$, and using the condition (2) of theorem 4.2 and the Gronwall Reid Bellman inequality, we obtain

$$\begin{aligned}
V(t,x,y,Z) &\leq V_0 + 3\Gamma_3\eta_3 + \frac{\Gamma_3}{\Gamma_1}\int_0^t V\big(s,x(s),y(s),Z(s)\big)J(s)ds \\
&\leq \Big(V_0+3\Gamma_3\eta_3\Big)e^{\frac{\Gamma_3}{\Gamma_1}\int_0^t J(s)ds} \\
&\leq \Big(V_0+3\Gamma_3\eta_3\Big)e^{\frac{\Gamma_3}{\Gamma_1}\eta_3} = K_1 < \infty, \qquad (4.10)
\end{aligned}$$

where $V_0 = V\big(0,x(0),y(0),Z(0)\big)$.

In view of inequalities (4.6) and (4.10), we get

$$(x^2+y^2+Z^2) \leq \frac{1}{\Gamma_1}V \leq K \qquad (4.11)$$

where $K = \dfrac{K_1}{\Gamma_1}$. Clearly (4.11) implies that

$$|x(t)| \leq \sqrt{K},\ |y(t)| \leq \sqrt{K},\ |Z(t)| \leq \sqrt{K}, \quad \text{for all} \quad t \geq 0.$$

Hence,

$$|x(t)| \leq \sqrt{K},\ |x'(t)| \leq \sqrt{K},\ |x''(t)+\rho_1 x''(t-r)| \leq \sqrt{K}, \quad \text{for all} \quad t \geq 0. \qquad (4.12)$$

Now, we define $F_t = F(t,x(t),y(t),Z(t))$ as

$$F_t = V + \varepsilon\int_0^t \left(y^2(s)+Z^2(s)\right)ds,$$

where $\varepsilon > 0$. It is esay to see that $F_t$ is positive definite,

$$\begin{aligned}\dot{F}_{t(4.8)} &\leq -\Gamma_2\Big(y^2(t)+Z^2(t)\Big)+\Big((1+\rho_2)|Z|+m|y|\Big)|\varphi(t,x(t),x'(t),x''(t))|+\varepsilon\Big(y^2(t)+Z^2(t)\Big)\\ &\leq \Big(\varepsilon-\Gamma_2\Big)\Big(y^2(t)+Z^2(t)\Big)+\Gamma_3\Big(|y(t)|+|Z(t)|\Big)J(t). \qquad (4.13)\end{aligned}$$

By choosing $\varepsilon = \Gamma_2$ we obtain

$$\begin{aligned}\dot{F}_{t(4.8)} &\leq \Gamma_3\Big(3+x^2+y^2(t)+Z^2(t)\Big)J(t)\\ &\leq \Gamma_3\Big(3+\frac{1}{\Gamma_1}V\Big)J(t)\\ &\leq \Gamma_4 J(t). \qquad (4.14)\end{aligned}$$

where $\Gamma_4 = \Gamma_3\Big(3+K\Big)$

Integrating the last inequality (4.14) from 0 to $t$, and using again the Gronwall Reid Bellman inequality and using the condition (2) of theorem 4.2, we get

$$\begin{aligned}F_t &\leq F_0+\Gamma_4\eta_3\\ \varepsilon\int_0^t\big(y^2(s)+Z^2(s)\big)\,ds &\leq V+\varepsilon\int_0^t\big(y^2(s)+Z^2(s)\big)\,ds \leq F_0+\Gamma_4\eta_3\end{aligned}$$

Therefore

$$\int_0^\infty y^2(s)ds < K_2 \ \text{ and} \int_0^\infty Z^2(s) < K_2.$$

such that $K_2 = \dfrac{F_0+\Gamma_4\eta_3}{\varepsilon}$

which implies that

$$\int_0^\infty x'^2(s)ds = \int_0^\infty y^2(s)ds \leq K_2 \qquad (4.15)$$

and

$$\int_0^\infty \Big(x''(s)+\rho_1 x''(s-r)\Big)^2 ds = \int_0^\infty Z^2(s) < K_2$$

The proof of Theorem 4.2 is completed. □

### 4.1.4 Example

We consider the following third order nonlinear integro-differential equation with variable delay

$$\Big[x(t)+0.25x(t-0.5)\Big]''' + \left(\frac{1}{\pi}\arctan(t)+\frac{161}{2}\right)x''(t) + \left(\frac{1}{t^2+2}+1\right)x'(t)$$

$$+\left(\frac{1}{t^2+4}+1\right)\left[\frac{x(t)}{12}+\frac{x(t)}{6+x^2(t)}+\frac{1}{6}\Big(\frac{x(t-0.25)}{12}+\frac{x(t-0.25)}{6+x^2(t-0.25)}\Big)\right]$$

$$=\frac{\sin(t)}{1+t^2}+\frac{1}{550}\int_0^t\frac{\left(e^{-t^2}\cos t-e^{-t^2}\sin t\right)e^{-s^2}}{4(1+s^2)}x'(s)ds.$$

For all $t \geq t_0 + r$, where $r = 0.5$, $\sigma = 0.25$ and $\rho = 0.25$

$(i)$ $80 = a_0 \leq a(t) = \frac{1}{\pi}\arctan(t) + \frac{161}{2} \leq a_1 = 81, \ \mid a'(t) \mid = \frac{1}{\pi(t^2+1)} \leq \frac{1}{\pi} = a_2,$

$1 = c_0 \leq c(t) = \frac{1}{t^2+4} + 1 \leq b(t) = \frac{1}{t^2+2} + 1 \leq \frac{3}{2} = b_1,$

*and* $b'(t) = \frac{-2t}{(t^2+2)^2} \leq c'(t) = \frac{-2t}{(t^2+4)^2} \leq 0$

$(ii)$ $\frac{1}{4} = \gamma \geq \frac{f(x)}{x} = \frac{1}{12} + \frac{1}{6+x^2} \geq \frac{1}{12} = \delta, \ \mid f'(x) \mid = \mid \frac{1}{2} + \frac{6-x^2}{6+x^2} \mid \leq \frac{1}{4} = A, \ \ f(0) = 0$

For m= 16

iii)

$$\begin{cases} A(1+\rho)^2 = 0.39 \leq m \leq \min\{a_0, \dfrac{2a_0}{A\sigma}\} = 80, \\ \\ c_0 = 1 \geq \dfrac{1}{2}a_2 + \dfrac{2Ab_1}{m} + \dfrac{A(4+m)}{2m}\sigma + \dfrac{\sigma A}{2} = 0.27 \end{cases}$$

H1) By taking $\eta_2 = \dfrac{1}{2}$ we obtain

$$\int_0^t |\Omega(t,s)|\,ds = \int_0^t \left| \frac{\left(e^{-t^2}\cos t - e^{-t^2}\sin t\right)e^{-s^2}}{4(1+s^2)} \right| ds$$
$$\leq \frac{1}{2}\int_0^t e^{-s^2}ds \leq \frac{1}{2}\int_0^{+\infty} e^{-s^2}ds = \frac{\sqrt{\pi}}{4} < \eta_2,$$

$$\alpha = c_0 m - \frac{1}{2}ma_2 - 2Ab_1 - \frac{A(4+m)}{2}\sigma - \frac{m\sigma A}{2} = 15.32$$
$$\beta = a_0 - m\frac{A\sigma}{2} = 79.5$$

$$\rho = \frac{1}{4} < \min\left\{ 1,\ \frac{\alpha}{b_1\Big(1+2A+\eta_2(2m+4)\Big)},\ \frac{2\beta}{a_0+a_1+2b_1+3A\sigma+8\eta_2+6-2m} \right\}.$$
$$= \min\left\{1,\ \frac{15.32}{29.25},\ \frac{159}{142.18}\right\}$$
$$= 0.52.$$

$$|\ \varphi(t,x(t),x'(t),x''(t))\ | = |\ \frac{\sin(t)}{1+t^2}\ | \leq \frac{1}{1+t^2} = J(t)$$
$$\int_{t_1}^t J(s)ds = \int_{t_1}^t \frac{1}{1+s^2}ds \leq \frac{\pi}{2} = \eta_3$$

$$\int_{t_1}^t \int_0^s |\Omega(s,\tau)|\,d\tau ds = \int_{t_1}^t \int_0^s \left| \frac{\left(e^{-s^2}\cos s - e^{-s^2}\sin s\right)e^{-\tau^2}}{4(1+\tau^2)} \right| d\tau ds$$
$$\leq \frac{1}{2}\int_0^t \int_0^s e^{-\tau^2}e^{-s^2}d\tau ds \leq \frac{1}{2}\int_0^{+\infty}\int_0^s e^{-\tau^2}e^{-s^2}d\tau ds = \frac{\pi}{8} = \eta_4.$$

## 4.2 Fourth Order Nonlinear Integro-Differential Equations With Delay of Neutral Type

In this section, we study the qualitative behavior of such equations by establishing conditions that ensure the boundedness and square integrability of their solutions. The analysis focuses on understanding how the interaction between nonlinear terms and delay components influences the stability and long-term behavior of the system. We develop conditions under which all the solutions of the following fourth order nonlinear neutral integro-differential equation are bounded and are square integrable

$$\Big(\varphi(t)\big(x'''(t)+\rho x'''(t-r)\big)\Big)'+L(t)\Big(\psi\big(x(t)\big)x''(t)\Big)'+R(t)\Big(\phi\big(x(t)\big)x'(t)\Big)'$$
$$+P(t)f\big(x(t)\big)x'(t)+Q(t)g\big(x(t)\big)=e(t)+\xi\int_0^t\Omega(t,s)x'(s)ds, \quad (4.16)$$

for all $t \geq t_1 = t_0 + r$ where $r$,$\xi$ and $\rho < 1$ are positive constants, the primes in (4.16) denote differentiation with respect to t; the functions $\varphi, L, R, P, Q, e : [0,\infty) \to (0,\infty)$ and $\psi, \phi, f, g : \mathbb{R} \to \mathbb{R}$ are continuous differentiable and $\Omega(t,s)$ is continuous for $0 \leq s \leq t < \infty$ and $g(0) = 0$.

By solution of (4.16) we mean a continuous function $x : [t_x,\infty) \to \mathbb{R}$ for $t_x \geq t_1$ which satisfied the equation (4.16) in $[t_x,\infty)$ and such that

$$x(t) + \rho x(t-r) \in \mathcal{C}^4([t_x,\ \infty),\ \mathbb{R}).$$

### 4.2.1 Basic Assumptions

Suppose that there are positive constants $L_0$, $R_0$, $P_0$, $Q_0$, $f_0$, $\varphi_0$, $\psi_0$,$\phi_0$, $L_1$, $R_1$, $P_1$, $Q_1$, $f_1$, $\varphi_1$,
$\psi_1$, $\phi_1$, $m$, $M$, $\delta$, and $\eta_1$, such that the following conditions are satisfied

i) $0 < L_0 \leq L(t) \leq L_1$, $0 < R_0 \leq R(t) \leq R_1$, $0 < P_0 \leq P(t) \leq P_1$,
$0 < Q_0 \leq Q(t) \leq Q_1$, $0 < \varphi_0 \leq \varphi(t) \leq \varphi_1 < 1$ and $Q'(t) \leq 0$ for $t \geq t_1$;

ii) $0 < f_0 \leq f(x) \leq f_1$, $0 < \psi_0 \leq \psi(x) \leq \psi_1$, $0 < \phi_0 \leq \phi(x) \leq \phi_1$ for $x \in \mathbb{R}$ and

$0 < m < \min\{f_0, \psi_0, 1\}$, $M > \max\{f_1, \psi_1, 1\}$;

iii) $\dfrac{g(x)}{x} \geq \delta > 0$ ( for $x \neq 0$) *and* $g(0) = 0$;

iv) $\displaystyle\int_{t_1}^{+\infty} \left(|L'(t)| + |R'(t)| + |P'(t)| + |\varphi'(t)| - Q'(t)\right) dt < \eta_1.$

### 4.2.2 Boundedness And Square Integrability

To prove the main results the following Lemma is needed.

**Lemma 4.1.** *[41] Let* $g(0) = 0$, $xg(x) > 0$ $(x \neq 0)$ *and* $\delta(t) - g'(x) \geq 0$ $(\delta(t) > 0)$, *then*

$$2\delta(t)G(x) \geq g^2(x) \qquad \text{where} \quad G(x) = \int_0^x g(s)ds.$$

**Theorem 4.3.** *In addition to conditions (i) ∼ (iv) being satisfied, suppose that there are positive constants* $g_0$, $\delta_0$, $\delta_1$, $\eta_2$, $\eta_3$ *and* $\eta_4$ *such that the following conditions hold:*

*H1)* $\displaystyle g_0 - \frac{L_0 m \delta_0}{Q_1} \leq g'(x) \leq \frac{g_0}{2}$ *for* $x \in \mathbb{R}$;

*H2)* $\displaystyle \delta_1 = \frac{Q_1 g_0 L_1 M}{P_0 m} + \frac{P_1 M + \delta_0}{L_0 m} < R_0 \phi_0$;

*H3)* $\displaystyle\int_{-\infty}^{+\infty} \left(|\psi'(s)| + |\phi'(s)| + |f'(s)|\right) ds < \eta_2$;

*H4)* $\displaystyle \max\left\{\int_{t_1}^{+\infty} |e(t)|\, dt,\ \int_0^t |\Omega(t,s)|ds,\ \int_t^{+\infty} |\Omega(\tau,t)|d\tau\right\} < \eta_3$;

*H5)* $\displaystyle\int_{t_1}^{t}\int_0^s |\Omega(s,u)|\, dsdu < \eta_4$.

*Then, there exists a finite positive constant $K$ such that every solution $x(.)$ of (4.16) and their derivatives $x'(.)$, $x''(.)$, $x'''(.)$ and $X'''(.)$ are satisfied*

1. $|x(t)| \leq \sqrt{K}$, $|x'(t)| \leq \sqrt{K}$, $|x''(t)| \leq \sqrt{K}$, $|X'''(t)| \leq \sqrt{K}$, *for all* $t \geq t_1$.

2. $\displaystyle\int_{t_1}^{\infty} \left(x^2(s) + x'^2(s) + x''^2(s) + x'''^2(s)\right) ds < \infty.$

**Proof.** First we prove the boundedness of solutions. The equation (4.16) is equivalent to the differential system

$$\begin{cases} x' = y \\ y' = z \\ z' = w \\ (\varphi(t)W)' = e(t) + \xi\int_0^t \Omega(t,s)y(s)ds - L(t)\psi(x)w - \Big(R(t)\phi(x) + L(t)\theta_1\Big)z \\ \qquad\qquad -\Big(R(t)\theta_2 + P(t)f(x)\Big)y - Q(t)g(x), \end{cases} \tag{4.17}$$

where $W(t) = x'''(t) + \rho x'''(t-r) = w(t) + \rho w(t-r)$ and

$$\theta_1(t) = \psi'(x(t))x'(t), \text{ and } \theta_2(t) = \phi'(x(t))x'(t).$$

It is esay to see from (4.17) that

$$\begin{cases} X'(t) = x'(t) + \rho x'(t-r) = y(t) + \rho y(t-r) = Y(t) \\ \\ X''(t) = x''(t) + \rho x''(t-r) = z(t) + \rho z(t-r) = Z(t) \\ \\ X'''(t) = x'''(t) + \rho x'''(t-r) = w(t) + \rho w(t-r) = W(t). \end{cases}$$

The proof of this theorem depends on properties of the continuously differentiable function
$U = U(t,x,y,z,w)$ defined as

$$U = e^{-\frac{1}{\eta}\int_{t_1}^t \gamma(s)\,ds}V, \tag{4.18}$$

where

$$\gamma(t) = |L'(t)| + |R'(t)| + |P'(t)| + |\varphi'(t)| - Q'(t) + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|,$$

$$\theta_3(t) = f'(x(t))x'(t)$$

and

$$\begin{aligned}
2V &= 2\beta Q(t)\,G(x) + P(t)\,f(x)\,y^2 + \alpha R(t)\,\phi(x)z^2 + L(t)\,\psi(x)z^2 + 2\beta L(t)\,\psi(x)yz \\
&\quad + [\beta R(t)\,\phi(x) - \alpha g_0 Q(t)]\,y^2 - \beta z^2 + \alpha\varphi(t)W^2 + 2Q(t)\,g(x)\,y + 2\alpha Q(t)\,g(x)\,Z \\
&\quad + \alpha\rho Q(t)\,(z(t-r))^2 + 2\alpha P(t)\,f(x)\,yz + 2\beta\varphi(t)yW + 2\varphi(t)zW + \mu_1\int_{t-r}^{t} z^2(s)\,ds \\
&\quad + \mu_2\int_{t-r}^{t} w^2(s)\,ds + \xi\mu_3\int_0^t\int_t^{+\infty} |\Omega(\tau,s)|y^2(s)d\tau ds,
\end{aligned}$$

with $\alpha = \dfrac{1}{L_0 m} + \varepsilon$ , $\beta = \dfrac{Q_1 g_0}{P_0 m} + \varepsilon$, $\varepsilon$, and $\eta$ are positive constants to be determined later in the proof.

We rewrite $2V$ as

$$\begin{aligned}
2V = {} & L(t)\,\psi(x)\left[\frac{\varphi(t)W}{L(t)\,\psi(x)} + z + \beta y\right]^2 + P(t)\,f(x)\left[\frac{Q(t)\,g(x)}{P(t)\,f(x)} + y + \alpha z\right]^2 \\
& + \frac{Q^2(t)\,g^2(x)}{P(t)\,f(x)} + \mu_1\int_{t-r}^{t} z^2(s)\,ds + \mu_2\int_{t-r}^{t} w^2(s)\,ds + \xi\mu_3\int_0^t\int_t^{+\infty} |\Omega(\tau,s)|y^2(s)d\tau ds \\
& + V_1 + V_2 + V_3 + V_4,
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= 2Q(t)\int_0^x g(s)\left[\frac{Q_1 g_0}{P_0 m} - 2\frac{Q(t)}{P(t)\,f(x)}g'(s)\right]ds, \\
V_2 &= \left[\alpha R(t)\,\phi(x) - \beta - \alpha^2 P(t)\,f(x)\right]z^2, \\
V_3 &= \left[\beta R(t)\,\phi(x) - \alpha g_0 Q(t) - \beta^2 L(t)\,\psi(x)\right]y^2 + \left[\frac{\alpha}{\varphi(t)} - \frac{1}{L(t)\,\psi(x)}\right]\varphi^2(t)W^2, \text{ and} \\
V_4 &= 2\varepsilon Q(t)\,G(x) + 2\alpha\rho Q(t)\,g(x)\,z(t-r) + \alpha\rho Q(t)\,(z(t-r))^2.
\end{aligned}$$

Now, we will prove that $V$ is positive definite. Take

$$\varepsilon < \min\left\{\frac{1}{L_0 m}\ ,\ \frac{Q_1 g_0}{P_0 m}\ ,\ \frac{R_0\phi_0 - \delta_1}{M(L_1 + P_1)}\right\}, \tag{4.19}$$

then

$$\frac{1}{L_0 m} < \alpha < \frac{2}{L_0 m}, \quad \frac{Q_1 g_0}{P_0 m} < \beta < 2\frac{Q_1 g_0}{P_0 m}. \tag{4.20}$$

Using conditions (i) $\sim$ (iii), (H1), (H2) and inequalities (4.19), (4.20) we get

$$\begin{aligned}
V_1 &\geq 4Q(t)\frac{Q_1}{P_0 m}\int_0^x g(s)\left[\frac{g_0}{2}-g'(s)\right]ds \geq 0,\\
V_2 &= \Big(\alpha\Big(R(t)\phi(x)-\beta L(t)-\alpha P(t)f(x)\Big)+\beta\big(\alpha L(t)-1\big)\Big)z^2\\
&\geq \alpha\Big(R_0\phi_0-\frac{Q_1 g_0 L_1}{P_0 m}-\frac{P_1 M}{L_0 m}-\varepsilon(L_1+P_1 M)\Big)z^2+\beta\big(\frac{1}{m}-1\big)z^2\\
&\geq \alpha\big(R_0\phi_0-\delta_1-\varepsilon M(L_1+P_1)\big)z^2 \geq 0,\\
V_3 &\geq \beta\Big(R_0\phi_0-\frac{\alpha}{\beta}g_0 Q_1-\beta L_1 M\Big)y^2+\Big(\alpha-\frac{1}{L_0 m}\Big)\varphi_0^2 W^2\\
&\geq \beta\Big(R_0\phi_0-\frac{P_0}{L_0}-L_1\frac{Q_1 g_0 M}{P_0 m}-\varepsilon(P_0 m+L_1 M)\Big)y^2+\varepsilon\varphi_0^2 W^2\\
&\geq \beta\big(R_0\phi_0-\delta_1-\varepsilon M(P_1+L_1)\big)y^2+\varepsilon\varphi_0^2 W^2 \geq 0
\end{aligned}$$

and by choosing $\rho < \frac{4\varepsilon}{\alpha g_0}$

$$\begin{aligned}
V_4 &= 2\varepsilon Q(t)\int_0^x g(\xi)d\xi+\alpha\rho Q(t)\left[(z(t-r)+g(x))^2-g^2(x)\right]\\
&\geq 2\varepsilon Q(t)\int_0^x g(\xi)d\xi-\frac{\alpha\rho}{2}Q(t)\int_0^x g'(\xi)g(\xi)d\xi\\
&\geq 2Q(t)\int_0^x\left(\varepsilon-\frac{\alpha\rho g_0}{4}\right)g(\xi)d\xi\\
&\geq 2Q_0\left(\varepsilon-\frac{\alpha\rho g_0}{4}\right)G(x).
\end{aligned}$$

Hence, it is evident from the terms contained in the last inequalities, that there exists a positive constant $\Gamma_0$ such that

$$2V \geq \Gamma_0\left(y^2+z^2+W^2+G(x)\right). \tag{4.21}$$

By Lemma 4.1 and conditions (iii) and (H1) it follows that there is a positive constant $\Gamma_1$ such that

$$2V \geq \Gamma_1\left(x^2+y^2+z^2+W^2\right). \tag{4.22}$$

Thus $V$ is positive definite. From (i)—(iii), it is not difficult to see that there is a positive constant $\Delta_1$ such that

$$V \leq \Delta_1\left(x^2+y^2+z^2+W^2\right).$$

By (H3), we have

$$\begin{aligned}\int_{t_1}^{t}\big(|\theta_1(s)|+|\theta_2(s)|+|\theta_3(s)|\big)ds &= \int_{\alpha_1(t)}^{\alpha_2(t)}\big(|\psi'(u)|+|\phi'(u)|+|f'(u)|\big)du\\ &\leq \int_{-\infty}^{+\infty}\big(|\psi'(u)|+|\phi'(u)|+|f'(u)|\big)du < \eta_2 < \infty,\end{aligned} \tag{4.23}$$

where $\alpha_1(t) = \min\{x(t_1),x(t)\}$, and $\alpha_2(t) = \max\{x(t_1),x(t)\}$. From inequalities (iv), (4.22), and (4.23), it follows that

$$U \geq \Gamma_2(x^2+y^2+z^2+W^2), \tag{4.24}$$

where $\Gamma_2 = \dfrac{\Gamma_1}{2}e^{-\frac{\eta_1+\eta_2}{\eta}}$. Also, it is esay to see that there is a positive constant $\Delta_2$ such that

$$U \leq \Delta_2(x^2+y^2+z^2+W^2), \tag{4.25}$$

for all $x$,$y$,$z$ and $w$, and all $t \geq t_1$.
Next we show that $\dot{U}$ is negative definite function. Using the following derivative

$$\frac{d}{dt}\big(\alpha\varphi(t)W^2(t)\big) = -\alpha\varphi'(t)W^2 + 2\alpha W(t)\frac{d}{dt}\big(\varphi(t)W(t)\big).$$

The derivative of the function $V$, along any solution $(x(t),y(t),z(t),w(t))$ of system (4.17), with respect to $t$ after simplifying is given by

$$2\dot{V}_{(4.17)} = V_5+V_6+V_7+V_8+V_9+V_{10}+2(\beta y+z+\alpha W)e(t),$$

where

$$
\begin{aligned}
V_5 =& -2\left(\frac{Q_1 g_0}{P_0 m} P\left(t\right) f(x) - Q\left(t\right) g'\left(x\right)\right) y^2 - 2\alpha Q\left(t\right)\left(g_0 - g'\left(x\right)\right) yz,\\
V_6 =& -2\big(R\left(t\right)\phi(x) - \alpha P\left(t\right) f(x) - \beta L\left(t\right)\psi(x)\big) z^2,\\
V_7 =& -2\big(\alpha L\left(t\right)\psi(x) - \varphi(t)\big) w^2,\\
V_8 =& -2\varepsilon P\left(t\right) f(x) y^2 - 2\alpha\rho L(t)\psi\left(x\right) w_t w - 2\alpha\rho R(t)\phi\left(x\right) z w_t - 2\alpha\rho P\left(t\right) f\left(x\right) y w_t\\
&+ 2\alpha\rho Q\left(t\right) g'\left(x\right) y z_t + \mu_1 z^2 + \mu_2 w^2 - \mu_1 z_t^2 - \mu_2 w_t^2 + 2\alpha\rho Q\left(t\right) z_t w_t + 2\rho w w_t\\
&+ 2\beta\rho z w_t + \xi\Big(2\beta y + 2z + 2\alpha w + 2\alpha\rho w_t\Big)\int_0^t \Omega(t,s) y(s) ds + \xi\mu_3 y^2(t)\int_t^{+\infty}|\Omega(\tau,t)| d\tau\\
&- \xi\mu_3\int_0^t |\Omega(t,s)| y^2(s) ds\\
V_9 =& -L(t)\theta_1\Big(z^2 + 2\alpha z W\Big) - R(t)\theta_2\Big(\beta y^2 + 2\alpha y W + 2yz - \alpha z^2\Big)\\
&+ P(t)\theta_3\Big(y^2 + 2\alpha yz\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
V_{10} =\;& Q'\left(t\right)\left[2\beta G\left(x\right) - \alpha g_0 y^2 + 2g\left(x\right) y + 2\alpha g\left(x\right) z\right] + P'\left(t\right)\left[f(x) y^2 + 2\alpha f(x) yz\right]\\
&+ R'\left(t\right)\left[\alpha\phi(x) z^2 + \beta\phi(x) y^2\right] + L'\left(t\right)\left[\psi(x) z^2 + 2\beta\psi(x) yz\right] - \alpha\varphi'(t) W^2\\
&+ \alpha\rho Q'\left(t\right)\left[z\left(t-r\right) + g(x)\right]^2 - \alpha\rho Q'\left(t\right) g^2(x).
\end{aligned}
$$

By conditions (i), (ii), (H1), (H2) and inequality (4.19)and (4.20), we obtain

$$
\begin{aligned}
V_5 &\le -2\left[Q\left(t\right) g_0 - Q\left(t\right) g'\left(x\right)\right] y^2 - 2\alpha Q\left(t\right)\left[g_0 - g'\left(x\right)\right] yz\\
&\le -2Q\left(t\right)\left[g_0 - g'\left(x\right)\right] y^2 - 2\alpha Q\left(t\right)\left[g_0 - g'\left(x\right)\right] yz\\
&\le -2Q\left(t\right)\left[g_0 - g'\left(x\right)\right]\left[\left(y + \frac{\alpha}{2} z\right)^2 - \left(\frac{\alpha}{2} z\right)^2\right]\\
&\le \frac{\alpha^2}{2} Q\left(t\right)\left[g_0 - g'\left(x\right)\right] z^2.
\end{aligned}
$$

Therefore,

$$\begin{aligned}V_5+V_6 &\le -2\left[R\left(t\right)\phi(x)-\alpha P\left(t\right)f(x)-\beta L\left(t\right)\psi(x)-\frac{\alpha^2}{4}Q\left(t\right)\left[g_0-g'\left(x\right)\right]\right]z^2\\ &\le -2\left[R_0\phi_0-\big(\frac{1}{L_0m}+\varepsilon\big)P_1M-\big(\frac{Q_1g_0}{P_0m}+\varepsilon\big)L_1M-\frac{\alpha^2}{4}\left(L_0m\delta_0\right)\right]z^2\\ &\le -2\left[R_0\phi_0-\frac{M}{L_0m}P_1-\frac{Q_1g_0L_1M}{P_0m}-\frac{\delta_0}{L_0m}-\varepsilon M\left(L_1+P_1\right)\right]z^2\\ &\le -2\left[R_0\phi_0-\delta_1-\varepsilon M\left(L_1+P_1\right)\right]z^2\le 0,\end{aligned}$$

$$V_7\le -2\left[\alpha L_0m-1\right]w^2=-2\varepsilon w^2\le 0$$

and

$$\begin{aligned}V_8 &\le -2\varepsilon P\left(t\right)f(x)y^2+\alpha\rho L_1Mw_t^2+\alpha\rho L_1Mw^2+\alpha\rho R_1\phi_1z^2+\alpha\rho R_1\phi_1w_t^2+\alpha\rho P_1My^2\\ &\quad+\alpha\rho P_1Mw_t^2+\alpha\rho Q_1\lambda_0y^2+\alpha\rho Q_1\lambda_0z_t^2+\mu_1z^2+\mu_2w^2-\mu_1z_t^2-\mu_2w_t^2\\ &\quad+\alpha\rho Q_1z_t^2+\alpha\rho Q_1w_t^2+\rho w^2+\beta\rho z^2+\rho w_t^2+\beta\rho w_t^2+2\rho|ww_t|-2\rho|ww_t|\\ &\quad+(\rho-\rho^2)w_t^2+\xi\Big(\beta y^2+z^2+\alpha w^2+\alpha\rho w_t^2\Big)\int_0^t|\Omega(t,s)|ds\\ &\quad+\xi\Big((1+\rho)\alpha+\beta+1-\mu_3\Big)\int_0^t|\Omega(t,s)|y^2(s)ds+\xi\mu_3y^2(t)\int_t^{+\infty}|\Omega(\tau,t)|d\tau\\ &\le -\left(2\varepsilon P_0m-\alpha\rho P_1M-\alpha\rho Q_1\lambda_0-\xi\beta\eta_3-\xi\mu_3\eta_3\right)y^2+\left(\alpha\rho R_1\phi_1+\beta\rho+\xi\eta_3+\mu_1\right)z^2\\ &\quad+\left(\alpha\rho L_1M+\alpha\rho R_1\phi_1+\alpha\rho P_1M+\alpha\rho Q_1+\beta\rho+3\rho+\xi\alpha\rho\eta_3-\mu_2\right)w_t^2\\ &\quad+\left(\alpha\rho Q_1\lambda_0+\alpha\rho Q_1-\mu_1\right)z_t^2-\rho^2w_t^2-2\rho|ww_t|+\left(\alpha\rho L_1M+2\rho+\xi\alpha\eta_3+\mu_2\right)w^2\\ &\quad+\xi\Big((1+\rho)\alpha+\beta+1-\mu_3\Big)\int_0^t|\Omega(t,s)|y^2(s)ds,\end{aligned}$$

where

$$\lambda_0=\max\left\{\frac{g_0}{2M},\left|\frac{g_0}{m}-\frac{L_0m\delta_0}{Q_1}\right|\right\}.$$

By taking

$$\begin{cases}\mu_1=\alpha\rho Q_1\lambda_0+\alpha\rho Q_1,\\ \mu_2=\alpha\rho L_1M+\alpha\rho R_1\phi_1+\alpha\rho P_1M+\alpha\rho Q_1+\beta\rho+3\rho+\xi\alpha\rho\eta_3,\\ \mu_3=(1+\rho)\alpha+\beta+1,\end{cases}$$

we obtain

$$\begin{aligned} V_8 \ \leq \ & -\left(2\varepsilon P_0 m - \alpha\rho P_1 M - \alpha\rho Q_1\lambda_0 - \xi\beta\eta_3 - \xi\mu_3\eta_3\right) y^2 + \left(\alpha\rho R_1\phi_1 + \beta\rho + \xi\eta_3 + \mu_1\right) z^2 \\ & + \left(\alpha\rho L_1 M + 2\rho + \xi\alpha\eta_3 + \mu_2\right) w^2 - \rho^2 w_t^2 - 2\rho|ww_t|. \end{aligned}$$

Then we have

$$\begin{aligned} V_5 + V_6 \quad +V_7 \quad +V_8 \leq & \\ & -\left(2\varepsilon P_0 m - \alpha\rho P_1 M - \alpha\rho Q_1\lambda_0 - \xi\big((1+\rho)\alpha + 2\beta + 1\big)\eta_3\right) y^2 \\ & -2\left[R_0\phi_0 - \delta_1 - \frac{1}{2}\xi\eta_3 - \varepsilon M\left(L_1 + P_1\right) - \frac{1}{2}\rho\left(\alpha R_1\phi_1 + \beta + \alpha Q_1\lambda_0 + \alpha Q_1\right)\right] z^2 \\ & -\left(2\varepsilon - \xi\alpha\eta_3 - \rho\left(2\alpha L_1 M + 5 + \alpha R_1\phi_1 + \alpha P_1 M + \alpha Q_1 + \beta + \xi\alpha\eta_3\right)\right) w^2 \\ & -\rho^2 w_t^2 - 2\rho|ww_t|, \end{aligned}$$

since $\xi \leq \rho < 1$ we obtain

$$\begin{aligned} V_5 + V_6 \quad +V_7 \quad +V_8 \leq & \\ & -\left(2\varepsilon P_0 m - \alpha\rho P_1 M - \alpha\rho Q_1\lambda_0 - \rho\big(2\alpha + 2\beta + 1\big)\eta_3\right) y^2 \\ & -2\left[R_0\phi_0 - \delta_1 - \varepsilon M\left(L_1 + P_1\right) - \frac{1}{2}\rho\left(\alpha R_1\phi_1 + \beta + \alpha Q_1\lambda_0 + \alpha Q_1 + \eta_3\right)\right] z^2 \\ & -\left(2\varepsilon - \rho\left(2\alpha L_1 M + 5 + \alpha R_1\phi_1 + \alpha P_1 M + \alpha Q_1 + \beta + 2\alpha\eta_3\right)\right) w^2 \\ & -\rho^2 w_t^2 - 2\rho|ww_t|, \end{aligned}$$

provided that

$$\begin{aligned} \rho < \min & \left\{1, \frac{4\varepsilon}{\alpha g_0}, \frac{2\varepsilon P_0 m}{\alpha P_1 M + \alpha Q_1\lambda_0 + \big(2\alpha + 2\beta + 1\big)\eta_3}, \right. \\ & \left. 2\frac{R_0\phi_0 - \delta_1 - \varepsilon M\left(L_1 + P_1\right)}{\alpha R_1\phi_1 + \beta + \alpha Q_1\lambda_0 + \alpha Q_1 + \eta_3}, \frac{2\varepsilon}{2\alpha L_1 M + 5 + \alpha R_1\phi_1 + \alpha P_1 M + \alpha Q_1 + \beta + 2\alpha\eta_3}\right\}. \end{aligned}$$

Hence, there exists a positive constant $\Gamma_3$ such that,

$$\begin{aligned} V_5 + V_6 + V_7 + V_8 &\leq -2\Gamma_3\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho|ww_t|\right) \qquad (4.26) \\ &\leq -2\Gamma_3\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho ww_t\right) = -2\Gamma_3\left(y^2 + z^2 + W^2\right). \end{aligned}$$

From (4.21), and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} V_9 &\leq L(t)|\theta_1|\Big(z^2+\alpha(z^2+W^2)\Big)+R(t)|\theta_2|\Big(\alpha z^2+\alpha(y^2+W^2)+\beta y^2+(y^2+z^2)\Big) \\ &\quad + P(t)|\theta_3|\Big(y^2+\alpha(y^2+z^2)\Big) \\ &\leq \lambda_1(|\theta_1|+|\theta_2|+|\theta_3|)\left(y^2+z^2+W^2+G\left(x\right)\right) \\ &\leq 2\frac{\lambda_1}{\Gamma_0}\Big(|\theta_1|+|\theta_2|+|\theta_3|\Big)V, \end{aligned}$$

where $\lambda_1 = \max\left\{L_1(1+\alpha),R_1(1+2\alpha+\beta),P_1(1+\alpha)\right\}$. Using condition (H1) and Lemma 4.1, we obtain

$$g^2(x) \leq g_0 G(x),$$

consequently

$$\begin{aligned} |V_{10}| &\leq -Q'\left(t\right)\left[2\beta G\left(x\right)+\alpha g_0 y^2+\left(g^2\left(x\right)+y^2\right)+\alpha\left(g^2\left(x\right)+z^2\right)+\alpha\rho g^2(x)\right] \\ &\quad +|P'\left(t\right)|M\left[y^2+\alpha\left(y^2+z^2\right)\right]+|R'\left(t\right)|M\left[\alpha z^2+\beta y^2\right]+\alpha|\varphi'(t)|W^2 \\ &\quad +|L'\left(t\right)|M\left[z^2+2\beta\left(y^2+z^2\right)\right] \\ &\leq \lambda_2\left[|L'\left(t\right)|+|R'\left(t\right)|+|P'\left(t\right)|+|\varphi'(t)|-Q'\left(t\right)\right]\left(y^2+z^2+W^2+G\left(x\right)\right) \\ &\leq 2\frac{\lambda_2}{\Gamma_0}\left[|L'\left(t\right)|+|R'\left(t\right)|+|P'\left(t\right)|+|\varphi'(t)|-Q'\left(t\right)\right]V, \end{aligned}$$

such that $\lambda_2 = \max\left\{2\beta+(\alpha\rho+\alpha+2\beta+1)g_0+1,\alpha g_0+\alpha+2\beta+1\right\}$. By taking $\frac{1}{\eta} = \frac{1}{\Gamma_0}\max\left\{\lambda_1,\lambda_2\right\}$, we obtain

$$\dot{V}_{(4.17)} \leq -\Gamma_3(y^2+z^2+W^2)+\frac{1}{\eta}\gamma\left(t\right)V+\big(\beta y+z+\alpha W\big)e(t). \tag{4.27}$$

From (iv), (H3), ( 4.23 ), ( 4.22 ), ( 4.27 ) and the estimate $x \leq\mid x \mid\leq x^2+1$, we get

$$\begin{aligned}
\dot{U}_{(4.17)} &= \left(\dot{V}_{(4.17)} - \frac{1}{\eta}\gamma(t)V\right) e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \\
&\leq \Big(-\Gamma_3\left(y^2+z^2+W^2\right) + \big(\beta y + z + \alpha W\big)e(t)\Big) e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \qquad (4.28)\\
&\leq (\beta|y| + |z| + \alpha|W|)\,|e(t)| \\
&\leq \Gamma_4\,(|y|+|z|+|W|)\,|e(t)| \\
&\leq \Gamma_4\left(3+y^2+z^2+W^2\right)|e(t)| \\
&\leq \Gamma_4\left(3+\frac{1}{\Gamma_2}U\right)|e(t)| \\
&\leq 3\Gamma_4|e(t)| + \frac{\Gamma_4}{\Gamma_2}U|e(t)|, \qquad (4.29)
\end{aligned}$$

where $\Gamma_4 = \max\{\alpha,\beta,1\}$. Integrating (4.29) from 0 to $t$, and using the condition (H4) and the Gronwall inequality, we obtain

$$\begin{aligned}
U(t,x,y,z,W) &\leq U\big(0,x(0),y(0),z(0),W(0)\big) + 3\Gamma_4\eta_3 \\
&\quad + \frac{\Gamma_4}{\Gamma_2}\int_{t_1}^{t} U\big(s,x(s),y(s),z(s),W(s)\big)|e(s)|ds \\
&\leq \Big(U\big(0,x(0),y(0),z(0),W(0)\big) + 3\Gamma_4\eta_3\Big) e^{\frac{\Gamma_4}{\Gamma_2}\int_{t_1}^{t}|e(s)|ds} \\
&\leq \Big(U\big(0,x(0),y(0),z(0),W(0)\big) + 3\Gamma_4\eta_3\Big) e^{\frac{\Gamma_4}{\Gamma_2}\eta_3} = K_1 < \infty. \qquad (4.30)
\end{aligned}$$

In view of inequalities (4.30) and (4.24),

$$(x^2+y^2+z^2+W^2) \leq \frac{1}{\Gamma_2}U \leq K \qquad (4.31)$$

where $K = \dfrac{K_1}{\Gamma_2}$. Clearly (4.31) implies that

$$|x(t)| \leq \sqrt{K},\ |y(t)| \leq \sqrt{K},\ |z(t)| \leq \sqrt{K},\ |W(t)| \leq \sqrt{K} \quad \text{for all} \quad t \geq t_1.$$

Hence,

$$|x(t)| \leq \sqrt{K},\ |x'(t)| \leq \sqrt{K},\ |x''(t)| \leq \sqrt{K},\ |X'''(t)| \leq \sqrt{K} \quad \text{for all} \quad t \geq t_1. \qquad (4.32)$$

We prove the square integrability of solutions and their derivatives.

First from (4.26 ) we have

$$V_5 + V_6 + V_7 + V_8 \leq -2\Gamma_3 \left( y^2 + z^2 + w^2 \right)$$

then,

$$\dot{V}_{(4.17)} \leq -\Gamma_3 (y^2 + z^2 + w^2) + \frac{1}{\eta} \gamma(t) V + \big( \beta y + z + \alpha W \big) e(t).$$

From (iv), (H3), ( 4.23 ), ( 4.24 ) and ( 4.26 ) we obtain

$$\begin{aligned} \dot{U}_{(4.17)} &= \Big( \dot{V}_{(4.17)} - \frac{1}{\eta} \gamma(t) V \Big) e^{-\frac{1}{\eta} \int_{t_1}^{t} \gamma(s)\, ds} \\ &\leq \Big( -\Gamma_3 \left( y^2 + z^2 + w^2 \right) + \big( \beta y + z + \alpha W \big) e(t) \Big) e^{-\frac{1}{\eta} \int_{t_1}^{t} \gamma(s)\, ds}. \end{aligned} \tag{4.33}$$

Now, we define $F_t = F(t,x(t),y(t),z(t),w(t))$ by

$$F_t = U + \sigma \int_{t_1}^{t} \left( y^2(s) + z^2(s) + w^2(s) \right) ds,$$

where $\sigma > 0$. It is esay to see that $F_t$ is positive definite, since $W = W(t,x,y,z,w)$ is already positive definite. Using the estimate $e^{-\dfrac{\eta_1 + \eta_2}{\eta}} \leq e^{-\frac{1}{\eta} \int_{t_1}^{t} \gamma(s)\, ds} \leq 1$ and (4.33), we have

$$\begin{aligned} \dot{F}_{t(4.17)} &\leq -\Gamma_3 \Big( y^2(t) + z^2(t) + w^2(t) \Big) e^{-\frac{\eta_1+\eta_2}{\eta}} \\ &\quad + \Gamma_4 \Big( |y(t)| + |z(t)| + |W(t)| \Big) |e(t)| \\ &\quad + \sigma \Big( y^2(t) + z^2(t) + w^2(t) \Big). \end{aligned} \tag{4.34}$$

By choosing $\sigma = \Gamma_3 e^{-\dfrac{\eta_1 + \eta_2}{\eta}}$ we obtain

$$\begin{aligned} \dot{F}_{t(4.17)} &\leq \Gamma_4 \Big( 3 + y^2(t) + z^2(t) + W^2(t) \Big) |e(t)| \\ &\leq \Gamma_4 \Big( 3 + \frac{1}{\Gamma_2} U \Big) |e(t)| \\ &\leq 3\Gamma_4 |e(t)| + \frac{\Gamma_4}{\Gamma_2} F_t |e(t)|. \end{aligned} \tag{4.35}$$

Integrating the last inequality (4.35) from 0 to $t$, and again using the Gronwall inequality and the condition (H4), we get

$$\begin{aligned}F_t &\leq F_0 + 3\Gamma_4\eta_3 + \frac{\Gamma_4}{\Gamma_2}\int_{t_1}^{t} F_s|e(s)|ds \\ &\leq \Big(F_0 + 3\Gamma_4\eta_3\Big)e^{\frac{\Gamma_4}{\Gamma_2}\int_{t_1}^{t}|e(s)|ds} \\ &\leq \Big(F_0 + 3\Gamma_4\eta_3\Big)e^{\frac{\Gamma_4}{\Gamma_2}\eta_3} = K_2 < \infty. \end{aligned} \tag{4.36}$$

Therefore

$$\int_{t_1}^{\infty} y^2(s)ds < K_2 \quad , \quad \int_{t_1}^{\infty} z^2(s) < K_2 \text{ and } \int_{t_1}^{\infty} w^2(s)ds < K_2,$$

which implies that

$$\int_{t_1}^{\infty} x'^2(s)ds \leq K_2 \quad , \quad \int_{t_1}^{\infty} x''^2(s)ds \leq K_2 \quad , \quad \int_{t_1}^{\infty} x'''^2(s)ds \leq K_2. \tag{4.37}$$

Next, multiplying (4.16) by $x(t)$ and integrating by parts from $t_1$ to $t$, we obtain

$$\int_{t_1}^{t} Q(s)x(s)g(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + N_0, \tag{4.38}$$

where

$$\begin{aligned}I_1(t) &= \varphi(t)x'(t)X''(t) - \varphi(t)x(t)X'''(t) - \int_{t_1}^{t}\varphi'(s)x'(s)x''(s)ds - \rho\int_{t_1}^{t}\varphi'(s)x'(s)x''(s-r)ds \\ &\quad - \int_{t_1}^{t}\varphi(s)x''^2(s)ds - \rho\int_{t_1}^{t}\varphi(s)x''(s)x''(s-r)ds, \\ I_2(t) &= -L(t)\psi(x(t))x(t)x''(t) + \int_{t_1}^{t} L'(s)\psi(x(s))x(s)x''(s)ds + \int_{t_1}^{t} L(s)\psi(x(s))x'(s)x''(s)ds, \\ I_3(t) &= -R(t)\phi(x(t))x(t)x'(t) + \int_{t_1}^{t} R'(s)\phi(x(s))x(s)x'(s)ds + \int_{t_1}^{t} R(s)\phi(x(s))x'^2(s)ds, \\ I_4(t) &= -\frac{1}{2}P(t)f(x(t))x^2(t) + \frac{1}{2}\int_{t_1}^{t} P'(s)f(x(s))x^2(s)ds + \frac{1}{2}\int_{t_1}^{t} P(s)f'(x(s))x'(s)x^2(s)ds, \\ I_5(t) &= \int_{t_1}^{t} e(s)x(s)ds, \\ I_6(t) &= \int_{t_1}^{t} x(s)ds\int_{t_1}^{s}\Omega(s,u)x'(u)\,du \end{aligned}$$

and

$$\begin{aligned}N_0 &= \varphi(t_1)x(t_1)X'''(t_1) - \varphi(t_1)x'(t_1)X''(t_1) + L(t_1)\psi(x(t_1))x(t_1)x''(t_1) \\ &\quad + R(t_1)\phi(x(t_1))x(t_1)x'(t_1) + \frac{1}{2}P(t_1)f(x(t_1))x^2(t_1).\end{aligned}$$

From (4.32)and (4.37) and conditions (i), (ii), (iv), (H3), (H4) and (H5), we have

$$\begin{aligned}
I_1(t) &\leq (2+\rho)\varphi_1 K + (1+\rho)K\int_{t_1}^{t}|\varphi'(s)|ds + \left(1+\frac{1}{2}\rho\right)\varphi_1\int_{t_1}^{t}x''^2(s)ds \\
&\quad +\frac{1}{2}\rho\varphi_1\int_{t_1}^{t}x''^2(s-r)ds, \\
&\leq (2+\rho)\varphi_1 K + (1+\rho)K\int_{t_1}^{t}|\varphi'(s)|ds + \left(1+\frac{1}{2}\rho\right)\varphi_1\int_{t_1}^{t}x''^2(s)ds \\
&\quad +\frac{1}{2}\rho\varphi_1\int_{t_1-r}^{t_1}x''^2(s)ds + \frac{1}{2}\rho\varphi_1\int_{t_1}^{t-r}x''^2(s)ds, \\
I_2(t) &\leq L_1MK + MK\int_{t_1}^{t}|L'(s)|ds + L_1M\int_{t_1}^{t}x'(s)x''(s)ds, \\
&\leq L_1MK + \frac{1}{2}L_1M\int_{t_1}^{t}x'^2(s)ds + \frac{1}{2}L_1M\int_{t_1}^{t}x''^2(s)ds + MK\int_{t_1}^{t}|L'(s)|ds, \\
I_3(t) &\leq R_1\phi_1 K + \phi_1 K\int_{t_1}^{t}|R'(s)|ds + R_1\phi_1\int_{t_1}^{t}x'^2(s)ds, \\
I_4(t) &\leq \frac{1}{2}P_1MK + \frac{1}{2}MK\int_{t_1}^{t}|P'(s)|ds, + \frac{1}{2}P_1K^{\frac{3}{2}}\int_{t_1}^{t}|f'(s)|ds, \\
I_5(t) &\leq \sqrt{K}\int_{t_1}^{t}|e(s)|ds. \\
I_6(t) &\leq \left|\int_{t_1}^{t}\int_{0}^{s}\Omega(s,u)x'(u)\,x(s)duds\right| \leq k\int_{t_1}^{t}\int_{0}^{s}|\Omega(s,u)|\,dsdu
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{t\to+\infty} I_1(t) \leq (2+\rho)\varphi_1 K + (1+\rho)K\eta_1 + (1+\rho)\varphi_1 K_2 + \frac{1}{2}\rho\varphi_1 K r = N_1, \\
&\lim_{t\to+\infty} I_2(t) \leq L_1MK + L_1MK_2 + MK\eta_1 = N_2, \\
&\lim_{t\to+\infty} I_3(t) \leq R_1\phi_1 K + \phi_1 K\eta_1 + R_1\phi_1 K_2 = N_3, \\
&\lim_{t\to+\infty} I_4(t) \leq \frac{1}{2}P_1MK + \frac{1}{2}MK\eta_1 + \frac{1}{2}P_1K^{\frac{3}{2}}\eta_2 = N_4, \\
&\lim_{t\to+\infty} I_5(t) \leq \sqrt{K}\eta_3 = N_5 \quad \text{and} \quad \lim_{t\to+\infty} I_6(t) \leq k\int_{t_1}^{t}\int_{0}^{s}|\Omega(s,u)|\,dsdu \leq K\eta_4 = N_6.
\end{aligned}$$

Thus

$$\lim_{t\to+\infty}\left(I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6\,(t)\right) \leq \sum_{i=1}^{6}N_i < \infty. \tag{4.39}$$

Consequently, (4.38)and (4.39) and condition iii) give

$$\int_{t_1}^{\infty}x^2(s)ds \leq \frac{1}{Q_0\delta}\int_{t_1}^{\infty}Q(s)x(s)g(x(s))ds \leq \frac{1}{Q_0\delta}\sum_{i=0}^{6}N_i < \infty,$$

which completes the proof of the theorem. □

**Remark 4.1.** If $e(t) = 0$ and $\xi = 0$, similarly to the proof above, the inequality (4.26 ) becomes

$$V_5 + V_6 + V_7 + V_8 \leq -2\Gamma_3 \left(y^2 + z^2 + (|w| + \rho|w_t|)^2\right)$$

then,

$$\begin{aligned}\dot{V}_{(4.17)} \leq & -\Gamma_3(y^2 + z^2 + (|w| + \rho|w_t|)^2) \\ & + \frac{1}{\eta}\Big( |L'(t)| + |R'(t)| + |P'(t)| + |\varphi'(t)| - Q'(t) + |\theta_1| + |\theta_2| + |\theta_3|\Big)V. \quad (4.40)\end{aligned}$$

From (iv), (H3), ( 4.23 ), ( 4.24 ) and ( 4.40 )

$$\begin{aligned}\dot{U}_{(4.17)} &= \left(\dot{V}_{(4.17)} - \frac{1}{\eta}\gamma(t)V\right) e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \\ &\leq -\Gamma_3\left(y^2 + z^2 + (|w| + \rho|w_t|)^2\right) e^{-\frac{1}{\eta}\int_{t_1}^{t}\gamma(s)\,ds} \\ &\leq -\mu\left(y^2 + z^2 + (|w| + \rho|w_t|)^2\right) \leq -\mu\left(y^2 + z^2 + W^2\right),\end{aligned}$$

where $\mu = \Gamma_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. It can also be seen that the only solution of system (4.17) for which $\dot{U}_{(4.17)}(t,x,y,z,W) = 0$ is the solution $x = y = z = w = 0$. The above discussion guarantees that the trivial solution of equation (4.16) is asymptotically stable, and the same conclusion as in the proof of Theorem 4.3 can be drawn for square integrability of solutions of equation (4.16).

### 4.2.3 Example

We consider the following fourth order non-autonomous differential equation of neutral type

$$\begin{aligned}&\left(\frac{4e^{2t} + e^t + 2}{2e^t + 1}\left(x'''(t) + \frac{1}{500}x'''(t-r)\right)\right)' + \left(e^{-t}\sin t + 2\right)\left(\left(\frac{x + 4e^x + 4e^{-x}}{4\left(e^x + e^{-x}\right)}\right)x''\right)' \\ + &\left(\frac{\cos t + 4t^2 + 4}{1 + t^2}\right)\left(\left(\frac{\sin x + 6e^x + 6e^{-x}}{e^x + e^{-x}}\right)x'\right)' + \left(e^{-2t}\sin^3 t + 2\right)\left(\frac{x\cos x + 5x^4 + 5}{5\left(1 + x^4\right)}\right)x' \\ + &\left(\frac{1}{20\cosh t} + \frac{1 + 2\left(1 + t^2\right)}{20\left(1 + t^2\right)}\right)\left(\frac{x}{x^2 + 1}\right) = \frac{2\sin t}{3t^2 + 2} \qquad (4.41) \\ + &\frac{1}{550}\int_0^t \frac{\left(e^{-t^2}\cos t - e^{-t^2}\sin t\right)e^{-s^2}}{1 + s^2}x'(s)ds.\end{aligned}$$

by taking
$\psi(x) = \dfrac{x + 4e^{x} + 4e^{-x}}{4(e^{x} + e^{-x})}$ , $\phi(x) = \dfrac{\sin x + 3e^{x} + 3e^{-x}}{e^{x} + e^{-x}}$ , $f(x) = \dfrac{x\cos x + 5x^{4} + 5}{5(1 + x^{4})}$ ,
$g(x) = \dfrac{x}{x^{2} + 1}$ , $L(t) = e^{-t}\sin t + 2$ , $R(t) = \dfrac{\cos t + 4t^{2} + 4}{1 + t^{2}}$ , $P(t) = e^{-2t}\sin^{3} t + 2$ ,
$Q(t) = \dfrac{1}{20\cosh t} + \dfrac{1 + 2(1 + t^{2})}{20(1 + t^{2})}$ , $e(t) = \dfrac{2\sin t}{3t^{2} + 2}$, and $\varphi(t) = \dfrac{4e^{2t} + e^{t} + 2}{2e^{t} + 1}$,
$\Omega(t,s) = \frac{\left(e^{-t^{2}}\cos t - e^{-t^{2}}\sin t\right)e^{-s^{2}}}{1+s^{2}}$.

It follows easily that
$m = \dfrac{9}{10}$ , $M = \dfrac{11}{10}$ , $\phi_0 = \dfrac{5}{2}$ , $\phi_1 = \dfrac{7}{2}$ , $g_0 = \dfrac{11}{5}$ , $\delta_0 = \dfrac{3}{2}$ , $L_0 = 1$ , $L_1 = 3$ , $R_0 = 3$ ,
$R_1 = 5$, $P_0 = 1$ , $P_1 = 3$ , $Q_0 = \dfrac{1}{10}$ , $Q_1 = \dfrac{1}{5}$ , $\varphi_0 = \dfrac{2}{5}$ , $\varphi_1 = \dfrac{4}{5}$.
By taking $\varepsilon = \dfrac{5}{100}$, we find $\alpha = \dfrac{1}{L_0 m} + \varepsilon = \dfrac{209}{180}$, $\beta = \dfrac{Q_1 g_0}{P_0 m} + \varepsilon = \dfrac{97}{180}$ ,
$g_0 - \dfrac{L_0 m \delta_0}{Q_1} = -4.55 \leq g'(x) \leq \dfrac{g_0}{2} = 1.1$,
$\lambda_0 = \max\left\{\dfrac{g_0}{2M}, \left|\dfrac{g_0}{m} - \dfrac{L_0 m \delta_0}{Q_1}\right|\right\} = \dfrac{155}{36}$,
$R_0\phi_0 = \dfrac{15}{2} > \dfrac{6\,946\,7}{10000} = \dfrac{Q_1 g_0 L_1 M}{P_0 m} + \dfrac{P_1 M + \delta_0}{L_0 m} = \delta_1$,

$$\rho = \frac{1}{500} < \min\left\{\frac{2\varepsilon P_0 m}{\alpha P_1 M + \alpha Q_1 \lambda_0 + \left(2\alpha + 2\beta + 1\right)\eta_3},\right.$$
$$2\frac{R_0\phi_0 - \delta_1 - \varepsilon M(L_1 + P_1)}{\alpha R_1 \phi_1 + \beta + \alpha Q_1 \lambda_0 + \alpha Q_1 + \eta_3}, \frac{2\varepsilon}{2\alpha L_1 M + 5 + \alpha R_1 \phi_1 + \alpha P_1 M + \alpha Q_1 + \beta + 2\alpha\eta_3},$$
$$\left. 1, \frac{4\varepsilon}{\alpha g_0}\right\}$$
$$= 2.368 \times 10^{-3}.$$

and we have

$$\begin{aligned}\int_{-\infty}^{+\infty} |\psi'(x)|\,dx &= \frac{1}{4}\int_{-\infty}^{+\infty}\left|\frac{1}{e^{x} + e^{-x}} + x\frac{e^{-x} - e^{x}}{(e^{x} + e^{-x})^{2}}\right| dx \\ &\leq \frac{1}{4}\int_{-\infty}^{0}\left(\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{(e^{x} + e^{-x})^{2}}\right) dx \\ &\quad + \frac{1}{4}\int_{0}^{+\infty}\left(\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{(e^{x} + e^{-x})^{2}}\right) dx = \frac{\pi}{4},\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|\phi'(x)\right|dx &= \int_{-\infty}^{+\infty}\left|\frac{(e^{x}+e^{-x})\cos x-(e^{x}-e^{-x})\sin x}{(e^{x}+e^{-x})^{2}}\right|dx\\ &\leq \int_{-\infty}^{+\infty}\left(\frac{1}{e^{x}+e^{-x}}+\frac{x}{(e^{x}+e^{-x})^{2}}\left(e^{x}-e^{-x}\right)\right)dx=\pi,\text{ and}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|f'(x)\right|dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{(\cos x-x\sin x)(x^{4}+1)-4x^{4}\cos x}{(x^{4}+1)^{2}}\right|dx\\ &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{\cos x}{x^{4}+1}-4x^{4}\frac{\cos x}{(x^{4}+1)^{2}}-x\frac{\sin x}{x^{4}+1}\right|dx\\ &\leq \frac{1}{5}\int_{-\infty}^{+\infty}\left(\frac{5}{x^{4}+1}+\frac{x^{2}}{x^{4}+1}\right)dx=\frac{6}{5}\sqrt{2}\pi.\end{aligned}$$

Consequently

$$\int_{-\infty}^{+\infty}\left(\left|\psi'(s)\right|+\left|\phi'(s)\right|+\left|f'(s)\right|\right)ds<\infty.$$

A simple computation gives

$$\int_{t_1}^{+\infty}\left|e(t)\right|dt=\int_{t_1}^{+\infty}\left|\frac{2\sin t}{3t^{2}+2}\right|dt\leq\int_{0}^{+\infty}\frac{2}{2t^{2}+2}dt=\frac{\pi}{2},$$

$$\begin{aligned}\int_{t_1}^{+\infty}\left|L'(t)\right|dt &= \int_{t_1}^{+\infty}\left|(\cos t)e^{-t}-(\sin t)e^{-t}\right|dt\leq\int_{0}^{+\infty}2e^{-t}dt=2,\\ \int_{t_1}^{+\infty}\left|R'(t)\right|dt &= \int_{t_1}^{+\infty}\left|-\frac{\sin t}{t^{2}+1}-2t\frac{\cos t}{(t^{2}+1)^{2}}\right|dt\leq\int_{0}^{+\infty}\left(\frac{1}{t^{2}+1}+\frac{2\left|t\right|}{(t^{2}+1)^{2}}\right)dt\\ &\leq \int_{0}^{+\infty}\left(\frac{1}{t^{2}+1}+\frac{t^{2}+1}{(t^{2}+1)^{2}}\right)dt=\int_{0}^{+\infty}\frac{2}{t^{2}+1}dt=\pi,\\ \int_{t_1}^{+\infty}\left|P'(t)\right|dt &= \int_{t_1}^{+\infty}\left|3\left(\cos t\sin^{2}t\right)e^{-2t}-2\left(\sin^{3}t\right)e^{-2t}\right|dt\\ &\leq \int_{0}^{+\infty}5e^{-2t}dt=\frac{5}{2}\end{aligned}$$

and

$$\begin{aligned}\int_{t_1}^{+\infty}\left(-Q'(t)\right)dt &\leq \int_{0}^{+\infty}\left(\frac{t+\sinh t+t\cosh 2t+2t^{2}\sinh t+t^{4}\sinh t}{10\cosh 2t+20t^{2}\cosh 2t+10t^{4}\cosh 2t+20t^{2}+10t^{4}+10}\right)dt\\ &= \frac{1}{10},\\ \int_{t_1}^{+\infty}\left|\varphi'(t)\right|dt &\leq \int_{0}^{+\infty}\left|\frac{e^{t}}{2e^{2t}+1}-\frac{4e^{3t}}{(2e^{2t}+1)^{2}}\right|dt=\frac{1}{3}.\end{aligned}$$

Therefore

$$\int_{t_1}^{+\infty} \left( |L'(t)| + |R'(t)| + |P'(t)| - Q'(t) + |\varphi'(t)| \right) dt < +\infty.$$

By taking $\eta_3 = 2$ we obtain

$$\begin{aligned} \int_0^t |\Omega(t,s)|\, ds &= \int_0^t \left| \frac{\left(e^{-t^2}\cos t - e^{-t^2}\sin t\right) e^{-s^2}}{1+s^2} \right| ds \\ &\leq 2\int_0^t e^{-s^2} ds \leq 2\int_0^{+\infty} e^{-s^2} ds = \sqrt{\pi} < \eta_3, \end{aligned}$$

$$\begin{aligned} \int_t^{+\infty} |\Omega(u,t)|\, du &= \int_t^{+\infty} \left| \frac{\left(e^{-u^2}\cos u - e^{-u^2}\sin u\right) e^{-t^2}}{1+t^2} \right| du \\ &\leq \int_t^{+\infty} 2e^{-u^2} du \leq \int_0^{+\infty} 2e^{-u^2} du = \sqrt{\pi} < \eta_3 \end{aligned}$$

and

$$\begin{aligned} \int_{t_1}^t \int_0^s |\Omega(s,u)|\, duds &= \int_{t_1}^t \int_0^s \left| \frac{\left(e^{-s^2}\cos s - e^{-s^2}\sin s\right) e^{-u^2}}{1+u^2} \right| duds \\ &\leq \int_0^t \int_0^s 2e^{-u^2} e^{-s^2} duds \leq \int_0^{+\infty} \int_0^s 2e^{-u^2} e^{-s^2} duds = \frac{\pi}{4} < \eta_3. \end{aligned}$$

Thus all the assumptions of Theorem (4.3) hold, so solutions of (4.41) are bounded and square integrable.

# Conclusion

In this thesis, we investigated several qualitative properties of solutions for higher-order nonlinear neutral differential equations, with emphasis on asymptotic, uniform and exponential stability, boundedness, and square integrability. Using various analytical techniques chiefly Lyapunov's second (direct) method via the construction of candidate Lyapunov functionals, we derive new sufficient conditions that improve and extend many existing results in the literature.

The main contributions are as follows:

• For certain third-order and fourth-order neutral delay differential equations, we establish criteria ensuring asymptotic, uniform, and exponential stability, as well as boundedness and square integrability of solutions.

• For two classes of third and fourth-order neutral integro differential equations, we obtain new inequality-type conditions that guarantee stability, boundedness, and square integrability.

Several illustrative (and numerical) examples validate the theoretical results. These findings enrich the theory of neutral differential equations and provide a mathematical foundation for future studies of more complex systems with multiple delays and/or time-varying coefficients.

Possible directions for future work include the development of suctue-preseving numerical methods that retain the qualitative properties established here, extensions to distributed or state-dependent delays and to uncertain/robust settings, and investigations of stochastic neutral delay (integro-) differential equations together with feedback-control designs based on Lyapunov functionals

•Formulate and solve application-driven problems that can be modeled by neutral delay differential equtions, leveraging the criteria developed in this word (e.g. population dynamics, epidemic spread, viscoelasticity, networked control).

• Extend the obtained results to neutral differential equations of different orders and stuctures (including distributed or state-dependent delays, time-varying coefficients, and uncertain parameters).

Finally, we hope that the results presented here will serve as a useful reference for both theoretical investigations and practical applications involving neutral delay differential equations.

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**ملخص :**

في هذه الرسالة، يُركز البحث على الاستقرار المقارب، والاستقرار المنتظم، والاستقرار الأسّي، والحدودية، وقابلية التكامل التربيعي لحلول بعض المعادلات التفاضلية غير الخطية ذات التأخير من رتبة أعلى للأنواع المحايدة. وتُستخدم تقنيات تحليلية متنوعة، والطريقة الرئيسية المستخدمة في هذه الأعمال هي طريقة ليابونوف الثانية، القائمة على بناء مرشح معين لدالة ليابونوف، حيث تم استنباط شروط كافية جديدة تُحسّن وتُوسّع وتُعزز العديد من النتائج الواردة في الأعمال السابقة المنشورة في الأدبيات.

وأخيرًا، قُدّمت أمثلة توضيحية عديدة لإثبات موثوقية النتائج النظرية. تُثري هذه النتائج الفهم النظري للمعادلات التفاضلية المحايدة، وتُوفر أساسًا رياضيًا للدراسات المستقبلية على أنظمة أكثر تعقيدًا ذات تأخيرات متعددة أو معاملات متغيرة زمنيًا.

**الكلمات المفتاحية :** المعادلات التفاضلية المحايدة، الاستقرار، المحدودية، التكامل التربيعي.

**Résume :**

Dans cette thèse,  l'accent est mis sur la stabilité asymptotique, la stabilité uniforme, la stabilité exponentielle,  la bournitude et carré  intégrabilité  des solutions pour certaines équations différentielles à retard non linéaires d'ordre supérieur de types neutre. Diverses techniques analytiques sont employées, la principale étant la seconde méthode de Lyapunov, basée sur la construction de certains fonctionnelle de Lyapunov candidates. De nouvelles conditions suffisantes, améliorant et étendant de nombreux résultats déjà publiés, sont établies.

Enfin, plusieurs exemples illustratifs sont présenter pour démontrer la valité des résultats théoriques. Ces résultats enrichissent la compréhension théorique des équations différentielles neutres et fournissent une base mathématique pour de futures études sur des systèmes plus complexes à retards multiples ou à coefficients variables dans le temps.

**Les mots clés** : Equations différentielles neutrale, stabilité, bournitude, carrée integrabilité.

**Abstract :**

This thesis investigates  asymptotic, uniform, and exponential stability, as well as boundedness and square integrability of solutions to certain higher-order nonlinear neutral delay  differential equations. Serval analytical techniques are employed, chiefly Lyapunov’s second (direct) method, based on construction candidate of the Lyapunov functionals. We derive new sufficient conditions that improve, extend and enhance many of the results exist in previous works in literature are derived.

Finally, several illustrative examples were provided to demonstrate the reliability  of the theoretical results. These findings enrich the theoretical understanding of neutral differential equations and provide a mathematical foundation for future studies on more complex systems with multiple delays or time-varying coefficients.

**Key words** : Neutral differential equations, stability, boundedness, square integrability.