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MOUCHIR Samiha

Contributions to the Analysis of a class of Stochastic Differential Equations and Applications

The thesis defended publicly on July 15/07/2025 in front of the committee:

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Dedicate

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To

my beloved

parents,

The reason I am who I

am today. Thank you for your

unwavering support and care.

To my sisters, brothers, and friends,

You have been my inspiration and soulmates.

Thank you for your encouragement

throughout my research.

Thank you.

May Allah

bless you

all.

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SAMIHA

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In the name of Allah, the Most Gracious and the Most Merciful

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Mouchir Samiha.

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General Notations

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$		set of natural numbers,set of rational numbers, set of real numbers.
$\mathbb{R}^+$		the non-negative real numbers , $\mathbb{R}^+ = [0, \infty[$.
$\mathbb{R}_+^*$	$=$	$\mathbb{R}^+ \setminus \{0\}$.
$\mathbb{R}^n$	$=$	$\{(x_i) : \ x_i \in \mathbb{R}, \ \ i = 1, \cdots, n\}$.
$\mathbb{R}^{n \times m}$	$=$	$\{(x_{i,j}) : \ x_{i,j} \in \mathbb{R}, \ \ i = 1, \cdots, n, \ j = 1, \cdots, m\}$.
$(a, b), [a, b]$		open, closed interval from a to b.
$:=$		equal to by definition.
$\emptyset$		the empty set.
$\in$		belongs to.
$(A_k)_{k \in \mathbb{N}}$		be a sequence.
$\limsup\limits_k A_k$	$:=$	$\bigcap\limits_{k=1}^{\infty} \bigcup\limits_{i=k}^{\infty} A_i$.
$\liminf\limits_k A_k$	$:=$	$\bigcup\limits_{k=1}^{\infty} \bigcap\limits_{i=k}^{\infty} A_i$.
1_A		the indicator function of a set A.
$\Gamma(.)$		Gamma function.
$\beta(.,.)$		Beta function.
A^c		the complement of A in Ω.
(x, y)		the scalar product of vectors x and y.
$tr(A)$		the trace of a square matrix $A = (a_{ij})_{d \times d}$, i.e. $tr(A) = \sum_{1 \le i \le d} a_{ii}$.

$f^{-1}: E_2 \to E_1$		f^{-1} is the inverse mapping from set E_2 to set E_1.
$\mathcal{B}$		the Borel σ-algebra.
$(\Omega, \mathcal{F}, P)$		probability space.
$\{\mathcal{F}_t\}_{t\geq 0}$		filtration.
$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$		a filtered Probability space or complete probability space.
$\mathbb{E}[x]$		expected value of x.
$P(a \leq X \leq b)$		probability of the event $a \leq X \leq b$.
$x, y, z, \cdots$ or $X, Y, Z, \cdots$		denote random variables or stochastic processes.
$Z \sim \mathcal{N}(\mu, \sigma^2)$		has a normal distribution with mean μ and variance σ^2.
$\{W(t)\}_{t\geq 0}$		brownian motion or $\mathcal{Q}$-Wiener process.
$\int_0^t h(s)dW(s)$		Itô integral of $h(s)$ with respect to $W(s)$.
$\{M_t\}_{t\geq 0}$		martingale.
$\left\{M_t^+\right\}_{t\geq 0}$	$:=$	$\max(0, M_t)$.
$\left\{M_t^-\right\}_{t\geq 0}$	$:=$	$\max(-M_t, 0)$.
Δt		stepsize for a numerical method.
t_n		meshpoint$n\,\Delta t$.
X_n		numerical approximation to solution of SDE at time t_n.

Λ	the space of elementary processes.
$(\mathbb{X}, \|.\|_{\mathbb{X}}), (\mathbb{Y}, \|.\|_{\mathbb{Y}})$	are Banach spaces with their respective norms.
$(\mathbb{H}, \|.\|_{\mathbb{H}}), (\mathbb{V}, \|.\|_{\mathbb{V}})$,	are real separable Hilbert space.
$L_b(\mathbb{X}, \mathbb{Y})$, $L_b(\mathbb{Y})$	are the spaces of bounded linear operators.
$\mathcal{C}([0,T], \mathbb{X})$	is the space of continuous functions on $[0,T]$ with values in $\mathbb{X}$.
$\mathcal{C}^1\big([0,T], \mathbb{X}\big)$	is the space of continuously differentiable functions from $[0,T]$ into $\mathbb{X}$.
$L^1([0,T], \mathbb{H})$	is the space of Bochner-integrable functions with values in $\mathbb{H}$.
$L^2(\Omega, \mathbb{R})$	is the space of square integrable random variables X with $\mathbb{E}\|X\|^2 < \infty$.
$L^2(\Omega, \mathbb{H})$	is a Banach space equipped with norm $\|X\|_{L^2} = \left(\mathbb{E}\|X\|^2\right)^{\frac{1}{2}}$.
$L^2([0,T] \times \Omega)$	the family of $\mathbb{R}^n$-valued random variables X with $\int_0^t \mathbb{E}\|X(t)\|^2 dt < \infty$.
$\mathcal{L}^2([0,T], \mathbb{R})$	the family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{X(t)\}_{t\in[0,T]}$ with $\int_0^T \|X(t)\|^2 dt < \infty\, a.s.$
$\mathcal{M}^2([0,T], \mathbb{R})$	the family of processes $\{X(t)\}_{t\in[0,T]}$ in $\mathcal{L}^2([0,T], \mathbb{R})$ with $\mathbb{E}\left[\int_0^T \|X(t)\|^2 dt\right] < \infty$.

List of Abbreviations

a.s.	almost surely, or $\mathbb{P}$-almost surely.
a.e.	almost everywhere.
s.p.	stochastic process.
MNC	measures of non-compactness.
IDEs	integro-differential equations.
SDEs	stochastic differential equations.
SIDEs	stochastic integro-differential equations.
FBM	fractional Brownian motion.
NII	non-instantaneous impulse.
R-L	Riemann–Liouville.
FDEs	fractional differential equations.
FSDEs	fractional stochastic differential equations.
FSIDEs	fractional stochastic integro-differential equations.
SVIDEs	stochastic Volterra integro-differential equations.
UH stability	Ulam-Hyers Stability.
UHR stability	Ulam-Hyers-Rassias Stability.
EM method	Euler–Maruyama method.
θ-EM method	θ-Euler–Maruyama method.

Lists of Publications and Communications

Publications:

1. **S.Mouchir** & A.Slama. "Convergence of the θ-Euler-Maruyama method for a class of Stochastic Volterra Integro-Differential Equations." Journal of Numerical Analysis and Approximation Theory 53.2 (2024): 298-323.

2. **S.Mouchir** & A.Slama. "Exact Controllability Results of Non-Instantaneous Impulsive Stochastic Integro-differential Equations Driven by a Fractional Brownian Motion." Filomat 39.2(2025): 381-406.

3. **S.Mouchir** & A.Slama. "Existence, uniqueness and UHR stability of non instantaneous impulsive fractional stochastic integro-differential equations". Arabian Journal of Mathematics (2025): 1–26.

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Abstract

This thesis investigates a class of stochastic integro-differential equations (SIDEs) with non-instantaneous impulses (INIs) and non-local conditions, focusing on the existence, uniqueness, controllability and stability of the solutions. Key results include the demonstration of existence and controllability for a class of SIDEs governed by fractional Brownian motion, using a generalized Darbo fixed point theory. In addition, the thesis explores a class of fractional order stochastic integral-differential equations (FSIDEs). Existence, uniqueness and stability results using Krasnoselski fixed point theory and the Banach contraction theorem are proven. Numerical methods for the approximation of the solutions are also studied. The numerical methods EM and Θ-EM have been used for the approximation of solutions of the stochastic integro-differential Volterra equations (SVIDEs). The existence, uniqueness and Hölder continuity of the solutions for the SVIDEs, as well as the strong convergence of the Θ-EM method have been studied.

Keywords:

Stochastic integral-differential equation, Volterra integral-differential equation, fractional integral-differential equation, stochastic integral-differential equation, non-instantaneous impulsive, nonlocal conditions, Wiener process, Brownian motion, fractional Brownian motion, C_0-semi-group, resolvent operator, mail solution, measure of non-compactness, fixed point theory, existence, uniqueness, Exact controllability, UHR stability, strong convergence, Numerical methods.

MSC 2020: 60J65, 60H20, 93B05, 60H10, 37H30, 26A33, 65C30, 60B10, 65L20.

Résumé

L'objet de cette thèse est l'étude d'une classe d'équations intégro-différentielles stochastiques (EIDSs) avec impulses non instantanées et des conditions non locales, en se concentrant sur l'existence, l'unicité, la contrôlabilité et la stabilité des solutions. Les résultats clés incluent la démonstration de l'existence et de la contrôlabilité pour une classe d'EIDSs gouvernées par un mouvement brownien fractionnaire, en utilisant une théorie généralisée des points fixes de Darbo. De plus, la thèse explore une classe équations intégro-différentielles stochastiques d'ordre fractionnaire (EIDSFs). Des résultats d'existence, d'unicité et de stabilité en utilisant la théorie des points fixes de Krasnoselski et le théorème de contraction de Banach ont été démontrés. Des méthodes numériques pour l'approximation des solutions sont également étudiées. Les méthodes numériques EM et Θ-EM ont été utilisées pour l'approximation de solutions des équations intégro-différentielles stochastiques de Volterra (EIDSVs). L'existence, l'unicité et la continuité de Hölder des solutions pour les EIDSVs, ainsi que la convergence forte de la méthode Θ-EM ont été étudiés.

Mots-clés : Équation intégro-différentielle stochastique, équation intégro-différentielle de Volterra, équation intégo-différentielle d'ordre fractionnaire, équation intégro-différentielle stochastique, impulsive non instantanée, conditions non locales, processus de Wiener, mouvement brownien, mouvement brownien fractionnaire, semi-groupe C_0, opérateur résolvant, solution mail, mesure de non-compacité, théorie des points fixes, existence, unicité, contrôlabilité exacte, stabilité UHR, convergence forte, méthodes numériques.

MSC 2020: 60J65, 60H20, 93B05, 60H10, 37H30, 26A33, 65C30, 60B10, 65L20.

الملخص

تتناول هذه الرسالة فئة من المعادلات التكاملية التفاضلية العشوائية ذات نبضات غير فورية وشروط غير المحلية، مع التركيز على وجود الحلول و وحدانيتها، وإمكانية التحكم فيها،واستقرارها. من النتائج الرئيسة إثبات وجود إمكانية التحكم لفئة من معادلات التكاملية التفاضلية المقادة بواسطة حركة براونية كسرية، وذلك باستخدام النظرية العامة لنقاط الثبات لداربو.

كما تستعرض الرسالة فئة من المعادلات التكاملية التفاضلية العشوائية ذات الرتب الكسرية، حيث يتم إثبات وجود الحلول وتفّردها واستقرارها باستخدام نظرية كراسنوستلزكي لنقاط الثبات ونظرية التقلص ِلبانخ.

وقد تم أيًضا دراسة الطرق العددية لتقريب الحلول، حيث استُخدمت طريقتا أويلر–ماروياما وطريقة– θ أويلر–ماروياما لتقريب حلول معادلات فولتيرا التكاملية التفاضلية العشوائية. وتمت دراسة وجود ووحدانية واستمرارية هولدر لحلول هذه المعادلات، بالإضافة إلى التقارب القوي لطريقة– θ أويلر–ماروياما.

الكلمات المفتاحية:

معادلة تكاملية تفاضلية عشوائية، معادلة فولتيرا التكاملية التفاضلية، معادلة تكاملية تفاضلية كسرية، معادلة تكاملية تفاضلية عشوائية، نبضات غير فورية، شروط غير محلية، عملية فينر، الحركة البراونية، الحركة البراونية الكسرية، شبه زمرة C_0، مؤثر الحل، الحل الرئيسي، مقياس اللاتراص، نظرية نقطة الثبات، الوجود، التفرد، إمكانية التحكم الدقيقة، استقرار أولام-هايرس-راسياس، التقارب القوي، الطرق العددية.

تصنيف الرياضيات الموضوعي:(MSC 2020)

.60J65, 60H20, 93B05, 60H10, 37H30, 26A33, 65C30, 60B10, 65L20

General introduction

Stochastic calculus is a branch of mathematics extending traditional calculus to systems affected by randomness. It was developed in the middle of the 20^{th} century by Andrey Kolmogorov [87] and Kiyoshi Itô, with its primary focus on random processes like Brownian motion. Over time, stochastic calculus has evolved into an essential tool for understanding and modeling systems affected by uncertainty, leading to significant advancements in both theoretical and applied mathematics. It has become essential in fields such as finance, physics, engineering, and biology (see [9, 11, 29, 56, 88, 93, 160, 182]).

1 Stochastic differential equations

Introduced in the 1950s, stochastic differential equations (SDEs) describe the evolution of systems under both deterministic forces and random noise. A typical form of an SDE is

$$dx(t) = \mu(t, x(t))dt + \sigma(t, x(t))dW(t),$$

where μ represents the deterministic drift, σ represents the randomness, and $W(t)$ denotes a Wiener process. In this context, several researchers have examined various qualitative aspects of these equations, particularly those driven by fractional Brownian motion (FBM) which continues to be a focal point of interest within academic circles (see [25, 36, 76, 98, 136, 149]).The capacity of stochastic differential equations to simulate complex natural phenomena makes them an effective tool for understanding dynamic behaviors across various systems. By employing quantitative techniques and mathematical concepts, scientists can develop models that help explain the random behaviors observed in diverse fields, including biology and physics, particularly in the Black-Scholes model for option pricing, in physics to model diffusion processes, Brownian motion that describes the random, erratic motion of particles suspended in a fluid and in biology for modeling gene expression or population dynamics (see [124, 125, 126, 138, 160, 180, 185]).

2 Stochastic Volterra integral equations

The Volterra integral equations (VIEs) were proposed by Vito Volterra, which Traian Lalescu later studied in his 1908 thesis "Sur les équations de Volterra", written under the direction of Émile Picard. Volterra integral equations find application in viscoelastic materials, fluid mechanics, and demography (see, [38, 91, 94, 107, 183]). Stochastic Volterra integral equations (SVIEs) are an extension of ordinary Volterra integral equations to include random noise, making them suitable for modeling systems with stochastic components. SVIEs find applications in various fields, including mathematical finance, biology, physics and engineering. For example, in finance, SVIEs are used to model the evolution of financial asset prices over time, taking into account the stochastic nature of market movements. In biology, they can be used to describe population dynamics subject to random environmental factors. Therefore, in recent years, SVIEs have attracted the attention of many researchers. For instance (see [91, 94, 107, 183]. The exact methods for solving stochastic differential equations (SDEs) involve addressing current challenges in the field (cf. [38, 44, 74, 91, 119, 170, 178, 183]). While there are many analytical methods available, the complexity of these equations makes it difficult to obtain exact solutions. Among the numerical methods are the Milstein method, Runge-Kutta method see [17], Euler-Maruyama method, stochastic theta method, and others (see [107, 108, 145, 183]). Zong *et al.* [191] conducted a study on two classes of theta-Milstein schemes for stochastic differential equations in terms of convergence and stability.

3 Fractional Differential Equations

Fractional calculus are a significant branch of applied mathematics that involves integrals and derivatives of arbitrary order. One of the key advantages of fractional calculus is that its fractional derivatives are well-suited for simulating the memory and inherited properties of various materials and processes.This powerful tool is particularly effective in characterizing long-memory processes, long-range interactions, and allometric scaling laws. Fractional differential equations have significant applications in several domains including physics, engineering, finance, and biology to study problems such as anomalous diffusion, control theory, and viscoelasticity (see [7, 8, 10, 14, 37, 69, 70, 85, 131, 141, 153, 155, 162]).

Fractional stochastic differential equations have been receiving increasing attention in recent years in both theoretical and applied aspects of several fields of science and engineering. For the general theory of stochastic differential equations, one can refer to the monograph [107] and for recent developments in the literature (see [18, 75, 105, 148, 161, 188] and their references). Guendouzi and Mehdi [65], Wang et al. [172], Song and Luo [152], Tian and Luo [161], and Ben Makhlouf et al. [20] explored a class of Caputo FSDEs. Guo et al. [66], El-Hady et al. [67], and Arioui [13] discovered a class of Riemann-Liouville FDEs. Blouhi et al. [23] discussed a new class of impulsive Hilfer fractional stochastic equations. Ben Makhlouf et al. [21] examined Hadamard FSDEs.

4 Non-Instantaneous Impulses

Many real-life phenomena and processes which evolve with respect to time are characterized by abrupt changes in the form of impulses. Based on the duration of these changes, there are two common types of impulses:

- Instantaneous Impulses: when the duration of the change is relatively short compared to the overall duration of the entire process. (see [5, 90, 116]).

  Let an increasing sequence of points $(t_i)_{i=1}^{n+1}$ be given such that $0 \leq t_i < t_{i+1}$, with $\lim_{k\to\infty} t_k = \infty$. Let $t_0 \in \mathbb{R}^n$ be an arbitrary point, and assume without loss of generality that $t_0 \in [0, t_1]$. Consider the nonlinear instantaneous impulsive differential equation.

$$\begin{cases} dx(t) = f(t, x(t))dt, & \text{for } t \neq t_k,\ k = 0, 1, 2, \cdots, \\ x(t_{k+1}) = x(t_k) + I_k\left(x(t_k)\right), & \text{for },\ k = 1, 2, \cdots, \\ x(t_0) = x_0. \end{cases}$$

The points t_k, $k = 1, 2, \cdots$, are called points of instantaneous impulses and the functions I_k, $k = 1, 2, \cdots$, are called instantaneous impulsive functions.

- Non-instantaneous impulses: When changes start at an arbitrary fixed point and remain active on a finite time interval (see [5, 72]).

  Let two increasing sequences of points $(t_i)_{i=1}^{\infty}$ and $(s_i)_{i=0}^{\infty}$ be given such that $0 < s_0 < t_1 \cdots < t_i \leq s_i < t_{i+1}$, with $\lim_{k\to\infty} t_k = \infty$. Let $t_0 \in [0, s_0) \cup \left(\bigcup\limits_{k=1}^{\infty} [t_k, s_k)\right)$ be an arbitrary point, and assume without loss of generality that $t_0 \in [0, s_0)$. Consider the nonlinear non-instantaneous impulsive differential equation.

$$\begin{cases} dx(t) = f(t, x(t))dt, & \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \cdots, \\ x(t) = \phi_k\left(t, x(s_k)\right), & \text{for } t \in (t_k, s_k],\ k = 1, 2, \cdots, \\ x(t_0) = x_0. \end{cases}$$

  The intervals $(t_k, s_k]$, for $k = 1, 2, \cdots$, are called the intervals of non-instantaneous impulses, and the functions $\phi_k(t, x)$, for $k = 1, 2, \cdots$, are referred to as the non-instantaneous impulsive functions.

Thus, impulsive differential equations has attracted considerable attention due to its numerous applications in fields including biology, ecology, electrical engineering, and mechanics (see ,[3, 12, 13, 23, 24, 65, 66, 68, 95, 109, 161]). Impulsive differential equations provide an effective model for studying the evolution of processes that are prone to sudden and abrupt changes in their states, making them a valuable tool in various scientific and engineering disciplines.

However, the action of instantaneous impulsive phenomena seen as do not describe some certain dynamics of evolution processes in pharmacotherapy [135]. A well-known application of non-instantaneous impulses is the process of inducing a vaccine and absorption of the drug by the body. The resulting absorption is gradual because it remains active for a finite time interval [95]. This process can be modelled mathematically by non-instantaneous impulsive differential and integro-differential equations. Recently, many authors have established results on non-instantaneous impulse differential equations (see [95, 98, 125]) and references therein.

5 Nonlocal Conditions

The nonlocal condition problem is a generalization of the classical condition problem, motivated by physical issues. The pioneering work on nonlocal conditions is attributed to Lakshmikantham and Byszewski [89]. The main rationale behind their creation is the anticipation that the application of these field theories will provide a more elegant and effective way to tackle problems in particle and high-energy physics, as has been achieved with local field theories up to now. Both space and time may experience nonlocal effects. For instance, the extension from local to nonlocal descriptions appears in the time domain as a memory effect, which essentially indicates that past events, as well as the system's current state, impact an object's actual behavior. As a result, stochastic differential equations with nonlocal conditions have been studied by numerous researchers, and several fundamental results on nonlocal problems have been established (see [13, 95, 102, 109, 148, 184] and the references contained therein).

6 Controllability of dynamic systems

Controllability is a key concept in modern control theory, referring to the ability to guide a dynamic system from an initial state to a desired final state through specific control inputs. The study of control theory began systematically in the 1960s, and since then, it has become a crucial area of research in engineering and applied mathematics. Controllability problems arise in various fields, includingmechanics, electricity (See [47, 103, 167]), electronics (See [34, 79, 168, 174, 177]), biology (See [61, 117, 139, 146, 157]), , and more. In many systems, control influences only a subset of the system's state variables, rather than the entire state of the system.

In many dynamic systems, control does not affect the entire system state but is limited to a specific part of it. This is also true in real-world industrial processes, where only part of the system's state may be observable. For instance, controllability can be observed in aircraft, where control involves processes like takeoff, landing, directing the aircraft, speed adjustment, and other operations. Controllability here is demonstrated by the ability to steer the aircraft from one state to another using appropriate control inputs, as illustrated in references [101, 115, 132].

From a mathematical viewpoint, a control system is a dynamic system dependent on a control parameter. These systems are typically expressed as a differential equation:

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \quad \text{for all } t \geq 0, \tag{1}$$

where x is the state of the system and u is the control. For finite-dimensional linear control systems, Kalman [81] provided a straightforward characterization of exact controllability. However, for nonlinear systems, the problem of controllability is significantly more complex. Several researchers have extended the concept of exact controllability to systems with unbounded operators in infinite dimensional Banach spaces [83, 151].

Let us consider a semilinear control system given by the form:

$$x'(t) = Ax(t) + Bu(t) + F(x(t)), \quad t \in [0, T], \tag{2}$$

where $x(t) \in \mathbb{X}$, $u(t) \in \mathbb{U}$, and $\mathbb{X}$ and $\mathbb{U}$ are Banach spaces. The operator $A : \mathbb{X} \to D(A) \to \mathbb{X}$ is a linear, generally unbounded operator that generates a strongly continuous semigroup of linear bounded operators $\mathcal{T} : \mathbb{X} \to \mathbb{X}$ for $t > 0$, and $B : \mathbb{U} \to \mathbb{X}$ is a linear bounded operator. The nonlinear operator $F : \mathbb{X} \to \mathbb{X}$ represents the nonlinear dynamics of the system. For the starting condition $x(0) = x_0$, a mild solution to the problem (2) satisfies the following nonlinear integral equation:

$$x(t) = \int_0^t \mathcal{T}(t-s)Bu(s)ds + \int_0^t \mathcal{T}(t-s)F(x(s))ds. \tag{3}$$

Controllability results for the semilinear system (2) depend heavily on the controllability properties of the linear part of the system given by:

$$x'(t) = Ax(t) + Bu(t). \tag{4}$$

Now, let us define the controllability operator for the linear system (4):

$$\mathcal{C} : L^p([0,T], \mathbb{U})$$

by

$$\mathcal{C}u = \int_0^t \mathcal{T}(T-s)Bu(s)ds. \tag{5}$$

The system (4) is said to be exactly controllable to a linear subspace $\mathbb{V} \subset \mathbb{X}$ on $[0,T]$ if $\mathbb{V} \subset \text{Range}(\mathcal{C})$.

If the linear system (4) is exactly controllable to the subspace $\mathbb{V}$, then we can assume, without loss of generality, that $\text{Range}(\mathcal{C}) = \mathbb{V}$. We can then construct an invertible operator $\mathcal{C}^{-1}$ defined on $\mathcal{C} : L^p([0,T], \mathbb{U})/\ker \mathcal{C}$, such that $\mathcal{C}^{-1}$ maps the target $x_1 \in \mathbb{V}$ to a suitable control function.

Now, consider the control law:

$$u(t) = \mathcal{C}^{-1}\left(x_1 - \int_0^T \mathcal{T}(T-s)F(x(s))ds\right)(t).$$

The control $u(t)$ given by this equation steers the semilinear control system (2) from the zero initial state to the state $x_1 \in \mathbb{V}$ in time T.

In [134], Quinn and Carmichael showed that a controllability problem can be transformed into a fixed point problem. Using this method, several authors have explored this subject, for example, one can consult [4, 51, 71, 76, 98, 125, 136, 149, 164, 186] and the references therein.

7 Stability of functional equations

The Ulam-Hyers stability is a concept that arises in the study of functional equations, particularly in the context of stability theory. It is named after the mathematicians S.M. Ulam and D.H. Hyers, who introduced it in the 1940s. S. M. Ulam gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin , where he discussed a variety of significant unsolved problems. Among those was the following question concerning the stability of homomorphisms [165]:

Let G_1 be a group and G_2 a metric group with a metric $d(.,.)$. For any $\varepsilon > 0$, there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta, \quad \text{for all} \quad x, y \in G_1,$$

then there exists a homomorphism $g : G_1 \to G_2$ such that

$$d(f(x), g(x)) < \varepsilon \quad \text{for all} \quad x \in G_1?$$

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

D. H. Hyers was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces [78].

In 1978, Th. M. Rassias addressed the Hyers's stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference

$$f(x+y) - f(x) - f(y),$$

and proved a considerably generalized result of Hyers by making use of a direct method.

HU stability, although often associated with dynamical systems, finds applications across various fields of science and mathematics, such as biology, economics, and nonlinear analysis (see [54, 117, 175]).

8 Description and Contributions of the Thesis

The aim of this thesis is to study the solutions of stochastic integro-differential equations and to prove some properties of these systems. Our results are based upon very recent fixed point theorems, semigroup theory, stochastic calculus, and numerical analysis. We consider the following problem:

In Chapter 1, we introduce the notations and some preliminary concepts. The first sections, 1.1, are devoted to notations and some fundamental definitions of functional analysis, measures of non-compactness, fixed point theorems, and properties of solutions. In the second Section 3, some properties of fractional derivatives and differential equations are introduced. In Section 1.3 is devoted to a brief presentation of the tools of stochastic analysis with respect to Wiener processes and fractional Brownian motion. Finally, Sections 1.4 are devoted to the introduction of some basic numerical methods.

In Chapter 2, by employing the resolvent operator and a generalized Darbo's fixed point theorem, we prove the existence of mild solutions and controlabillty to a class of SIDEs with NII driven by fractional Brownian motion with the Hurst index $H > 1/2$ in a Hilbert space. The system is defined by:

$$\begin{cases} dy(t) = Ay(t)dt + \int_0^t h(t-s)y(t)dsdt + \varphi(t, y(t))dt + F(t)dW^H(t), \quad t \in \bigcup\limits_{k=0}^{N} (s_k, t_{k+1}], \\ y(t) = g_k(t, y(t_k^-)), \quad t \in \bigcup\limits_{k=1}^{N} (t_k, s_k], \\ y(0) + \psi(y) = y_0 \in \mathbb{H}. \end{cases}$$

Where $A : \mathbb{D}(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of bounded linear operators on a separable Hilbert space $\mathbb{H}$, Let $h : \mathbb{D}(h) \to \mathbb{H}$ be a closed linear operator on $\mathbb{H}$, with domain $\mathbb{D}(A) \subset \mathbb{D}(h)$, φ, ψ and $F : [0, T] \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ are appropriate functions, where $\mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ denotes the space of Hilbert-Schmidt operators from $\mathbb{V}$ into $\mathbb{H}$.

Let $0 = s_0 = t_0 < t_1 \leq s_1 < t_2 \leq ... < t_N \leq s_N < t_{N+1} = T$, where $T > 0$,g_k : $(t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is define NII function, for each $k = 1, 2, ..., N$. The state $y(.)$ takes values in $\mathbb{H}$.The initial value y_0 is an $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\|y_0\|^2 < \infty$.

In Chapter 3, we study the existence, uniqueness, UH, and UHR stability of mild solutions for the following class of fractional stochastic integro-differential equations with non-instantaneous impulsive and nonlocal conditions using contraction mapping theory and Krasnoselskii's fixed point theorem :

$$\begin{cases} {}_0^cD_t^\beta y(t) &= Ay(t) + \hbar\Big(t, y(t), K_1y(t)\Big) + \aleph\Big(t, y(t), K_2y(t)\Big)\dfrac{dW(t)}{dt}, \quad t \in \bigcup\limits_{k=0}^{m} (s_k, t_{k+1}], \\ y(t) &= H_k(t, y(t_k^-)), \quad t \in \bigcup\limits_{k=1}^{m} (t_k, s_k], \\ y(0) + \gamma(y) &= y_0 \in \mathbb{H}, \end{cases}$$

where ${}^{c}_{0}D^{\beta}_{t}$ is the Caputo fractional derivative of order β, $\beta \in (\frac{1}{2}, 1)$, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of bounded linear operators on a separable Hilbert space $\mathbb{H}$. Let $J = [0, T]$, $T > 0$ is a constant, $\hbar : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, $\aleph : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ and $\gamma : \mathcal{PC}\big(J, \mathbb{H}\big) \to \mathbb{H}$ are given functions satisfying certain assumptions $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq \cdots < t_m \leq s_m < t_{m+1} = T$, $H_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ are non-instantaneous impulsive functions for all $k = 1, 2, \cdots, m$, the terms $K_i y(t)$ is given by $K_i y(t) = \int_0^t \wp_i(t,s) y(s) ds$, where $\wp_i \in \mathcal{C}\big(D, \mathbb{R}^+\big)$, for $i = 1, 2$ are the set of all positive continuous functions on $D := \Big\{(t, s) : 0 \leq s < t \leq T\Big\}$, and y_0 is an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|y_0\|^2 < \infty$.

In Chapter 4, we investigate the strong convergence of the θ-Euler-Maruyama method for a class of stochastic Volterra integro-differential equations (SVIDEs), as presented in the following:

$$\begin{cases} dY(t) = f\Big(Y(t), \int_0^t \sigma_1(t,s) Y(s) dW(s)\Big) dt \\ \qquad\qquad\qquad + g\Big(Y(t), \int_0^t \sigma_2(t,s) Y(s) ds\Big) dW(t), \quad t \in [0, T], \\ Y(0) = Y_0 \in \mathbb{R} \end{cases}$$

with the initial condition Y_0, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions. The kernels $\sigma_i : D \to \mathbb{R}$ are continuous on $D := \{(t, s) : 0 \leq s \leq t \leq T\}$ with the norm $\|\sigma_i\|_\infty = \max_{(t,s)\in D} |\sigma_i(t, s)|$ for $i = 1, 2$. $Y(t)$ is a stochastic processes defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $B(t)$ is a standard Brownian motion defined on the same probability space. Additionally, we assume that $\mathbb{E}\|Y_0\|^2 < \infty$.

Chapter 1

Preliminaries

This chapter provides an overview of classic results in functional analysis, fractional calculus, stochastic analysis, and numerical methods for stochastic differential equations, which are essential for the subsequent developments of the thesis. The fundamentals of functional analysis, fractional calculus, stochastic analysis, and numerical methods for stochastic differential equations are discussed. Classical results in functional analysis, as cited in [6, 15, 16, 30, 33, 63, 71, 142], are recalled, including fixed point theorems, non-compact measure theory, generalities on semigroups, and resolvent operators. This is followed by the presentation of mild solutions to the Cauchy problem, along with a verification of controllability and stability properties.

In fractional calculus, as referred to in [8, 20, 37, 85, 87, 95, 131, 140, 144, 148], the fractional derivatives of Riemann-Liouville and the Riemann-Liouville and Caputo integrals are introduced, along with a class of fractional stochastic differential equations. In stochastic analysis, we cite [9, 28, 29, 86, 110, 182], exploring concepts of stochastic processes, stochastic integrals, and stochastic differential equations. Finally, in the context of numerical methods, we mention [17, 77, 108, 158, 169, 170, 171, 178, 180, 183], where some classical numerical techniques are discussed, and their convergence properties are examined in relation to stochastic differential equations.

1.1 Some Elements of Functional Analysis

In this section, we will introduce some fundamental concepts of functional analysis that are essential for the development of this thesis. These concepts provide the theoretical foundation upon which the rest of the work is built. We begin with the basic properties of measures of non-compactness, which are critical in understanding the structure of bounded sets in Banach spaces [15, 16] . Next, we introduce fixed point theory, which provides a powerful set of tools for analyzing the existence and behavior of solutions to equations in functional spaces [60, 62, 121, 143, 156]. Finally, we present the classical Cauchy problem and explore some quantitative and qualitative properties of its solutions [43, 107, 122, 129].

The concept of the measure of noncompactness emerged as a response to the limitations of classical compactness, particularly in infinite-dimensional spaces, where compactness does not always provide the necessary tools for analysis. As early as the 1920s and 1930s, mathematicians began investigating compactness in infinite-dimensional spaces. However, it wasn't until the 1950s that the measure of noncompactness was formally introduced. In 1953, the Russian mathematician Krasnoselskii made the first systematic development of this measure in the context of fixed point theorems for Banach spaces. The introduction of the measure allowed for more refined analysis of operator behavior, especially concerning nonlinear operators. This concept was later expanded by Darbo in 1955, who applied the measure to prove the existence of fixed points for certain classes of condensing operators. Since then, the measure of noncompactness has become an essential tool for extending classical theorems, such as Schauder's fixed point theorem and Banach's contraction principle, to more general settings. This development has significantly enriched the theory of fixed points.

1.1.1 Some fixed point theorems

In this section, we present several fixed point theorems, some of which will be referenced in later discussions. For further details, refer to [6, 43, 58, 60, 62, 99, 125, 128, 156].

The fixed point theory is an effective tool for solving nonlinear problems (such as differential equations, integral differential equations, functional equations, and stochastic equations...). The origins of this theory can be traced back to the works of Brouwer (1912) and Banach (1922). Cauchy (1844) was the first mathematician to prove the existence and uniqueness of solutions for differential equations of the form $\dfrac{dx}{dt} = f(t,x)$, $x(t_0) = x_0$. Later, Lipschitz (1977) extended Banach's proof by introducing the Lipschitz condition. Since then, this theory has become a thriving field, with many researchers contributing numerous research papers on the subject. The development of this theory has been closely linked to the advancement of functional analysis in the 1950s. In (1953) Karsnoselskii introduced a new fixed point theory that combines the fixed point theory of Schauder and Banach's contraction principle. In addition to this work, the Italian mathematician Darbo (1955) introduced a result that ensures the existence of a fixed point for a certain class of condensing operators,using a measure of non-compactness. This result has been the subject of numerous applications in both linear and nonlinear analysis. contexts.

We begin by introducing the concept of a fixed point.

Definition 1.1. Let X be any space and f a map from X to X. A point $x \in X$ is called a fixed point if $f(x) = x$.

Theorem 1.1. *[150] (Banach's fixed point theorem (1922)) Let C be a nonempty closed subset of a Banach space $\mathbb{X}$, then any contraction mapping f from C into itself has a unique fixed point.*

Theorem 1.2. *[62](Schauder's fixed point theorem (1930)) Let $\mathbb{X}$ be a Banach space, C be a bounded closed convex subset of $\mathbb{X}$ and $f : C \to C$ be a compact and continuous map. Then f has at least one fixed point in C.*

Theorem 1.3. *[62](Schaefer's fixed point theorem (1955)) Let $\mathbb{X}$ be a Banach space, and $f : \mathbb{X} \to \mathbb{X}$ completely continuous operator. If the set $E = \big\{x = \lambda f(x), \quad \text{for some } \lambda \in (0,1)\big\}$ is bounded, then f has fixed points.*

Krasnoselskii proved the following theorem, which combines the contraction principle of Banach with the Schauder theorem.

Theorem 1.4. *[60] (Krasnoselskii's fixed point theorem (1953)) Let C be a bounded closed and convex subset in a Banach space $\mathbb{X}$, let, and let f_1, f_2 be maps of C into $\mathbb{X}$ such that* *(i)* $f_1x + f_2x \in C$ *for all* $x \in C$,

(ii) f_1 *is a contraction,*

(iii) f_2 *is completely continuous.*

Then the equation $f_1x + f_2x = x$ *has a solution on* C.

Definition 1.2. [43] A continuous map $\Theta : D \subset \mathbb{X} \to \mathbb{X}$ is called an α-contraction if there exists a constant $K \in [0, 1)$ such that for every bounded set $\Omega \subset D$

$$\alpha(\Theta(\Omega)) \leq K\alpha(\Omega).$$

Theorem 1.5. *[62](Darbo's Fixed Point Theorem (1955)) Let* $\mathbb{X}$ *be a Banach space and* C *be a bounded, closed, convex and nonempty subset of* $\mathbb{X}$. *Suppose a continuous mapping* $f : C \to C$ *is such that for all closed subsets D of C,*

$$\alpha\big(f(D)\big) \leq K\alpha(D), \tag{1.1}$$

where $0 \leq K < 1$, *and* α *is the Kuratowski measure of noncompactness. Then f has a fixed point in C.*

Remark 1.1. Mappings satisfying the condition (1.1) have subsequently been called K-set contractions.

Theorem 1.6. *[99, 156] (Generalized Darbo's fixed point theorem) Let* C *be a closed and convex subset of a real Banach space* $\mathbb{X}$. *Suppose that* $Q : C \to C$ *is a continuous operator and* $Q(C)$ *is bounded, for any bounded subset* $D \subset C$

$$Q^1(D) = Q(D), \quad Q^n(D) = Q\big(\overline{co}(Q^{n-1}(D))\big), \ n = 2, 3 \cdots N.$$

If there exists a constant $0 \leq \lambda < 1$, *and a positive integer* n_0 *such that for any bounded subset* $D \subset C$.

$$\alpha\big(Q^{n_0}(D)\big) \leq \lambda\alpha\big(Q(D)\big). \tag{1.2}$$

Then Q has at least one fixed point in D.

Theorem 1.7. *[111](Mönch's fixed point principle) Let C be a closed, convex and bounded subset of a Banach Space $\mathbb{X}$, such that $C \subset \mathbb{X}$ and $0 \in C$. Assume that $\digamma : C \to C$ is a continuous map which satisfies the condition, for every subset $G \subset C$*

$$G = \overline{co}\big(\{0\} \cup \digamma(G)\big) \quad \Longrightarrow \overline{G} \quad \text{is compact.}$$

Then $\digamma$ has at least one fixed point in $\mathbb{X}$.

In 1967, Sadovskii [143] defined a new fixed point principle as follows

Theorem 1.8. *[143] (Sadovskii's fixed point principle) Let C be a closed, convex and bounded subset of a Banach Space $\mathbb{X}$, such that $C \subset C$. If $\digamma : C \to C$ is a α-condensing map.*

Then $\digamma$ has a fixed point in $\mathbb{X}$.

1.1.2 Measures of non-compactness

In this part of the thesis, we present key definitions and properties related to the Kuratowski and Hausdorff measures of non-compactness (MNC) in Banach spaces $\mathbb{X}$, essential for understanding bounded sets in infinite dimensional spaces see [15, 16, 121] and the references therein.

Definition 1.3. [15] Let $\mathbb{X}$ be a Banach space and $\Omega_{\mathbb{X}}$ the bounded subsets of $\mathbb{X}$.

The Kuratowski measure of non-compactness is the map $\alpha : \Omega_{\mathbb{X}} \to [0, +\infty]$ defined by

$$\alpha(D) = \inf \left\{ \varepsilon > 0 : \ D \subseteq \bigcup_{i=1}^{n} D_i \ \text{ and } diam(D_i) \leq \varepsilon \right\}, \quad \text{where } \ D \in \Omega_{\mathbb{X}},$$

where

$$diam(D_i) = \sup \big\{ \|u - v\|_{\mathbb{X}} : u, v \in D_i \big\}.$$

This measure of noncompactness satisfies some important properties .

Theorem 1.9. *[15, 16, 121] Let $\mathbb{X}$ be a Banach space, $C, D \subset \mathbb{X}$ be bounded sets, then the following properties are satisfied:*

(1) *Regular: if the condition $\alpha(D) = 0 \iff \overline{D}$ is compact (D is relatively compact), where $\overline{D}$ denotes the closure of D.*

(2) $\alpha(D) = \alpha(\overline{D}) = \alpha(co(D))$, *where* $co(D)$ *means the convex hull of D.*

(3) *Algebraically semiadditive, if* $\alpha(C + D) \leq \alpha(C) + \alpha(D)$,

(4) *Monotone, if* $B \subset D \Longrightarrow \alpha(C) \leq \alpha(D)$,

(5) *Semi-homogeneity:* $\alpha(kD) = |k|\alpha(D)$, *for any* $\beta \in \mathbb{R}$,

(6) *Semi-additivity:* $\alpha(C \cup D) = \max\{\alpha(C), \alpha(D)\}$

(7) $\alpha(C \cap D) = \min\{\alpha(C), \alpha(D)\}$

(8) *Nonsingular, if* $\alpha(D + x) = \alpha(D)$, *for all* $x \in \mathbb{X}$,

(9) *Lat* $\digamma : D \subset \mathbb{X} \longrightarrow \mathbb{X}$ *a continuous bounded mapping. We say that* $\digamma$ *is* α*-Lipschitz if there exists* $k \geq 0$ *such that*

$$\alpha(\digamma(D)) \leq K\alpha(D), \qquad \text{for any bounded subset, } D \subset \Omega.$$

If, in addition, $k < 1$, *then we say that* $\digamma$ *is* α*-condensing if* $\digamma : D \subset \mathbb{X} \longrightarrow \mathbb{X}$ *is said to be* α*-condensing if and only if*

$$\alpha\big(\digamma(D)\big) < \alpha(D), \qquad \text{for any } D \subset \Omega \text{ bounded .}$$

(10) *if* $D \subset \mathcal{PC}([0, T], \mathbb{H})$ *is bounded, then* $\alpha(D(t)) \leq \alpha_{\mathcal{PC}}(D)$ *for all* $t \in [0, T]$ *where* $D(t) = \{x(t) : x \in D\} \subseteq \mathbb{H}$ *Furthermore, if D is equicontinuous on* $[0, T]$, *then* $D(t)$ *is continuous for* $t \in [0, T]$, *and* $\alpha_{\mathcal{PC}}(D) = \sup_{t\in[0,T]} \alpha(D(t))$.

The notation $\alpha(.)$, $\alpha_{\mathcal{C}}(.)$, $\alpha_{\mathcal{PC}}(.)$ *are the Kuratowskii measure of non-compactness on the bounded set of* $\mathbb{H}$, $\mathcal{C}([0, T], \mathbb{H})$ *and* $\mathcal{PC}([0, T], \mathbb{H})$, *respectively. For more details see [15].*

Proof. See [16].

Lemma 1.1. *[96] Let* $\mathbb{X}$ *be a Banach space, be bounded* $D \subset \mathbb{X}$. *Then there exist a countable set* $D_0 \subset D$, *such that*

$$\alpha(D) \leq 2\alpha(D_0). \tag{1.3}$$

Lemma 1.2. *[15, 97] If G is a bounded subset in $\mathcal{C}([0,T],\mathbb{X})$, then $G(t)$ is bounded in $\mathbb{X}$, and $\alpha\big(G(t)\big) \leq \alpha(G)$. Furthermore, if G is also equicontinuous for all $[0,T]$, then $\alpha\big(G(t)\big)$ is continuous for $t \in [0,T]$ and*

$$\alpha(G) = \sup_{t\in[0,T]} \alpha(G(t)), \quad \text{where} \quad G(t) = \Big\{x(t) \,:\, x \in G\Big\} \subseteq \mathbb{X}.$$

Lemma 1.3. *[15, 122] If $(x^n)_{n=1}^{+\infty}$ is a sequence of Bochner integrable functions from $[0,T]$ into $\mathbb{H}$ with the estimation $\|x^n(t)\| \leq \vartheta(t)$ for almost all $t \in [0,T]$ and every $n \geq 0$, where $\vartheta \in L^1([0,T],\mathbb{R})$ and satisfies*

$$\alpha\bigg(\bigg\{\int_0^t x^n(s)ds \,:\, n \geq 0\bigg\}\bigg) \leq 2\int_0^t \alpha\big(\omega(s)\big)ds,$$

where the function $\omega(t) = \alpha\Big(\big\{x^n(t) \,:\, n \geq 0\big\}\Big)$ belongs to $L^1([0,T],\mathbb{R})$.

Lemma 1.4. *[45]. If $\sigma \subset \mathcal{C}\big([0,T], L_2^0(\mathbb{K},\mathbb{H})\big)$, W is a standard Wiener process, then*

$$\alpha\Big(\int_0^t \sigma(s)dW(s)\Big) \leq \sqrt{T}\alpha\Big(\sigma(s)\Big),$$

where

$$\int_0^t \sigma(s)dW(s) = \bigg\{\int_0^t \vartheta(s)dW(s) \,:\, \text{for all } \vartheta \in \sigma\bigg\}.$$

In the definition of the Kuratowski MNC, we can consider balls instead of arbitrary sets, which leads to the definition of the Hausdorff MNC:

Definition 1.4. [16] Let $\mathbb{X}$ be a metric space and D be a bounded subset of $\mathbb{X}$. Then the Hausdorff measure of noncompactness $\mu(D)$ of the set D is the infimum of the numbers ε such that D admits a finite covering by balls of radius smaller than ε. This is defined by the formula:

The Hausdorff measure of non-compactness μ defined on each bounded subset D of $\mathbb{X}$ by

$$\mu(D) = \inf\left\{\varepsilon > 0 : \, D \subseteq \bigcup_{i=1}^{n} D_{\varepsilon}\left(x_i, r_i\right)\right\},$$

where $D_{\varepsilon}\left(x_i\right)$ is a ball of radius $\leq \varepsilon$ centered at x_i, $i = 1, 2, \cdots, n$, $n \in \mathbb{N}$.

Remark 1.2. The Hausdorff MNC retains the same properties as the Kuratowski MNC, as discussed in Lemma 1.1, 1.3, 1.4 and Theorem 1.9.

The next theorem shows that the both measures α, μ are in some sense equivalent.

Theorem 1.10. *[16] Let $\mathbb{X}$ be a metric space and D be a bounded subset of $\mathbb{X}$, since the following inequality holds*

$$\mu(D) \leq \alpha(D) \leq 2\mu(D).$$

In some Banach spaces that possess a well-defined geometric structure, the inequality above can be strengthened. For example in Hilbert spaces

$$\sqrt{2}\mu(D) \leq \alpha(D) \leq 2\mu(D).$$

1.1.3 Some general theory of semigroups

In this part of the thesis, we provide a reminder of the concepts and certain properties of semigroup theory.

Definition 1.5. [1, 130] Let $\mathbb{X}$ be a Banach space. A one-parameter family $\mathcal{T}(t)$, $0 \leq t < \infty$, of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$ is called a semigroup of bounded linear operators on $\mathbb{X}$, if

(i) $\mathcal{T}(0) = \mathcal{X}$, ($\mathcal{X}$ is the identity operator on $\mathbb{X}$).

(ii) $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators, $\mathcal{T}(t)$, is uniformly continuous if

$$\lim_{t\to 0}\|\mathcal{T}(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{\mathcal{T}(t)x - x}{t} \text{ exists,} \right\}$$

and

$$Ax = \lim_{t\to 0} \frac{\mathcal{T}(t)x - x}{t} = \frac{d^+\mathcal{T}(t)x}{dt}\bigg|_{t=0} \text{ for } x \in D(A),$$

is the infinitesimal generator of the semigroup $\mathcal{T}(t), D(A)$ is the domain of A. From the definition it is clear that if $\mathcal{T}(t)$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{t\to s}\|\mathcal{T}(t) - \mathcal{T}(s)\| = 0.$$

Theorem 1.11. *[130] A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is bounded and linear.*

Proof. See [130].

Definition 1.6. [130] A semigroup $\mathcal{T}(t), 0 \leq t < \infty$, of bounded linear operators on $\mathbb{X}$ is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0} \mathcal{T}(t)x = x \quad \text{for every} \quad x \in \mathbb{X}.$$

A strongly continuous semigroup of bounded linear operators on $\mathbb{X}$ will be called a semigroup of class C_0 or simply a C_0-semigroup.

Theorem 1.12. *[130, 181] Let $\mathcal{T}(t)$ be a C_0-semigroup. There exist constants $a \geq 0$ and $M \geq 1$ such that*

$$\|\mathcal{T}(t)\| \leq Me^{at} \quad \textit{for every} \quad 0 \leq t < \infty.$$

Proof. See [130], [181].

Theorem 1.13. *[181] assume that an operator A with the domain $D(A)$ generates a semigroup $\mathcal{T}(t), t \geq 0$. Then for arbitrary $x \in D(A)$ and $t \geq 0$,*

(i)

$$\mathcal{T}(t)x \in D(A),$$

(ii)

$$\frac{d}{dt}\mathcal{T}(t)x = A\mathcal{T}(t)x = \mathcal{T}(t)Ax,$$

(iii)

$$\mathcal{T}(s)x - x = \int_0^t \mathcal{T}(s)Ax\, ds.$$

Moreover,

(iv) the domain $D(A)$ is dense in $\mathbb{X}$ and the operator A is closed (i.e., its graph $\{(x, A(x)), x \in D(A)\}$ is a closed subset of $\mathbb{X} \times \mathbb{X}$).

Proof. See [130, 181].

1.1.4 Some properties of resolvent operators

In this part of the thesis, we will introduce the fundamental definitions and theory of the resolvent operator and provide applications in differential and integro-differential equations. Let A and $h(t)$ are closed linear operators on $\mathbb{X}$ and $\mathbb{Y}$ represents the Banach space $\mathcal{D}(A)$ equipped with the graph norm defined by

$$\|y\|_{\mathbb{Y}} = \|Ay\|_{\mathbb{X}} + \|y\|_{\mathbb{X}}, \;\; y \in \mathbb{Y}.$$

Let us consider the following Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + \int_0^t h(t-s)y(s)ds \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{X}. \end{cases} \tag{1.4}$$

Definition 1.7. [64] A resolvent operator for (1.4) is a bounded linear operator valued function $\mathcal{R}(t) \in L_b(\mathbb{X})$ for $t \geq 0$, which satisfies the following properties

(i). $\mathcal{R}(0) = \mathcal{X}$ (The Identity operator of $\mathbb{X}$) and $\|\mathcal{R}(t)\| \leq Me^{\delta t}$ for some constants $k > 0$ and $\delta \in \mathbb{R}$.

(ii). For each $y \in \mathbb{X}$, $\mathcal{R}(t)y$ is strongly continuous for $t \geq 0$.

(iii). For $y \in \mathbb{Y}$, $\mathcal{R}(.)y \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{X}) \cap \mathcal{C}(\mathbb{R}^+, \mathbb{Y})$ and

$$\begin{aligned} \mathcal{R}'(t)y =& A\mathcal{R}(t)y + \int_0^t h(t-s)\mathcal{R}(s)y\,ds \\ =& \mathcal{R}(t)Ay + \int_0^t \mathcal{R}(t-s)h(s)y\,ds, \quad \text{for } \; t \geq 0. \end{aligned} \tag{1.5}$$

The resolvent operators have a great importance in obtaining variation of constants formula for nonlinear systems and in studying the existence of solutions, see [46, 64].

Now, we make the following assumptions:

(H_1) The operator A is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $\mathbb{X}$.

(H_2) For all $t \geq 0$, $h(t)$ is a closed linear operator from $\mathbb{D}(h)$ to $\mathbb{X}$ and $h(t) \in L_b(\mathbb{Y}, \mathbb{X})$. For any $y \in \mathbb{Y}$ the map $t \to h(t)y$ is bounded, differentiable and the derivative $t \to h'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Lemma 1.5. *[64] Assume that (H_1) and (H_2) hold. Then there exists a unique resolvent operator to the Cauchy problem(1.4).*

We consider the integro-differential equation (IDE) defined as follows:

$$\begin{cases} y'(t) = Ay(t) + \int_0^t h(t-s)y(s)ds + f(t), \quad t > 0 \\ y(0) = y_0 \in \mathbb{X}. \end{cases} \tag{1.6}$$

where $f : [0, +\infty[\to \mathbb{H}$ is a continuous function.

Definition 1.8. [64] A continuous function $y : \mathbb{R}^+ \to \mathbb{X}$ is said to be a strict solution of (1.6) if

(i) $y \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{X}) \cap \mathcal{C}(\mathbb{R}^+, \mathbb{Y})$,

(ii) y satisfies (1.6) for $t \geq 0$.

Theorem 1.14. *[63, 64] Suppose $\mathcal{R}(t)$ is a resolvent operator for (1.6), $y_0 \in \mathbb{Y}$ and $f \in \mathcal{C}([0, \infty), \mathbb{Y})$. If $y(t)$ is a solution of (1.6), then*

$$y(t) = \mathcal{R}(t)y_0 + \int_0^t \mathcal{R}(t-s)f(s)\, ds. \tag{1.7}$$

Proof. Consider

$$\begin{aligned} w(t) =& y(t) - \mathcal{R}(t)y_0 - \int_0^t \mathcal{R}(t-s)f(s)\, ds \\ =& \int_0^t \frac{\partial}{\partial s}\left(\mathcal{R}(t-s)y(s)\right) ds - \int_0^t \mathcal{R}(t-s)f(s)\, ds. \end{aligned}$$

From Definition 1.7 (iii) it follows that for $x \in \mathbb{Y}$,

$$\mathcal{R}'(t-s)\, y = \mathcal{R}(t-s)\, Ay + \int_s^t \mathcal{R}(t-r)h(r-s)\, x\, dr.$$

Using the fact that $Y(t)$ is a solution of (1.6) we see that

$$\begin{aligned} w(t) =& \int_0^t \int_0^s \mathcal{R}(t-s)h(v-s)\, dv\, ds \\ & - \int_0^t \int_s^t \mathcal{R}(t-r)h(r-s)\, x(s) dr\, ds \\ =& 0. \end{aligned}$$

Hence, by Fubini's theorem the result follows.

since $\mathcal{R}(t-r)h\,(r-s)\,x(s)$ is integrable.

This result is the standard variation of parameters formula for (1.6). Because our problem is infinite dimensional, however, (1.7) may not, in fact, be a solution. However, the following result is an easy consequence of the existence of a resolvent operator.

□

Definition 1.9. A continuous function $y : \mathbb{R}^+ \to \mathbb{X}$ is said to be a strict solution of (1.6) if $y \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{X}) \cap \mathcal{C}(\mathbb{R}^+, \mathbb{Y})$ and satisfies (1.7) for $t \geq 0$.

Definition 1.10. [185] A semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ in $\mathbb{X}$ is called equicontinuous if the operator $\mathcal{T}(t)$ is uniformly continuous with respect to the operator norm for every $t > 0$.

Lemma 1.6. *[51] Let A be the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ and that $\{h(t)\}_{t\geq 0}$ satisfies condition (H_2). Then the resolvent operator $\{\mathcal{R}(t)\}_{t>0}$ is continuous in the operator norm (or continuous in the uniform operator topology) for $t > 0$ if and only if $\{\mathcal{T}(t)\}_{t\geq 0}$ is continuous with respect to the operator norm for $t \geq 0$.*

1.1.5 Mild Solutions of Abstract Cauchy Problem

In this part, we will present mild solutions to the abstract Cauchy problem. To begin, we first define the abstract Cauchy problem, which forms the foundation for analyzing these solutions and their properties.

Let A be a linear operator on a real separable Hilbert space $\mathbb{H}$, and consider the abstract Cauchy problem:

$$\begin{cases} x'(t) = Ax(t), \quad 0 < t < T \\ x(0) = x_0 \in \mathbb{H}. \end{cases} \tag{1.8}$$

Definition 1.11. [57]A function $x : [0, T) \to \mathbb{H}$ is a (classical) solution of the problem (1.8) on $[0, T)$ if x is continuous on $[0, T)$, continuously differentiable and $x \in D(A)$ for $t \in [0, T)$, and (1.8) is satisfied on $[0, T)$.

Argue why if $x_0 \notin \overline{D(A)}$ then (1.8) cannot have a solution. If A is an infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, then for any $x_0 \in D(A)$ the function $x(t) = \mathcal{T}(t)x_0, t \geq 0$, is a solution of (1.8). On the other hand, if $x \notin D(A)$ then the continuity at zero may not be a problem, but $x(t)$ does not have to be differentiable, unless the C_0-semigroup has additional properties, for example, when it is a differentiable semigroup.

In this case, $x(t) = \mathcal{T}(t)x_0$ is not a solution in the usual sense, but it can be viewed as a generalized solution, which will be called a mild solution. In fact, the concept of mild solution can be introduced to study the following nonhomogeneous initial-value problem:

$$\begin{cases} x(t) = Ax(t) + f(t), \quad 0 < t < T \\ X(0) = x_0 \in \mathbb{H}, \end{cases} \tag{1.9}$$

where $f : [0, T) \to \mathbb{H}$.

We assume that A is an infinitesimal generator of a C_0-semigroup so that the homogeneous equation (1.8) has a unique solution for all $x_0 \in D(A)$ The definition of a classical solution, Definition 1.11, extends to the case of the nonhomogeneous initial-value problem by requiring that in this case, the solution satisfies (1.9). We now define the concept of a mild solution.

Definition 1.12. Let A be an infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$ on $\mathbb{H}$, $x_0 \in \mathbb{H}$ and $f \in L^1([0,T], \mathbb{H})$. The function $x \in \mathcal{C}([0,T], \mathbb{H})$ given by

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)ds, \;\; 0 \leq t \leq T,$$

is the mild solution of the initial-value problem (1.9) on $[0, T]$.

1.1.6 Some Fundamental Principles of Controllability

In this subsection, we will discuss the fundamental properties of the controllability of system.

Consider the linear control system

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{1.10}$$

where A generates a strongly continuous semigroup of bounded linear operators $\mathcal{T}(t)$ on a Banach space $\mathbb{X}$ and B is a bounded linear operator from a Banach space $\mathbb{U}$ into $\mathbb{X}$. Now, if $x(.)$ is a classical solution of (1.10), then $x(t) \in D(A)$, for all $t \in [0, T]$.

So, in the general case when A is unbounded, $D(A) \neq \mathbb{X}$, which means that the system cannot be steered to all of $\mathbb{X}$. Therefore, only the mild solution

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)Bu(s)ds, \tag{1.11}$$

will be considered in this paper with the following definition of exact controllability.

Definition 1.13. [104, 164] The system (1.10) is said to be exactly controllable on $J = [0, T]$ if, for any two points $x_0, x_1 \in \mathbb{X}$, there exists a control $u \in L^2(J, \mathbb{U})$ such that $x(T) = x_1$.

The following hypothesis is assumed throughout the paper:

(H_0) The linear operator $\mathcal{W}$ from $L^2(J, \mathbb{U})$ into $\mathbb{X}$, defined by

$$\mathcal{W}u = \int_0^T \mathcal{T}(T-s)Bu(s)\,ds,$$

induces a bounded inverse operator $\mathcal{W}^{-1}$ defined $L^2(J, \mathbb{U})/\ker(\mathcal{W})$.

Then, by using the control

$$u(t) = \mathcal{W}^{-1}\Big(x_1 - \mathcal{T}(T)x_0\Big)(t),$$

in (1.11), Hypothesis (H_0) yields $x(T) = x_1$, and so (1.10) is exactly controllable on J.

The construction of $\mathcal{W}^{-1}$ is outlined as follows (See [104, 134]). Let $U = L^2(J, \mathbb{U})/\ker(\mathcal{W})$. Since $\ker(\mathcal{W})$ is closed, U is a Banach space under the norm

$$\begin{aligned}\Big\| \, [u] \, \Big\|_U &= \inf_{u \in [u]} \|u\|_{L^2(J,\mathbb{U})} \\ &= \inf_{\mathcal{W}\hat{u}=0} \|u + \hat{u}\|_{L^2(J,\mathbb{U})},\end{aligned}$$

Remark 1.3. Triggiani [164] proved that, if $\mathbb{U}$ is infinite dimensional, then the system (1.10) is never exactly controllable when B is compact or when $\mathcal{T}(t)$ is compact for all $t > 0$. Louis and Wexler [100] proved that the system (1.10) is not exactly controllable when there exists $t_0 > 0$ such that $\mathcal{T}(t_0)$ is compact or $\mathcal{T}(t)$ is differentiable. Thus, the concept of exact controllability is too strong. At times, approximate controllability is more useful, leading to the following definition

Definition 1.14. [104] The system (1.10) is said to be approximately controllable on $[0, T]$ if $\overline{Range(\mathcal{W})} = \mathbb{X}$. Hence, (1.10) is approximately controllable if, for any $x_1 \in \mathbb{X}$ and any $\epsilon > 0$, there exists a control $u \in L^2(J, \mathbb{U})$ such that $\|x(T) - x_1\| < \epsilon$.

Several authors have studied the controllability of linear systems and linear delay systems in Banach spaces (See Triggiani [164], Zhou [186], Henriquez [71]). It should be noted that, if X is finite dimensional, then the exact controllability and approximate controllability of a linear system are equivalent. Next, the above problems will be discussed for nonlinear systems in Banach spaces. The following fixed point theorems are used widely in establishing the controllability results for different types of nonlinear systems.

On the other hand, we are also concerned with understanding how to apply optimal control theory to specific systems. Optimal control theory is a mathematical framework used to determine the best possible control inputs that guide a dynamic system from an initial state to a desired final state (see [52, 55, 157, 173, 174] and refernces contained therein). This is achieved while minimizing or maximizing a specific performance measure, often referred to as the cost functional. The cost functional typically accounts for factors such as energy consumption, time efficiency, or overall system performance, based on a given criterion.

Our primary focus will be on control systems (1.10) and the associated criteria, which are commonly known as cost functionals. Specifically, we define the cost functional as follows:

$$J_T\left(x, u\left(.\right)\right) = \int_0^T g\left(y(t), u(t)\right) dt + G\left(y(T)\right), \tag{1.12}$$

when $T < +\infty$. If the duration of the control interval is $[0, +\infty]$, then the cost functional

$$J\left(x, u\left(.\right)\right) = \int_0^{+\infty} g\left(y(t), u(t)\right) dt. \tag{1.13}$$

Our aim will be to find a control $\widehat{u}\left(.\right)$ such that for all admissible controls $u\left(.\right)$

$$J_T\left(x, \widehat{u}\left(.\right)\right) \leq J_T\left(x, u\left(.\right)\right), \tag{1.14}$$

or

$$J\left(x, \widehat{u}\left(.\right)\right) \leq J\left(x, u\left(.\right)\right). \tag{1.15}$$

There are two main methods for minimizing cost functionals (1.12) or (1.13):

Bellman's equation: This method involves embedding the minimization problem into a parametrized family of similar problems. The minimal value, as a function of the parameter, satisfies an analytic relation. If the parameter is the initial state and the length of the control interval, the minimal value of the cost functional is called the value function, and the corresponding analytic relation is Bellman's equation. Solving this equation allows finding the optimal strategy in the form of a closed loop control.

Pontriagin's maximum principle: This method provides necessary conditions for optimal open-loop control. It is derived using classical calculus and involves considering a parametrized family of controls and the corresponding values of the cost functional (see [12, 157] and refernces contained therein).

1.1.7 Some Fundamental Concepts of Ulam Stability

In this section, we discuss the stability of functional equations in Banach spaces. The concept of stability was introduced by Ulam, Hyers, and Rassias (for more details, see [2, 78, 142, 165, 166]). Now, we consider the following:

Denote by $\mathcal{BC} := \mathcal{BC}\left(\mathbb{R}^+\right)$ the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ into $\mathbb{R}$. Let $C \subset \mathcal{BC}$ and let $N : C \to C$. We now examine the solutions of the equation:

$$(Nx)(t) = x(t). \tag{1.16}$$

We introduce the following concept of the Ulam stability for (1.16). Let $\Phi : J \to [0, \infty)$ be a continuous function. We consider the following inequalities:

$$|x(t) - (Nx)(t)| \leq \varepsilon, \quad t \in J, \tag{1.17}$$

$$|x(t) - (Nx)(t)| \leq \Phi(t), \quad t \in J, \tag{1.18}$$

$$|x(t) - (Nx)(t)| \leq \varepsilon\Phi(t), \quad t \in J, \tag{1.19}$$

Definition 1.15. [2, 166] The problem (1.16) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{BC}$ of the inequality

(1.17) there exists a solution $y \in \mathcal{BC}$ of (1.16) with

$$|x(t) - y(t)| \leq \varepsilon C_f, \quad t \in J,$$

Definition 1.16. [2, 142] The problem (1.16) is generalized Ulam–Hyers stable if there exists continuous functions $C_f : \mathbb{R}^+ \to \mathbb{R}^+$ with $C_f(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{BC}$ of the inequality (1.17) there exists a solution $y \in \mathcal{BC}$ of (1.16) with

$$|x(t) - y(t)| \leq C_f(\varepsilon), \quad t \in J,$$

Definition 1.17. [2, 142] The problem (1.16) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $C_{f,\Phi} > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{BC}$ of the inequality (1.19) there exists a solution $y \in \mathcal{BC}$ of (1.16) with

$$|x(t) - y(t)| \leq \varepsilon C_{f,\Phi}\Phi(t), \quad t \in J,$$

Definition 1.18. [2, 142] The problem (1.16) is generalized Ulam– Hyers–Rassias stable with respect to Φ if there exists a real number $C_{f,\Phi} > 0$ such that for each solution $x \in \mathcal{BC}$ of the inequality (1.18) there exists a solution $y \in \mathcal{BC}$ of (1.16) with

$$|x(t) - y(t)| \leq C_{f,\Phi}\Phi(t), \quad t \in J,$$

Remark 1.4. It is clear that

(i) Definition 1.15 implies Definition 1.16,

(ii) Definition 1.17 implies Definition 1.18,

(iii) Definition 1.17 for $\Phi(.) = 1$ implies Definition1.15.

One can have similar remarks for the inequalities (1.17)- (1.19).

1.2 Fractional calculus

In this section, we consider some basic concepts of fractional integrals and different types of fractional differential equations used in the context of this thesis (see [85, 144, 159, 187]).

1.2.1 Riemann-Liouville fractional integral

Two of the most known definitions of fractional integrals are the so-called Riemann-Liouville and Liouville integrals. First, let us consider the definition of Riemann-Liouville fractional integration on a finite interval of the real line. More detailed information can be found in Samko et al [144], Kilbas et al [85].

If $f(t)$ is a continuous function on the real line, then we can form the definite integral from a to t:

$$\left(I^1 f\right)(t) = \int_a^t f(t_1)dt_1.$$

Repeating this process gives:

$$\left(I^2 f\right)(t) = \int_a^t \left(I^1 f\right)(t_1)dt_1 = \int_a^t dt_1 \int_a^{t_1} dt_2 f(t_2),$$

and this can be extended arbitrarily

$$\left(I^n f\right)(t) = \int_a^t dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} dt_n f(t_n).$$

The Cauchy formula for repeated integration is

$$\left(I^n f\right)(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)ds. \tag{1.20}$$

The proof is given by induction. Equation (1.20) can be generalized for non-integer n, using the Gamma function $\Gamma(n) = (n-1)!$, to remove the discrete nature of the fractional, we obtain a fractional generalization of the integration.

$$\left(I^\beta f\right)(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s)ds, \quad t > a, \quad \alpha > 0.$$

This equation can be considered as a definition of fractional integration.

Definition 1.19. [187] **(Left and right Riemann-Liouville fractional integrals)**

Let $[a, b]$ be a finite or infinite interval of $\mathbb{R}$. The left and right RiemannLiouville fractional integrals ${}_aI_t^\beta f(t)$ and ${}_tI_b^\beta f(t)$ of order $\beta \in \mathbb{R}^+$ are defined by

$${}_aI_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s)ds, \quad t > a, \quad \alpha > 0, \tag{1.21}$$

and

$${}_tI_b^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_t^b (s-t)^{\beta-1} f(s)ds, \quad t < b, \quad \alpha > 0. \tag{1.22}$$

We denote by $L^p([a,b])$, where $p \geq 1$, the set of those Lebesgue real-valued measurable functions f defined on $[a,b]$, for which:

$$\left(\int_a^b |f(x)|^p dx\right)^{1/p} < \infty.$$

If $f(x)$ is a function on a finite interval $[a,b]$ such that $f \in L^1([a,b])$.

1.2.2 Riemann-Liouville fractional derivatives

Definition 1.20. [187]**(Left and right Riemann-Liouville fractional derivatives)**.

The left and right Riemann-Liouville fractional derivatives ${}_aD_t^\beta f(t)$ and ${}_tD_b^\beta f(t)$ of order $\beta \in \mathbb{R}^+$ are defined by

$$\begin{aligned} {}_aD_t^\beta f(t) &= \frac{d^n}{dt^n}\, {}_aD_t^{-(n-\beta)} f(t) \\ &= \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \left(\int_a^t (t-s)^{n-\beta-1} f(s)ds\right), \quad t > a, \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} {}_tD_b^\beta f(t) &= (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{-(n-\beta)} f(t) \\ &= \frac{1}{\Gamma(n-\beta)} (-1)^n \frac{d^n}{dt^n} \left(\int_t^b (s-t)^{n-\beta-1} f(s)ds\right), \quad t < b, \end{aligned} \tag{1.24}$$

respectively, where $n = [\beta] + 1$, $[\beta]$ means the integer part of α.

In particular, when $n = \beta$, then

$${}_aD_t^0 f(t) =_t D_b^0 f(t) = f(t),$$

$${}_aD_t^n f(t) = f^{(n)}(t), \qquad \text{and} \qquad {}_tD_b^0 f(t) = (-1)^n f^{(n)}(t),$$

where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n. If $0 < \beta < 1$, then

$${}_aD_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\beta} f(s)ds\right), \quad t > a,$$

and

$${}_tD_b^\beta f(t) = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left(\int_t^b (s-t)^{-\beta} f(s)ds\right), \quad t < b.$$

1.2.3 Caputo fractional derivatives

The left and right Caputo fractional derivatives are defined via above RiemannLiouville fractional derivatives.

Definition 1.21. [187]**(Left and right Caputo fractional derivatives)**

The left and right Caputo fractional derivatives ${}_a^CD_t^\beta f(t)$ and ${}_t^CD_b^\beta f(t)$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$ {}_a^CD_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)}\left(\int_a^t (t-s)^{n-\beta-1} f^{(n)}(s)ds\right), \quad t > a, \tag{1.25}$$

and

$$ {}_t^CD_b^\beta f(t) = \frac{1}{\Gamma(n-\beta)}(-1)^n\left(\int_t^b (s-t)^{n-\beta-1} f^{(n)}(s)ds\right), \quad t < b, \tag{1.26}$$

respectively, where $n = [\beta] + 1$. In particular, when $0 < \beta < 1$

$$ {}_a^CD_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)}\left(\int_a^t (t-s)^{n-\beta-1} f'(s)ds\right), \quad t > a,$$

and

$$ {}_t^CD_b^\beta f(t) = \frac{1}{\Gamma(n-\beta)}(-1)^n\left(\int_t^b (s-t)^{n-\beta-1} f'(s)ds\right), \quad t < b,$$

If $n = \beta$ then, in particular,

$$ {}_a^CD_t^0 f(t) = {}_t^C D_b^0 f(t) = f(t).$$

1.2.4 Fractional differential equations

In this part, we present the concept of fractional differential equations, which are an extension of traditional differential equations involving derivatives of non-integer orders, typically formulated using fractional calculus. These equations are used to model complex systems with memory and hereditary properties, commonly observed in many natural and engineering processes. We now consider the following nonlocal fractional problem in a Banach space $\mathbb{X}$.

$$\begin{cases} {}_0^CD_t^\beta X(t) = f\left(t, X(t)\right), \quad 0 \le t \le T, \\ \\ X(0) + g(X) = X_0. \end{cases} \tag{1.27}$$

where${}_{0}^{C}D_{t}^{\beta}$ is Caputo fractional derivative of order $\beta \in (0,1)$, $t \in J := [0,T]$ X_0 is an element of $\mathbb{X}$, $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous. The nonlocal term $g : \mathcal{C}(J \times \mathbb{X}) \to \mathbb{X}$ is a given function, here $\mathcal{C}(J \times \mathbb{X})$ is the Banach space of all continuous functions from J into $\mathbb{X}$ with the norm $\|X\| := \sup_{t\in[0,T]} \left|X(t)\right|$ for $X \in \mathcal{C}(J \times \mathbb{X})$.

Definition 1.22. [189] A function $X \in \mathcal{C}^1(J \times \mathbb{X})$ is said to be a solution of the nonlocal problem (1.27) if X satisfies the equation ${}_{0}^{C}D_{t}^{\beta}X(t) = f(t, X(t))$, *a.e.* on J, and the condition $X(0) + g(X) = X_0$.

Lemma 1.7. *[189] A function $X \in \mathcal{C}(J \times \mathbb{X})$is a solution of the fractional integral equation*

$$X(t) = X_0 - g(X) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, X(s))\, ds, \tag{1.28}$$

if and only if X is a solution of the nonlocal problem (1.27).

1.2.5 Fractional abstract evolution equations

In this subsection, we study the appropriate definition of mild solutions for fractional differential equations (specifically, the problem (1.29)). We provide basic preliminary and results. Consider the following Cauchy problems of fractional evolution equation with Caputo derivative

$$\begin{cases} {}_{0}^{c}D_{t}^{\beta}y(t) = Ay(t) + f(t, y(t)), \quad t \in [0,T], \\ y(0) + g(y) = y_0 \in \mathbb{X}, \end{cases} \tag{1.29}$$

where ${}_{0}^{c}D_{t}^{\beta}$ is Caputo derivative of order a, $0 < \beta < 1$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{\mathcal{T}(t)\}_{t\geq 0}$ in Banach space $\mathbb{X}$, $f : J \times \mathbb{X} \to \mathbb{X}, g : \mathcal{C}(J, \mathbb{X}) \to L(J, \mathbb{X})$ are given operators satisfying some assumptions and y_0 is an element of the Banach space $\mathbb{X}$.

The following lemma is essential for introducing mild solutions to equation (1.29).

Lemma 1.8. *[187] If*

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\beta)} + \int_0^t (t-s)^{\beta-1} \Big(Ay(s) + f(s, y(s)) \Big) ds, \quad for\, t \geq 0, \tag{1.30}$$

holds, then we have

$$y(t)=U(t)\left(y_0-g(y)\right)+\int_0^t (t-s)^{\beta-1}V(t-s)f(s,y(s))ds, \tag{1.31}$$

where

$$U(t)=\int_0^\infty \sigma_\beta(r)T(t^\beta r)dr, \quad V(t)=\beta\int_0^\infty r\sigma_\beta(r)T(t^\beta r)dr,$$

$$\sigma_\beta(r)=\frac{1}{\beta}r^{-1-\frac{1}{\beta}}\varrho_p\big(r^{-\frac{1}{\beta}}\big) \text{ and } \varrho_p(r)=\frac{1}{\pi}\sum_{n=1}^{\infty}\big(-1\big)^{n-1}r^{-\beta n-1}\frac{\Gamma\big(\beta n+1\big)}{n!}\sin(n\pi\beta),$$

$\sigma_\beta(r)$ is a probability density function defined on $(0,\infty)$.

For more details of the Lemma 1.8 proof, see [187]

Remark 1.5. $\sigma_\beta(r)\geq 0,\ r\in(0,\infty),\ \int_0^\infty \sigma_\beta(r)dr=1, \qquad \int_0^\infty r\sigma_\beta(r)dr=\frac{1}{\Gamma(1+\beta)}$

Due to Lemma 1.8, we give the following definition of the mild solution of (1.29)

Definition 1.23. [187]By the mild solution of the nonlocal Cauchy problem (1.29), we mean that the function $X\in\mathcal{C}(J,\mathbb{X})$ which satisfies

$$y(t)=U(t)\left(y_0-g(y)\right)+\int_0^t (t-s)^{\beta-1}V(t-s)f(s,y(s))ds, \qquad \text{for } t\in[0,T].$$

Suppose that A is the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of uniformly bounded linear operators on Banach space $\mathbb{X}$. This means that there exists $M>1$ such that $M=\sup_{t\in[0,T]}\|\mathcal{T}(t)\|<\infty$.

Lemma 1.9. *[188] The operators $\{U(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$ satisfy the following properties:*

(i) *For any fixed $t\geq 0, U(t)$ and $V(t)$ are strongly continuous.*

(ii) *For any fixed $t\geq 0, U(t)$ and $V(t)$ are linear bounded operators, moreover for any $y\in\mathbb{H}$,*

$$\Big\|U(t)y\Big\|\leq M\|y\|, \qquad \Big\|V(t)y\Big\|\leq\frac{M}{\Gamma(\beta)}\|y\|.$$

(iii) *If $\{\mathcal{T}(t)\}_{t\geq 0}$ is an equicontinuous semigroup, then $U(t)$ and $V(t)$ are continuous for $t>0$ by the operator norm, which means that for $0<\tau_1<\tau_2\leq T$, we have*

$$\Big\|U(\tau_2)-U(\tau_1)\Big\|\to 0 \quad \text{and} \quad \Big\|V(\tau_2)-V(\tau_1)\Big\|\to 0 \quad \text{as} \quad \tau_2\to\tau_1.$$

(vi) *For every $t \geq 0$, $U(t)$ and $V(t)$ are also compact operators if $\mathcal{T}(t)$is compact for every $t > 0$.*

Lemma 1.10. *[188] If the assumption (H_1) is satisfied, then $U(t)$ and $V(t)$ are also compact operators for every $t \geq 0$,.*

1.3 Some Elements of Stochastic Analysis

In this section, we reviewed the key concepts related to stochastic analysis and probability theory. We introduced the stochastic process involving the concept of Brownian motion, also known as the Wiener process. We also explored its fractional case (fractional Brownian motion) with the Hurst parameter. Then, we defined the Itô stochastic integral. Finally, we presented the stochastic differential equation, and for more details, refer to [11, 25, 27, 84, 86, 107, 110, 118, 120, 123].

Let $\mathbb{X}$ be Banach Spaces, and $\{W(t)\}_{t\geq 0}$ be a Wiener process (Brownian motion) on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

1.3.1 Basic definitions and concepts

In this part, we introduce some fundamental concepts of probability theory.

Definition 1.24. [27, 107](σ-algebra)

If Ω is a given set, then a σ-algebra $\mathcal{F}$ is a family of subsets of Ω, should have the following properties:

(i) $\emptyset, \Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$, where $A^c = \Omega - A$ is the complement of A in Ω,

(iii) $\{A_i\}_{i\geq 1} \subset \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 1.25. [27, 107]A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \longrightarrow [0, 1]$ such that

(i) $P(\emptyset) = 0,\ P(\Omega) = 1,$

(ii) for any disjoint sequence $(A_i)_{i\geq 1}$ $\big($i.e. $A_i \cap A_j = \emptyset$ if $i \neq j\big)$ then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Definition 1.26. [107]$(\Omega, \mathcal{F}, P)$ is a probability space, we set

$$\tilde{\mathcal{F}} = \Big\{A \subset \Omega :\ B, C \in \mathcal{F} \text{ such that} B \subset A \subset C,\ P(B) = P(C)\Big\}.$$

Then $\tilde{\mathcal{F}}$ is a σ-algebra and is called the completion of $\mathcal{F}$.

Now $\big(\Omega, \tilde{\mathcal{F}}, P\big)$ is a complete probability space.

Definition 1.27. [123] Given any family $\mathcal{U}$ of subsets of Ω there is a smallest σ-algebra $\mathcal{H}_{\mathcal{U}}$ containing $\mathcal{U}$, namely

$$\mathcal{H}_{\mathcal{U}} = \bigcap \Big\{\mathcal{H},\ \mathcal{H}\, \sigma\text{-algebra of } \Omega, \mathcal{U} \subset \mathcal{H}\Big\}.$$

$\mathcal{H}_{\mathcal{U}}$ is called the σ-algebra generated by $\mathcal{U}$.

Definition 1.28. [123] If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $Y : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$Y^{-1}(V) := \Big\{\omega \in \Omega;\ \ Y(\omega) \in V\Big\} \in \mathcal{F},$$

for all open sets $V \in \mathbb{R}^n$ (or, equivalently, for all Borel sets $V \subset \mathbb{R}^n$).

Definition 1.29. [86] Every random variable X defines a σ-algebra

$$\mathcal{H}_X = \Big\{X^{-1}(D), \quad D \in \mathcal{B}\Big\},$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$, We denote this algebra by $\mathcal{H}_X$ and call it the σ-algebra generated by X.

Definition 1.30. [86] A function $X \in \Omega \to \mathbb{R}^n$ is called a random variable, if it is a measurable map from $(\Omega, \mathcal{F})$ to $(\mathbb{R}^n, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^n$.

Definition 1.31. [107]A random variable X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. Every random variable induces a probability measure μ_X on $\mathbb{R}^n$, defined by

$$\mu_X(D) = P\big(X^{-1}(D)\big),$$

μ_X is called the distribution of X.

Definition 1.32. [123]Given X in $L^1(\Omega, \mathcal{F}, P, \mathbb{X}) := L^1$, we denote by $\mathbb{E}[X]$ the Bochner integral

$$\mathbb{E}[X] := \int_\Omega |X| \, dP = \int_{\mathbb{R}^n} x d\mu_X(x)$$

is called to be the expectation (or the mean) of X

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable and $\int_\Omega |f| \, dP < \infty$ then we have

$$\mathbb{E}[f(X)] := \int_\Omega |f(X)| \, dP = \int_{\mathbb{R}^n} f(x) d\mu_X(x).$$

The vector space L^1 is called the space of integrable random variables. Similarly, for $p \geq 1$, denote L^p as the set of random variables X for which $\mathbb{E}[|X|^p] < \infty$.

Let X and X_k, for $k \geq 1$, be $\mathbb{R}^n$-valued random variables. s. The following four concepts of convergence are crucial in probability theory:

(i) Almost Sure Convergence: If there exists a P-null set $\Omega_0 \in \mathcal{F}$ (i.e., $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 0$) such that for every $\omega \notin \Omega_0$, the sequence $\{X_k(\omega)\}$ converges to $X(\omega)$ in the usual sense in $\mathbb{R}^n$, then we say that the sequence $\{X_k(\omega)\}$ converge to X almost surely, or with probability 1, and we write

$$\lim_{k\to\infty} X_k = X \quad \text{a.s.}$$

(ii) Convergence in Probability: If for every $\epsilon > 0$,

$$P\Big\{\omega : \big|X_k(\omega) - X(\omega)\big| > \epsilon\Big\} \to 0 \quad \text{as} \quad k \to \infty,$$

then we say that the sequence $\{X_k\}$ converge to X in probability. This is also called stochastic convergence.

(iii) Convergence in L^p (Moment Convergence): If X_k and X belong to L^p and

$$\mathbb{E}\left[\left|X_k - X\right|^p\right] \to 0,$$

then we say that $\{X_k\}$ converge to X in p th moment or in L^p.

(iv) Convergence in Distribution: If for every real-valued continuous bounded function f defined on $\mathbb{R}^n$,

$$\lim_{k\to\infty} \mathbb{E}f\big(X_k\big) = \mathbb{E}f\big(X\big),$$

then we say that the sequence $\{X_k\}$ converge to X in distribution.

These convergence concepts have the following relationship:

$$\begin{tikzcd}
 & \text{Convergence in } L^p \arrow[d] \\
\text{Almost Sure Convergence} \arrow[r] & \text{Convergence in Probability} \arrow[d] \\
 & \text{Convergence in Distribution}
\end{tikzcd}$$

We now state some very important integration convergence theorems.

Theorem 1.15. *[86, 107](Monotone convergence theorem, Beppo Lévi 1906) Let (X_k) be an increasing sequence of nonnegative random variables in L^p, then*

$$\lim_{k\to\infty} \mathbb{E}\left(X_k\right) = \mathbb{E}\big(\lim_{k\to\infty} X_k\big)$$

Proof. Can be found in [86].

Theorem 1.16. *[86](Fatou lemma, 1906) Let (X_k) be a sequence of random variables in L^p with $|X_k| \leq X$ a.s., where $X \in L^p$. Then*

$$\mathbb{E}\left[\liminf_{k\to\infty} X_k\right] \leq \liminf_{k\to\infty} \mathbb{E}\left(X_k\right) \leq \limsup_{k\to\infty} \mathbb{E}\left[X_k\right] \leq \mathbb{E}\left[\limsup_{k\to\infty} X_k\right].$$

Proof. See [86].

Theorem 1.17. *[86, 107](Lebesgue's dominated convergence theorem, 1902) Let $p \geq 1$, $\{X_k\}$ be a sequence in L^p with $|X_k| \leq Y$ a.s., where $Y \in L^p$, and $X_k \to X$ almost everywhere. Then $X \in L^p$, (X_k) converges to X in L^p, and*

$$\lim_{k\to\infty} \mathbb{E}\left(X_k\right) = \mathbb{E}\left(X\right).$$

When Y is bounded, this is also known as the bounded convergence theorem.

Proof. See [86].

Let $\{A_k\}$ be a sequence of sets in $\mathcal{F}$. The upper limit of the sets is defined as:

$$\limsup_{k\to\infty} A_k = \Big\{\omega \,:\, \omega \in A_k \text{ for infinitely many } k\Big\} = \bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty} A_k.$$

This set belongs to $\mathcal{F}$, and its probability can be analyzed using the Borel-Cantelli Lemma.

Lemma 1.11. *[107](Borel-Cantelli's lemma)*

(i) Let $(A_k)_{k\in\mathbb{N}} \subset \mathcal{F}$ be a sequence of events. If :

$$\sum_{k=1}^{\infty} P(A_k) < \infty,$$

then

$$P\Big(\limsup_k A_k\Big) = 0.$$

That is, there exists a set $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ and an integer-valued random variable k_0 such that for every $\omega \in \Omega_0$ we have $\omega \notin A_k$ for all $k \geq k_0(\omega)$.

(ii) If the sequence $(A_k)_{k\in\mathbb{N}} \subset \mathcal{F}$ be a sequence of independent events. If $\sum\limits_{k=1}^{\infty} P(A_k) = \infty$, then

$$P\Big(\limsup_k A_k\Big) = 1.$$

Proof. See [27]

The mathematical model for independence is the following:

Definition 1.33. [123]

- Two subsets $A, B \in \mathcal{F}$ are called independent if

$$P\big(A \cap B\big) = P\big(A\big) \cdot P\big(B\big).$$

- A collection $\mathcal{A} = \Big\{\mathcal{H}_i;\ \ i \in I\Big\}$ of families $\mathcal{H}_i$ of measurable sets is independent if

$$P\Big(H_{i_1} \cap \cdots \cap H_{i_k}\Big) = P\big(H_{i_1}\big)\cdots P\big(H_{i_k}\big),$$

for all choices of $H_{i_1} \in \mathcal{H}_{i_1}, \cdots, H_{i_k} \in \mathcal{H}_{i_k}$ with diferent indices $i_1, \cdots, i_k$.

- A collection of random variables $\{X_i\}_{i\in I}$ is independent if the collection of generated σ-algebras $\mathcal{H}_{X_i}$ is independent.

  If two random variables $X, Y : \Omega \to \mathbb{R}$ are independent then

  $$\mathbb{E}\big[XY\big] = \mathbb{E}\big[X\big] \cdot \mathbb{E}\big[Y\big].$$

  provided that $\mathbb{E}\big[X\big] < \infty$ and $\mathbb{E}\big[Y\big] < \infty$.

Definition 1.34. [84]A filtered an increasing collection of sub-σ-algebras of $\mathcal{F}$ (i.e. $\mathcal{F}_t \subset \mathcal{F}_s$ for all $0 \leq t < s < \infty$) in complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the condition

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$,

(ii) $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $0 \leq t < \infty$ that is, the filtration $\mathcal{F}_t$ is right continuous.

Definition 1.35. [84]A stopping time τ is a function from Ω to $\mathbb{N} \cup \{\infty\}$ such that

$$\{\tau = n\} \in \mathcal{F}_n, \quad \forall n < \infty.$$

Definition 1.36. [133]A random variable $\tau : \Omega \to [0, \infty]$ is a stopping time if the event $\{\tau \leq t\} \in \mathcal{F}_t$, every t, $0 \leq t \leq \infty$.

One important consequence of the right continuity of the filtration is the following theorem.

Theorem 1.18. *[133] The event $\{\tau < t\} \in \mathcal{F}_t$, $0 \leq t \leq \infty$, if and only if τ is a stopping time.*

Proof. Since $\{\tau \leq t\} = \bigcap_{t<u<t+\epsilon} \{\tau < u\}$, any $\epsilon > 0$, we have $\{\tau \leq t\} = \bigcap_{u>t} \mathcal{F}_u = \mathcal{F}_t$, so τ is a stopping time. For the converse, $\{\tau < t\} = \bigcup_{t>\epsilon>0} \{\tau \leq t - \epsilon\}$, and $\{\tau \leq t - \epsilon\} \in \mathcal{F}_{t-\epsilon}$, hence also in $\mathcal{F}_t$.

Definition 1.37. [53] Let τ be a stopping time and X an adapted process. The stopped random variable X_τ is defined by

$$X_\tau\left(\omega\right) = \sum_{n=0}^{\infty} X_n\left(\omega\right) 1_{\{\tau=n\}}.$$

Note that by definition, $X_\tau = X_\tau 1_{\{\tau<\infty\}}$.

1.3.2 Stochastic processes

In this section, we define stochastic processes and discuss their key properties, starting with Kolmogorov's approach, which views stochastic processes as families of random variables defined on a probability space, indexed by time. For further details, see [11, 84, 86, 110, 123].

Definition 1.38. [28, 86, 123]

A family $\{X_t\}_{t\in J}$ of $\mathbb{R}^d$-valued random variables is called a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with parameter set J and state space $\mathbb{R}^d$. The parameter set J is usually the halfline $\mathbb{R}^+ = [0, \infty)$, but it may also be an interval $[a, b]$, the non-negative integers and even subsets of $\mathbb{R}^d$ for $d \geq 1$.

Sometimes it is convenient to write $X(t, \omega)$ instead of $X_t(\omega)$. Thus we may also regard the process as a function of two variables

$$(t, \omega) \to X(t, \omega),$$

from $J \times \Omega$ into $\mathbb{R}^d$.

Whenever convenient the notation $\{X(t, \omega)\}$ or simply $\{X(t)\}$ will be used.

Note that for each $t \in J$ fixed we have a random variable

$$\omega \to X_t(\omega); \quad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \to X_t(\omega), \quad t \in J,$$

which is called a path of X_t.

Definition 1.39. [86]

Let $\{X_t\}_{t\geq 0}$ be a stochastic process taking values in $\mathbb{R}^d$. The process is said to be **continuous** (resp. **right continuous, left continuous**) if for almost all $\omega \in \Omega$ function $X_t(\omega)$ is **continuous** (resp. **right continuous, left continuous**) on $t \geq 0$.

Definition 1.40. [107] Let $\{X_t\}_{t\geq 0}$ be a stochastic process taking values in $\mathbb{R}^d$. The process is said to be cadlag (right continuous and left limit) if it is right continuous and for almost all $\omega \in \Omega$ the left limit $\lim\limits_{s\to t} X_s(\omega)$ exists and is finite for all $t > 0$.

Definition 1.41. [127]

A stochastic process $\{X_t\}_{t\geq 0}$ is called integrable if for every $t \geq 0$, X_t, is an integrable random variable.

Definition 1.42. [86] A stochastic process $\{X_t\}_{t\geq 0}$ taking values in $\mathbb{R}^d$ is said to be measurable if the stochastic process regarded as a function of (t, ω) from $\mathbb{R}^+ \times \Omega$ to $\mathbb{R}^n$ is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{R}^+)$ is the family of all Borel subsets of $\mathbb{R}^+$.

Definition 1.43. [107] A stochastic process is progressively measurable or progressive if, for every $T \geq 0$, $\{X_t\}_{0\leq t\leq T}$, is measurable with respect to $\mathcal{B}([0, T]) \times \mathcal{F}_T$, where $\mathcal{B}([0, T])$ is the family of all Borel subsets of $[0, T]$.

Definition 1.44. [84]A stochastic process $X = \{X_t\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ is a collection of $\mathbb{R}$-valued or $\mathbb{R}^d$-valued random variables. It is said to be $\mathcal{F}_t$ -adapted (or adapted) if, for each t, X_t is $\mathcal{F}_t$-measurable.

Definition 1.45. [107] Let $\mathcal{O}$ (resp. $\mathcal{P}$) denote the smallest σ-algebra on $\mathbb{R}^+ \times \Omega$ with respect to which every cadlag adapted process (resp. left continuous process) is a measurable function of (t, ω).

Definition 1.46. [107] A stochastic process is said to be optional (resp. predictable) if it is measurable with respect to $\mathcal{O}$ (resp. $\mathcal{P}$).

Definition 1.47. [107] A process $\{X_t\}_{t\geq 0}$ is called an increasing if, for almost all $\omega \in \Omega$, $X_t(\omega)$ is non-negative non-decreasing right continuous for $t \geq 0$.

Definition 1.48. [107] A stochastic process is called to be of finite variation if $X_t = \bar{X}_t - \tilde{X}_t$ with both $\{\bar{X}_t\}$ and $\{\tilde{X}_t\}$ are increasing processes.

These processes are cadlag, and thus, adapted processes of finite variation are optional.

The relations among the various stochastic processes are summarised below:

$$
\begin{tikzcd}
\text{continuous adapted} \arrow[d] & \text{continuous adapted} \arrow[d] & \text{adapted increasing} \arrow[d] \\
\text{left continuous adapted} \arrow[d] & \text{cadlag adapted} \arrow[d] & \text{adapted finite variation} \arrow[l] \\
\text{predictable} \arrow[r] & \text{optional} \arrow[d] & \\
 & \text{progressive} \arrow[r] \arrow[d] & \text{adapted} \\
 & \text{measurable} &
\end{tikzcd}
$$

Definition 1.49. [123] Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that $\{X_t\}_{t\geq 0}$ is a version of (or a modification) of $\{Y_t\}_{t\geq 0}$ if for all $t \geq 0$, $X_t = Y_t$ a.s.

i.e.

$$P\Big(\Big\{\omega : \;\; X_t(\omega) = Y_t(\omega)\Big\}\Big) = 1, \quad \forall t \geq 0.$$

Note that if X_t is a version of Y_t, then X_t and Y_t have the same finite dimensional distributions.

Definition 1.50. [127] Two stochastic processes $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ are said to be indistinguishable if for almost all $\omega \in \Omega$, we have $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$ i.e.

$$P\Big(\Big\{\omega : \;\; X_t(\omega) = Y_t(\omega), \;\; \text{for all } t \geq 0\Big\}\Big) = 1.$$

To illustrate that the (finite-dimensional) distributions alone do not give all the information regarding the continuity properties of a process, consider the following example:

Example 1.1. [123]Let $(\Omega, \mathcal{F}, P) = ([0, \infty), \mathcal{B}, \mu)$ where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$ and μ is a probability measure on $[0, \infty)$ with no mass on single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_t(\omega) = 0 \quad \text{for all } (t, \omega) \in [0, \infty) \times [0, \infty).$$

Prove that $\{X_t\}$ and $\{Y_t\}$ have the same distributions and that X_t is a version of Y_t. And yet we have that $t \to Y_t(\omega)$ is continuous for all ω, while $t \to X_t(\omega)$ is discontinuous for all ω.

Definition 1.51. [107] An adapted integrable process $\{M_t\}_{t\geq 0}$is called a martingale with respect to $\mathcal{F}_t$ (or martingale) if

$$\mathbb{E}\big(M_t|\mathcal{F}_s\big) = M_s \text{ a.s. for all } 0 \leq s < t < \infty.$$

Definition 1.52. [57] A process $\{M_t\}_{t\geq 0}$ which is adapted and integrable, is called a supermartingale with respect to $\mathcal{F}_t$ (or supermartingale) if

$$\mathbb{E}\big(M_t|\mathcal{F}_s\big) \leq M_s \text{ a.s. for all } 0 \leq s < t < \infty.$$

Definition 1.53. [107] A process $\{M_t\}_{t\geq 0}$ which is adapted and integrable,is called a submartingale with respect to $\mathcal{F}_t$ (or submartingale) if

$$\mathbb{E}\big(M_t|\mathcal{F}_s\big) \geq M_s \text{ a.s. for all } 0 \leq s < t < \infty.$$

The following is the well-known Doob martingale stopping theorem.

Theorem 1.19. *[57, 107](Doob's martingale stopping theorem) Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}_n$-valued martingale with respect to $\mathcal{F}_t$, and let ς, ϱ be two finite stopping times. Then*

$$\mathbb{E}\big(M_\varsigma|\mathcal{F}_\varrho\big) = M_{\varsigma\wedge\varrho} \text{ a.s.}$$

In particular, for any stopping time τ, we have

$$\mathbb{E}\big(M_{\tau\wedge t}|\mathcal{F}_s\big) = M_{\tau\wedge s} \text{ a.s., } \textit{ for all } 0 \leq s < t < \infty.$$

This implies that the stopped process $\{M_{\tau\wedge t}\}$ remains a martingale with respect to the same filtration $\mathcal{F}_t$.

1.3.3 Brownian Motion: Wiener Process

Brownian motion refers to the irregular movement of pollen grains suspended in water, first observed by the Scottish botanist Robert Brown in 1828. This motion was later explained by the random collisions of the pollen with water molecules. To mathematically describe this phenomenon, we use the concept of a stochastic process $W(t)$, which represents the position of the pollen grain at time t.

Definition 1.54. [107, 127] Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (one-dimensional) standard Brownian motion (or Wiener process) is a continuous stochastic process $W : \Omega \times [0, \infty[\to \mathbb{R}$ such that:

(i) $W_0 = 0$ a.s.,

(ii) for any $0 \leq s < t < \infty$, the increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$, i.e., $B_t - B_s \rightsquigarrow \mathcal{N}(0, t - s)$,

(iii) for each $0 = t_0 < t_1 < \cdots < t_k$ and $k \geq 2$, the increment $W_{t_1} - BW_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}}$ are independent random variables.

$\{W_t\}_{t\geq 0}$ is called a standard Brownian motion.

Definition 1.55. [129] A continuous stochastic process $\{W_t\}_{t\geq 0}$ is called a n-dimensional Brownian motion with values in $\mathbb{R}^n$ if one of the following three equivalent properties (i)–(iii) holds:

(i) its components $\{W_t^1\}_{t\geq 0}, \cdots, \{W_t^n\}_{t\geq 0}$ are mutually independent scalar Brownian motions,

(ii) (1) $W_0 = 0$ a.s.,

(2) $W_t - W_s \rightsquigarrow \mathcal{N}(0, t - s)1_{n\times n}$, for any $0 \leq s < t < \infty$,

(3) $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}}$ are independent random variables for each $0 = t_0 < t_1 < \cdots < t_k$ and $k \geq 2$.

(iii) $\{W_t\}_{t\geq 0}$ is a n-dimensional continuous centered Gaussian process whose covariance function is given by

$$\mathbb{E}(W_t W_s^T) = \big(t \wedge s\big) 1_{n\times n}, \quad t, s \geq 0.$$

Theorem 1.20. *[127] (Kolmogorov's continuity theorem)Given a stochastic process $\{X_t\}_{t\in[a,b]}$ if there exist constants $p, r, K > 0$ satisfies*

$$\mathbb{E}\left(|X_t - X_s|^p\right) \leq K\,|t-s|^{1+r}\,, \quad \text{for } s,t \in [a,b]\,,$$

, then there exists a modification (or version) of $\{X_t\}_{t\in[a,b]}$ that is a continuous process and whose paths are Hölder continuous for every $\delta \in [0, r/p]$.

Proof. For a proof see [154].

1.3.4 Stochastic Integral

The Stochastic Integral(Itô integral), a key concept in stochastic calculus, was introduced in the 1940s by Japanese mathematician Kiyosi Itô. It allows the integration of functions with respect to stochastic processes, such as Brownian motion. This innovation has become foundational in many fields, particularly in the modeling of systems with random behavior. Itâ's work laid the groundwork for stochastic differential equations (SDEs) and transformed the mathematical treatment of randomness in various disciplines [40, 107].

Let us first introduce the concept of elementary (simple) processes.

Definition 1.56. [107] A stochastic process $g = \{g(t)\}_{t\in[0,T]}$is called an elementary process if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0,T]$, and bounded random variables e_i, $0 \leq i \leq n-1$ such that e_i is $\mathcal{F}_{t_i}$- measurable and

$$g(t) = \sum_{i=0}^{n-1} e_i 1_{[t_i,t_{i+1})}(t). \tag{1.32}$$

where $\mathcal{X}_{[t_i,t_{i+1})}(t)$ is the indicator function on the interval $[t_i, t_{i+1})$.

Let Λ denote the family of all elementary processes.

Clearly, $\Lambda \subset \mathcal{M}^2\left([0,T]\,,\mathbb{R}\right)$. We now give the definition of the Itô integral for elementary processes.

Definition 1.57. [107] For a elementary process g with the form of (1.32) in Λ, define

$$\int_0^T g(t)dW(t) = \sum_{i=0}^{n-1} e_i\big(W(t_{i+1}) - W(t_i)\big),$$

is call it the stochastic integral of g with respect to the Brownian motion $\{W(t)\}$ or the Itô integral.

stochastic integral $\int_0^T g(t)dW_t$ is $\mathcal{F}_{t_i}$-measurable. We shall now show that it belongs to $L^2(\Omega, \mathbb{R})$.

Example 1.2. Consider the function g given by

$$g(t) = \sum_{i\geq 0} e_i 1_{[i\cdot 2^{-n},(i+1)\cdot 2^{-n})}(t), \tag{1.33}$$

where $\mathcal{X}_{[a,b]}(t)$ denotes the indicator function and $n \in \mathbb{N}$.

For functions of this form, it is reasonable to define the Itô integral $\int_s^t g(r)\,dW(r)$, where $W(t)$ is a standard Brownian motion. The integral is given by:

$$\int_s^t g(r)dW(r) = \sum_{i=0}^{n-1} e_i\big(W(r_{i+1}) - W(r_i)\big),$$

where

$$r_k = r_k^{(n)} = \begin{cases} k\cdot 2^{-n}, & \text{if } s \leq k\cdot 2^{-n} \leq t, \\ s, & \text{if } k\cdot 2^{-n} \leq s, \\ t, & \text{if } k\cdot 2^{-n} \geq t. \end{cases}$$

Next, we choose the following two functions: Choose

$$g_1(r) = \sum_{i\geq 0} W_{i\cdot 2^{-n}} 1_{[i\cdot 2^{-n},(i+1)\cdot 2^{-n})}(r),$$

where $W_{i\cdot 2^{-n}}$ represents the value of the Brownian motion $W(t)$ at time $i\cdot 2^{-n}$.

$$g_2(r) = \sum_{i\geq 0} W_{(i+1)2^{-n}} 1_{[i\cdot 2^{-n},(i+1)\cdot 2^{-n})}(r).$$

where $W_{(i+1)\cdot 2^{-n}}$ represents the value of $W(t)$ at time $(i+1)\cdot 2^{-n}$.

Then

$$\mathbb{E}\left[\int_0^t g_1(r)dW(r)\right] = \sum_{i\geq 0} \mathbb{E}\left[W(t_i)\left(W(t_{i+1}) - W(t_i)\right)\right],$$

since $\{W(t)\}_{t\geq 0}$ has independent increments. Since $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion, the increments $W(t_{i+1}) - W(t_i)$ are independent of $W(t_i)$, and have mean zero. Hence:

$$\mathbb{E}\left[\int_0^t g_1(r)dW(r)\right] = 0,$$

But

$$\begin{aligned}\mathbb{E}\left[\int_0^t g_2(r)dW(r)\right] &= \sum_{i\geq 0}\mathbb{E}\left[W(t_{i+1})\left(W(t_{i+1})-W(t_i)\right)\right] \\ &= \sum_{i\geq 0}\mathbb{E}\left[\left(W(t_{i+1})-W(t_i)\right)^2\right] = t,\end{aligned}$$

above the function g_1 is elementary while g_2 is not.

Lemma 1.12. *[107] Let $g, h \in \Lambda$ be piecewise functions. Then they have the properties that*

(i)
$$\mathbb{E}\left[\int_0^T g(t)dW(t)\right] = 0,$$

(ii)
$$\mathbb{E}\left[\left(\int_0^T g(t)dW(t)\right)^2\right] = \mathbb{E}\left[\int_0^T (g(t))^2\, dt\right],$$

(iii)
$$\mathbb{E}\left[\int_0^T g(t)dW(t)\int_0^T h(t)dW(t)\right] = \int_0^T \mathbb{E}\big[g(t)h(t)\big]dt.$$

Proof. See [35, 107]

For functions $f \in \mathcal{M}^2([0,T], \mathbb{R})$ we will now show how to define the Itô integral

$$\mathcal{I}(f) = \int_0^T f(t)dW(t),$$

where $W(t)$ is standard Brownian motion.

We define $\mathcal{I}(g_n)$ for a elementary functions g_n. Then we show that each $f \in \mathcal{M}^2([0,T], \mathbb{R})$, we approximate them by elementary processes $g_n(t)$. This means we can express any square-integrable processes $f(t)$ as a limit of elementary processes $g_n(t)$. In this case, the Itô integral for $f(t)$ is defined as:

$$\int_0^T f(t)\, dW(t) = \lim_{n\to\infty}\int_0^T g_n(t)\, dW(t).$$

This ensures that the Itô integral is well-defined for square-integrable functions.

The Itô integral has several important properties. We first observe the following:

Lemma 1.13. *[107] Let $f, f_1, f_2 \in \mathcal{M}^2([0,T],\mathbb{R})$, and $C_1, C_2 \in \mathbb{R}$. Then*

(i)

$$\int_0^T f(t)dW(t) \text{ is } \mathcal{F}_t\text{- measurable,}$$

(ii)

$$\mathbb{E}\left[\int_0^T f(t)dW(t)\right] = 0,$$

(iii)

$$\mathbb{E}\left[\left(\int_0^T f(t)dW(t)\right)^2\right] = \mathbb{E}\left[\int_0^T (f(t))^2\,dt\right], \quad \text{(Itô isometry),}$$

(vi)

$$\int_0^T \Big[C_1 f_1(t) + C_2 f_2(t)\Big] dW(t) = C_1 \int_0^T f_1(t)dW(t) + C_2 \int_0^T f_2(t)dW(t).$$

Theorem 1.21. *[107] Let $p \le 2$, $f \in \mathcal{M}^2([0,T],\mathbb{R})$ such that*

$$\mathbb{E}\int_0^T |f(t)|^p\,dt < \infty.$$

Then

$$\mathbb{E}\left|\int_0^T f(t)dW(t)\right|^p \le \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\int_0^T |f(t)|^p\,dt.$$

In particular, for $p = 2$, there is equality.

Proof. See [107].

Now, we proceed to define the stochastic integrals with stopping time. We observe that if τ is an $\{\mathcal{F}_t\}$-stopping time, then the process $\left\{1_{[0,\tau]}(t)\right\}_{t\le 0}$ is a bounded right continuous $\{\mathcal{F}_t\}$-adapted process.

Definition 1.58. [107] Let $f \in \mathcal{M}^2([0,T],\mathbb{R})$, and let τ be an $\{\mathcal{F}_t\}$-stopping time such that $0 \le \tau \le T$. Then, $\left\{1_{[0,\tau]}(t)f(t)\right\}_{0\le t\le T} \in \mathcal{M}^2([0,T],\mathbb{R})$. Thus, we define the stochastic integral with stopping time as:

$$\int_0^\tau f(t)dW(t) = \int_0^T 1_{[0,\tau]}(t)f(t)dW(t).$$

1.3.5 Fractional Brownian Motion

Fractional Brownian motion (FBM), introduced by Andrey Kolmogorov in 1940 [87], generalizes standard Brownian motion by incorporating memory effects and long-range dependence, characterized by the Hurst exponent. It was further studied by Yaglom in [176]. The term fractional Brownian motion was coined by Mandelbrot and Van Ness in 1968 [106], where they provided a stochastic integral representation of the process in terms of standard Brownian motion. For more details, see [4, 41, 42, 118, 136].

Definition 1.59. [106, 118] A fractional Brownian motion (FBM) $\{W^H(t)\}_{t\geq 0}$ of Hurst parameter $H \in (0,1)$, is a continuous and centered Gaussian process with covariance function

$$R_H(t,s) = \mathbb{E}(W^H(t)W^H(s)) := \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \quad \text{for} \quad t,s \geq 0.$$

Remark 1.6. For $H = \frac{1}{2}$, fractional Brownian motion becomes the standard Brownian motion.

By Definition 1.59 we obtain that a standard FBM $\{W^H(t)\}_{t\geq 0}$ has the following properties:

Now, we introduce the Wiener integral with respect to the one-dimensional FBM. Let $T > 0$ and denote by $\varLambda$ the linear space of $\mathbb{R}$- valued step functions on $[0,T]$, that is $\phi \in \varLambda$ if

$$\phi(t) = \sum_{i=1}^{m-1} y_i \mathcal{X}_{[t_i,t_{i+1})}(t), \quad \text{for all } t \in [0,T]\,,$$

where $\mathcal{X}_{[t_i,t_{i+1})}$ is the indicator function, $y_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdots t_m = T$.

For $\phi \in \varLambda$ we define its Wiener integral with respect to β^H by

$$\int_0^T \phi(s)d\beta^H(s) := \sum_{i=1}^{m-1} y_i\big(\beta^H(t_{i+1}) - \beta^H(t_i)\big).$$

Let $\mathbb{H}$ be the Hilbert space defined as the closure of $\varLambda$ with respect to the scalar product

$$\big(\mathcal{X}_{[0,t]}, \mathcal{X}_{[0,s]}\big)_{\mathbb{H}} = R_H(t,s).$$

Then the mapping

$$\phi = \sum_{i=1}^{m-1} y_i \mathcal{X}_{[t_i,t_{i+1})} \to \int_0^T \phi(s)d\beta^H(s),$$

(i) For $H > \frac{1}{2}$, the covariance of the FBM can be written as

$$R_H\left(t,s\right) = H\left(2H-1\right)\int_0^t\int_0^s \left|r-u\right|^{2H-2} du\, dr.$$

We can rewrite

$$\left|r{-}u\right|^{2H-2} = \frac{\left(ru\right)^{H-1/2}}{\beta\big(2-2H,\, H-1/2\big)}\cdot\int_0^{r\wedge u} v^{1-2H}\left(r-v\right)^{H-3/2}\left(u-v\right)^{H-3/2} dv, \quad (1.34)$$

with $\beta\left(z_1,z_2\right)=\Gamma\left(z_1\right)\Gamma\left(z_2\right)/\Gamma\left(z_1+z_2\right)$ ia the Beta function and $\Gamma\left(z\right)=\int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function.

Consider now the deterministic kernel

$$K_H(t,s) = C_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad (1.35)$$

where $C_H=\sqrt{H(2H-1)/\beta(2-2H,H-\frac{1}{2})}$, $t>s$. Then we have that

$$R_H(t,s) = \int_0^{t\wedge s} K_K(t,u)K_K(s,u)du,$$

since by (1.34) it follows that

$$\begin{aligned}\int_0^{t\wedge s} K_K(t,u)K_K(s,u)du =& C_H^2\int_0^{t\wedge s}\left(\int_u^t (x-u)^{H-\frac{3}{2}}x^{H-\frac{1}{2}}dx\right)\\ &\cdot\left(\int_u^s (y-u)^{H-\frac{3}{2}}y^{H-\frac{1}{2}}dy\right)u^{1-2H}\, du\\ =& C_H^2\int_0^t\int_0^s (xy)^{H-\frac{1}{2}}\left(\int_0^{t\wedge s} u^{1-2H}(x-u)^{H-\frac{3}{2}}(y-u)^{H-\frac{3}{2}}du\right)dy\, dx\\ =& C_H^2\beta\left(2-2H, H-\frac{1}{2}\right)\int_0^t\int_0^s (x-y)^{2H-2}\, dy\, dx = R_H(t,s).\end{aligned}$$

Note also that with a change of variable in (1.35), $K_K(t,s)$ can be expressed equivalently as

$$K_H(t,s) = C_H\left(t-s\right)_+^{H-1/2}\int_0^t u^{H-\frac{3}{2}}\left(1-\left(1-\frac{t}{s}\right)u\right)^{H-\frac{1}{2}} du.$$

For further details see [41, 42]

(ii) For $H < \frac{1}{2}$, the kernel

$$K_K(t,s) = C_H\left[\frac{t}{s}^{H-1/2}\left(t-s\right)^{H-1/2} - \left(H-\frac{1}{2}\right)s^{1/2-H}\int_s^t \left(u-s\right)^{H-1/2}u^{H-3/2}\right],$$

with $C_H = \sqrt{2H/\big((1-2H)\,\beta\,(1-2H, H+1/2)\,\big)}$, and $t > s$ satisfies

$$R_H(t,s) = \int_0^{t\wedge s} K_K(t,u)K_K(s,u)du. \tag{1.36}$$

For a detailed proof of equation (1.36), see [42] and the fact that

(iii) For $H = \frac{1}{2}$, we have $K_{\frac{1}{2}}(t,s) = \mathcal{X}_{[0,t]}(s)$,

In order to define the Wiener integrals with respect to W^H, we introduce the so-called reproducing kernel Hilbert space denoted by $\mathcal{H}$.

Definition 1.60. [22] The reproducing kernel Hilbert space, denoted by $\mathcal{H}$, associated to W^H for every $H \in (0,1)$, is defined as the closure of the vector space spanned by the set of functions $R_H(t,.)$, $t \in [0,T]$ with respect to the scalar product

$$\big(R_H(t,.), R_H(s,.)\big) = R_H(t,s), \qquad \forall t,s \in [0,T]\,. \tag{1.37}$$

In the case of standard Brownian motion, there exists a nice characterization of $\mathcal{H}$, which coincides with the space of absolutely continuous functions, vanishing at 0, with square-integrable derivative.

Consider the operator K_H induced by the kernel $K_H(t,s)$ on $L^2([0,T])$ for $H \geq \frac{1}{2}$ as follows:

$$\big(K_H f\big)(t) := \int_0^t K_H(t,s)f(s)ds.$$

is an isometry between Λ and the linear space span $\{\beta^H(t)\}_{t\in[0,T]}$, which can be extended to an isometry between $\mathbb{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{\beta^H(t)\}_{t\in[0,T]}$(see [163]). The image of an element $\phi \in \mathbb{H}$ by this isometry is called the Wiener integral of ϕ with respect to β^H. Our next goal is to give an explicit expression for this integral. To this end, we consider the square integrable kernel with $H \in \big(\frac{1}{2}, 1\big)$.

$$K_H(t,s) = C_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du,$$

where $C_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$, $t > s$ and $\beta(.,.)$ signifies the Beta function.

Observe that by representation for the square integrable kernel $K_H(t,s)$, we obtain

$$\frac{\partial K_H}{\partial t}(t,s) = C_H\bigg(\frac{t}{s}\bigg)^{\frac{1}{2}-H}(u-s)^{H-\frac{3}{2}}.$$

Now, we present the linear operator $K_H^*:\Lambda\to L^2([0,T])$, which is defined as follows:

$$\big(K_H^*\Upsilon\big)(s)=\int_s^T\Upsilon(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Then

$$\left(K_H^*\mathcal{X}_{[0,T]}\right)(s)=K_H(t,s)\mathcal{X}_{[0,t]}(s), \tag{1.38}$$

by equation (1.38) it follows that the operator (K_H^*) is an isometry between the space Λ of elementary functions and $L^2([0,T])$ that can be extended to the Hilbert space $\mathcal{H}$ (see [11]). This is because

$$\begin{aligned}\left(K_H^*1_{[0,t]},K_H^*1_{[0,s]}\right)_{L^2[0,T]}&\left(K_H\left(t,.\right)1_{[0,t]},K_H\left(s,.\right)1_{[0,s]}\right)_{L^2[0,T]}\\&=\int_0^{t\wedge s}K_H\left(t,u\right)K_H\left(s,u\right)du\\&=R_H\left(t,s\right)=\left(1_{[0,t]},1_{[0,s]}\right)_H.\end{aligned}$$

The operator K_H^* can be rewritten by using the means of fractional calculus.

Taking $\{W(t)\}_{t\in[0,T]}$ defined by

$$W(t)=\beta^H\big((K_H^*)^{-1}1_{[0,T]}\big),$$

W is a Brownian motion, β^H and has the Wiener integral form shown below

$$\beta^H(t)=\int_0^t K_H(t,s)dW(s).$$

Furthermore, for any $\phi\in\Lambda$,

$$\int_0^T\phi(s)d\beta^H(s)=\int_0^T(K_H^*\phi)(t)dW(t),$$

if and only if $K_H^*\phi\in L^2([0,T])$. Also

$$L^2_{\mathbb{H}}([0,T])=\big\{\phi\in\Lambda,\ \ K_H^*\phi\in L^2([0,T])\big\},$$

for all $H>\frac{1}{2}$ we can observe

$$L^{1/H}([0,T])\subset L^2_{\mathbb{H}}([0,T]).$$

see [110]. Furthermore, the following beneficial finding holds:

Lemma 1.14. *[120] For $\phi \in L^{\frac{1}{H}}([0,T])$*

$$H(2H-1)\int_0^T\int_0^T |\phi(s)||\phi(t)||t-s|^{2H-2}dtds \leq c_H\|\phi\|^2_{L^{\frac{1}{H}}([0,T])}. \tag{1.39}$$

Next we are interested in considering a FBM with values in a Hilbert space and giving the Definition of the corresponding stochastic integral.

Let $\phi \in L_b(\mathbb{V},\mathbb{H})$ be a non-negative self-adjoint operator. Defined by $L^0_\phi(\mathbb{V},\mathbb{H})$ the space of all $\xi \in L_b(\mathbb{V},\mathbb{H})$ such that $\xi\phi^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\xi|^2_{L^0_\phi(\mathbb{V},\mathbb{H})} = tr(\xi\phi\xi^*).$$

Let $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\mathcal{P})$. When one considers the following series

$$\sum_{i=1}^\infty \beta_i^H(t)e_i, \quad t \geq 0,$$

where $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis in $\mathbb{K}$ does not necessarily converge in the space $\mathbb{K}$ Thus, we consider a $\mathbb{H}$-valued stochastic process $W^H(t)$ given formally by the following series:

$$W^H(t) = \sum_{i=1}^\infty \beta_i^H(t)\phi^{\frac{1}{2}}e_i, \quad t \geq 0,$$

which is well-defined as a $\mathbb{V}$-valued ϕ-cylindrical fractional Brownian motion. Let ϕ : $[0,T] \mapsto L^0_\phi(\mathbb{V},\mathbb{H})$ such that

$$\sum_{i=1}^\infty \|K_H^*(\upsilon\phi^{\frac{1}{2}}e_i)\|_{L^{\frac{1}{H}}([0,T],\mathbb{H})} < \infty. \tag{1.40}$$

Definition 1.61. [36] Let $\upsilon : [0,T] \to L^0_\phi(\mathbb{V},\mathbb{H})$ be a given function, satisfy (1.40). The stochastic integral of υ with respect to W^H is defined by

$$\int_0^t \upsilon(s)dW^H(s) := \sum_{i=1}^\infty \int_0^t \upsilon(s)\phi^{\frac{1}{2}}e_i d\beta_i^H(s) = \sum_{i=1}^\infty \int_0^t \left(K_H^*(\upsilon\phi^{\frac{1}{2}}e_i)\right)(s)dW(s).$$

Notice that if

$$\sum_{i=1}^\infty \|\upsilon(s)\phi^{\frac{1}{2}}e_i\|_{L^{\frac{1}{H}}(([0,T]),\mathbb{H})} < \infty, \tag{1.41}$$

then in particular (1.40) holds, which follows immediately from (1.39).

Lemma 1.15. *[25] If $H \in (\frac{1}{2}, 1)$, then for any $\phi : [0,T] \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ satisfies*

$$\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0(\mathbb{V},\mathbb{H})} ds < \infty,$$

then the series in (1.41) is well defined as a $\mathbb{H}-$valued random variable and we have

$$\mathbb{E}\left\| \int_0^t \phi(s) dW^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0(\mathbb{V},\mathbb{H})} ds.$$

1.3.6 Stochastic differential equations

In this section, we introduce the concept of a Stochastic Differential Equation (SDE)

We consider semilinear SDEs on $[0, T]$ in $\mathbb{H}$. The general form of such SDE is

Problem 1.1. Consider the simple population growth model

$$\frac{dN}{dt} = a(t)N(t), \quad N(0) = N_0 \ \text{(constant)} \tag{1.42}$$

where $N(t)$ is the size of the population at time t, and $a(t)$ is the relative rate of growth at time t. It might happen that $a(t)$ is not completely known, but subject to some random environmental efiects, so that we have

$$a(t) = r(t) + \text{ "noise" },$$

where we do not know the exact behaviour of the noise term, only its probability distribution. The function $r(t)$ is assumed to be nonrandom. How do we solve (1.42) in this case?

We now turn to the question of flnding a reasonable mathematical interpretation of the "noise" term in the equation of **Problem** 1.1 in the Introduction:

$$\frac{dN}{dt} = \Big(r(t) + \text{ "noise" }\Big)N(t),$$

or more generally in equations of the form

$$\frac{dN}{dt} = \mu(t, X_t) + \sigma(t, X_t). \text{ “noise”}, \tag{1.43}$$

where b and σ are some given functions. Let us first concentrate on the case when the noise is 1-dimensional. It is reasonable to look for some stochastic process $W(t)$ to represent the noise term, so that

$$\frac{dN}{dt} = \mu(t, X_t) + \sigma(t, X_t).W(t). \tag{1.44}$$

Based on many situations, for example in engineering, one is led to assume that $W(.)$ has, at least approximately, these properties:

(i) $t_1 \neq t_2 \Longrightarrow W_{t_1}$ and W_{t_2} are independent.

(ii) $\{W_t\}$ is stationary, i.e. the (joint) distribution of $\{W_{t_1+t}, \cdots, W_{t_k+t}\}$ does not depend on t.

(iii) $\mathbb{E}[W_t] = 0$ for all t.

Definition 1.62. Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $\sigma(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function

$$\omega \to \sigma(t,\omega)$$

is $\mathcal{N}_t$-measurable.

We now describe our class of functions for which the Itô integral will be defined: [40] To this purpose, stochastic differential equations (SDEs)

$$\begin{aligned} dX =& \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \quad t \geq 0, \\ X(0) =& X_0, \end{aligned} \tag{1.45}$$

are characterized by a right-hand side depending on two terms:

the function $\mu : \mathbb{R}^d \to \mathbb{R}^d$, well-known as drift of the problem, that is the coefficient of its deterministic part;

the function $\mu : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, denoted in the literature as diffusion of the problem, that is the coefficient of its stochastic part.

The term. $W(t)$ in (1.45) is a m-dimensional standard Wiener process and, due to its nowhere differentiability (with probability 1), the representation given in (1.45) is only a shorthand notation for its integral counterpart

$$X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) \tag{1.46}$$

If the stochastic integral in the right-hand side of (1.46) is the Itô integral.

1.4 Numerical Methods for Stochastic Differential Equations

In this section, we introduce some numerical methods originally developed for deterministic differential equations and later extended to stochastic differential equations. We focus on the fundamental issues related to accuracy, stability, and conservation in the numerical dynamics used in this thesis. For further reading, comprehensive references such as [17, 32, 40, 44, 91, 171, 183], among others, are useful.

we aim to provide a discretization of the Wiener process. $W(t)$ evaluated in a discrete set of points in .$[0, T]$. Therefore, we provide a partition of the interval .$[0, T]$ in N subintervals of equal length $h = \frac{T}{N}$ and the corresponding set of sampled values is then given by

$$\Big\{W_0 = W(t_0) = 0\Big\} \cup \Big\{W_j = W(t_j),\ \ j = 1, \cdots, N\Big\},$$

with $t_j = j.h, j = 1, \cdots, N$. As a consequence, according to Definition 1.54, the following recursion is established

$$\begin{aligned} W_0 =& 0, \\ \Delta W_j =& W_{j+1} - W_j,\ \ j = 1, \cdots, N-1, \end{aligned}$$

with $\Delta W_j \sim \mathcal{N}(0, h)$.

We now aim to define the integral with respect to a Wiener process, in analogous way as in the case of deterministic Riemann integration. In other terms, for a given scalar function $f : [0, T] \to \mathbb{R}$, we aim to define the integral

$$\mathcal{I}(f) = \int_0^T f(s) dW(s). \tag{1.47}$$

The construction of this integral is now provided in two steps. We first define (1.47) when f is a step function

$$f(t) = f_j \in \mathbb{R},\ \ t \in [t_j, t_{j+1}),\ \ j = 1, \cdots, N-1,$$

In this case, we define the Itô integral of the step function f by

$$\mathcal{I}(f) = \sum_{j=1}^{N-1} f_j \big(W_{j+1} - W_j\big),$$

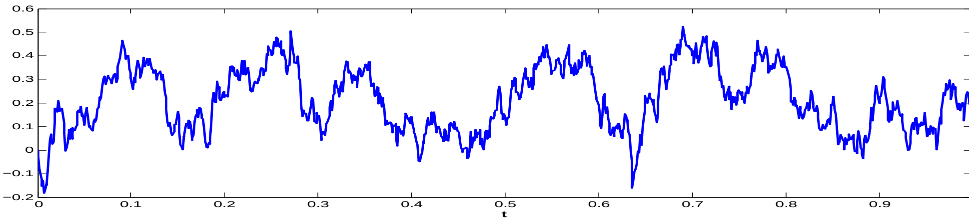


Figure 1.1: An illustration of the sample path of Brownian motion generated by the cumulative summation of increments.

with $0 = t_0 < t_1 \cdots < t_N = T$.

In other terms, given a discretization of the integration interval $0 = t_0 < t_1 \cdots < t_N = T$, the Itô integral of a continuous function can be approximated by the following Itô quadrature formula

$$\int_0^T v(s)dW(s) \approx \sum_{j=1}^{N-1} v_j\big(W_{j+1} - W_j\big),$$

involving a linear combination of values of the integrand function evaluated in the left-hand endpoint of each subinterval of the domain discretization.

An alternative definition of integral with respect to a Brownian motion is given by the so-called Stratonovich integral, defined as the limit of the following quadrature formula

$$\int_0^T v(s) \circ dW(s) \approx \sum_{j=1}^{N-1} v\left(\frac{t_j + t_{j+1}}{2}\right) \cdot \big(W_{j+1} - W_j\big),$$

and is usually denoted by the symbol $\circ$ close to $.dW(s)$ in the integral. Unlike Itô case, Stratonovich quadrature formula involves evaluations of the integrand function in the midpoint of each subinterval of the domain discretization.

1.4.1 Euler-Maryama Method

In this subsection, we present the Euler-Maruyama method, a numerical technique used to approximate the solution of Stochastic Differential Equations (SDEs). The method is an adaptation of the Euler method, originally developed for deterministic differential equations, and was later extended to stochastic processes by Gisiro Maruyama in 1955 (see, for instance, [40, 73] the references therein).

Now, we propose studying the stochastic differential equations (1.45) over the closed interval $[0, T]$ and partitioning it into N subintervals of equal length.

$$\Delta t = \frac{T}{N},$$

intercepting the corresponding set of grid points

$$I_{\Delta t} = \Big\{ t_j = j.\Delta t, \;\; j = 0, 1, \cdots, N \Big\}. \tag{1.48}$$

Let us now understand how to advance from a given point .tn to the subsequent point $.t_{n+1}$. To this purpose, we first consider the integral formulation (1.46) for $t \geq t_n$, i.e.,

$$X(t) = X(t_n) + \int_{t_n}^{t} f\left(X(s)\right) ds + \int_{t_n}^{t} g\left(X(s)\right) dW(s).$$

Let us truncate both equations to the first order, i.e., approximate

$$f\left(X(s)\right) \approx f\left(X(t_n)\right), \quad g\left(X(s)\right) \approx g\left(X(t_n)\right),$$

and replace these values in (1.46) obtainin

$$\begin{aligned} X(t) \approx & X(t_n) + \int_{t_n}^{t} f\left(X(t_n)\right) ds + \int_{t_n}^{t} g\left(X(t_n)\right) dW(s) \\ \approx & X(t_n) + f\left(X(t_n)\right)(t - t_n) + g\left(X(t_n)\right)\left(W(t) - W(t_n)\right). \end{aligned}$$

We finally evaluate last relation for $t = t_{n+1}$, leading to

$$X(t_{n+1}) \approx X(t_n) + f\left(X(t_n)\right) \Delta t + g\left(X(t_n)\right) \Delta W_{n+1},$$

having denoted $\Delta W_{n+1} = W(t_{n+1}) - W(t_n)$. Finally, defining $X_n :\approx X(t_n)$ yields

$$X_{n+1} = X_n + f\left(X_n\right) \Delta t + g\left(X_n\right) \Delta W_{n+1}, \tag{1.49}$$

To conclude, we note that, like deterministic methods, additional formulas for approximating SDEs (1.45) can be derived by including higher-order Itô-Taylor terms. An example is the Milstein method, given for scalar problems by:

$$X_{n+1} = X_n + f(X_n)\Delta t + g(X_n)\Delta W_{n+1} + \frac{1}{2}g'(X_n)g(X_n)\left(\Delta W_{n+1}^2 - \Delta t\right), \tag{1.50}$$

1.4.2 θ-Euler-Maryama Methods

In this part of the thesis, we discuss the θ-Euler-Maruyama (θ-EM) method is a generalization of the Euler-Maruyama(EM) method, with the parameter θ. The family of θ-EM methods shares the same diffusion term as the EM method (1.49) and is derived from the following quadrature formula to approximate the drift term of the equation:

$$\int_{t_n}^{t_{n+1}} f(X(t))\,dt \approx \left((1-\theta) f(X(t_n)) + \theta f(X(t_{n+1}))\right)\Delta t,$$

with $\theta \in [0,1]$. In other terms, the integral of the drift in $[t_n, t_{n+1}]$ is a convex combination of the value of $f(X(t_n))$ and $f(X(t_{n+1}))$, equivalently, the integrand is approximated by a linear interpolant. The corresponding numerical method is then given by

$$X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + g(X_n)\Delta W_{n+1}. \tag{1.51}$$

This family consists of one-step methods parameterized by $\theta \in [0,1]$. When $\theta = 0$, it reduces to the Euler-Maruyama method (1.49), which is the only explicit θ-Maruyama method. For $\theta \neq 0$, the methods (1.51) are implicit, and two key cases deserve attention:

(i) for $\theta = \frac{1}{2}$ we obtain the stochastic trapezoidal method

$$X_{n+1} \approx X_n + \left(f(X_n) + f(X_{n+1})\right)\frac{\Delta t}{2} + g(X_n)\Delta W_{n+1}, \tag{1.52}$$

(ii) for $\theta = 1$ we obtain the implicit Euler-Maruyama method

$$X_{n+1} = X_n + f(X_{n+1})\Delta t + g(X_n)\Delta W_{n+1}. \tag{1.53}$$

We also observe that if the diffusion $g = 0$ in (1.51), the stochastic θ-Maruyama method perfectly overlaps with the family of deterministic θ-methods.

$$X_{n+1} = X_n + \left((1-\theta) f(X_n) + \theta f(X_{n+1})\right)\Delta t.$$

We conclude this part by presenting the family of stochastic θ-Milstein method developed in [32] that, for the scalar case, reads

$$\begin{aligned} X_{n+1} = X_n + (1-\theta)\, \Delta t f\left(X_n\right) + \theta \Delta t f\left(X_{n+1}\right) + g\left(X_n\right) \Delta W_{n+1} \\ + \frac{1}{2} g'\left(X_n\right) g\left(X_n\right) \left(\Delta W_{n+1}^2 - \Delta t\right), \end{aligned} \tag{1.54}$$

with $\theta \in [0,1]$, sharing the diffusion part with Milstein method (1.50).

1.4.3 Example

Let's start with the following equation:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t). \tag{1.55}$$

Which is commonly used to model stock prices, this model is known as the Black-Scholes model. From this, it follows that the model (1.55) can be rewritten as:

$$\int_0^t \frac{dX(t)}{X(t)} = \mu t + \sigma W(t).$$

By applying Itô's Lemma to the function $g(t,x) = \ln(x)$, we obtain:

$$d(\ln X_t) = \frac{dX(t)}{X(t)} - \frac{1}{2}\mu^2 dt = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t).$$

The corresponding solution can be expressed explicitly as follows:

$$X(t) = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \alpha W(t)\right).$$

For $\sigma = 0$, we recover the classical deterministic solution. The solution (1.55) is also called a geometric Brownian motion.

A solution to an SDE is itself a stochastic function, which means that its value $W(t)$ at any given time t is a random variable. We'll begin by simulating four sample paths for $X(t)$ to illustrate this point. Sample paths differ because of different realizations of the Brownian motion term. We first discretize a given time interval $[0,T]$ into N chunks: $0 = t_0 < t_1 < \cdots < t_N = T$, and compute the value of X at each point t_i. For convenience, we'll assume that the gap between each time point is the same (i.e., $t_{n+1} - t_n = \Delta t$ for all n, although this isn't necessary. To compute the value of W at those same points, we first construct a sample path of increments $\Delta W(t_n) = W(t_{n+1}) - W(t_n)$ by recognizing that an increment is distributed according to $\sqrt{\Delta t} \cdot N(0,1)$ (this is an important property of Brownian motion). We can then create the sample path of Brownian motion W by summing the increments, and compute the exact solution X using the analytical solution above. We can rewrite the integral form of Geometric Brownian Motion (GBM) as:

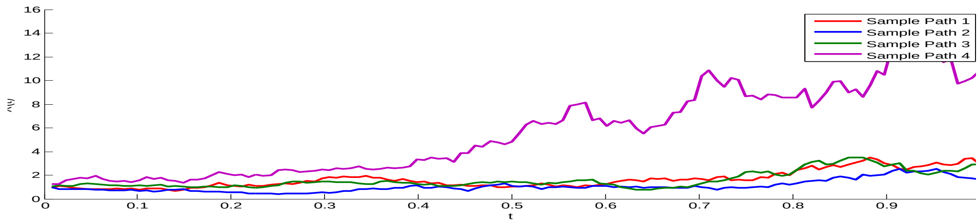


Figure 1.2: Sample Solution Paths for Geometric Brownian Motion.

$$X(t_{n+1}) - Y(t_n) = \mu \int_{t_n}^{t_{n+1}} X(s)\, ds + \sigma \int_{t_n}^{t_{n+1}} X(s)\, dW(s)$$

The Euler-Maruyama method approximates this integral by discretizing the time step, and we can express the update as:

$$X_{n+1} - X_n = \mu X_n \Delta t + \sigma X_n \Delta W_n.$$

Where

μ is the drift term (rate of growth), σ is the diffusion term (volatility), Δt is the time step, ΔW_n is the increment in Brownian motion (typically modeled as $\sqrt{\Delta t} \cdot Z_n$, where $Z_n \sim N(0,1)$).

This method approximates the continuous-time process in discrete steps, and its accuracy depends on the time step Δt.

Here is an example for simulating the Geometric Brownian Motion (GBM) using the Euler-Maruyama method, and comparing it to the exact solution.

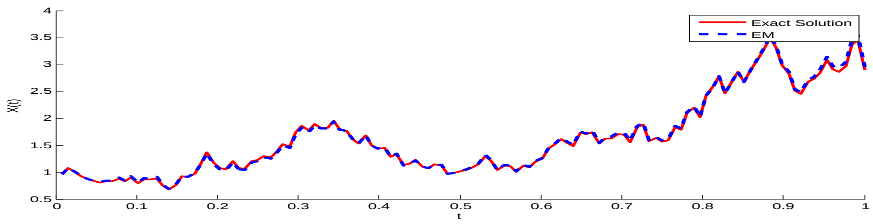


Figure 1.3: Euler-Maruyama Approximation vs Exact Solution.

Chapter 2

Exact controllability results of NII stochastic integro-differential equations driven by a FBM

In this chapter, we present the main results obtained in our article [113].

2.1 Introduction

This chapter is concerned with the existence of mild solution and the exact controllability for a class of SIDEs driven by a fractional Brownian motion (FBM) accompanied by nonlocal conditions and non-instantaneous impulsive (NII). The system under discussion takes the following form:

$$\begin{cases} dy(t) = Ay(t)dt + \int_0^t h(t-s)y(t)dsdt \\ \qquad\qquad +\varphi(t, y(t))dt + F(t)dW^H(t), \quad t \in \cup_{k=0}^{N}(s_k, t_{k+1}], \\ y(t) = G_k(t, y(t_k^-)), \quad t \in \cup_{k=1}^{N}(t_k, s_k], \\ \\ y(0) + \psi(y) = y_0 \in \mathbb{H}. \end{cases} \tag{2.1}$$

Where A is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of bounded linear operators on a separable Hilbert space $\mathbb{H}$ with domain $\mathbb{D}(A)$, $h : \mathbb{D}(h) \to \mathbb{H}$ is a closed linear operator on $\mathbb{H}$ with domain $\mathbb{D}(A) \subset \mathbb{D}(h)$, φ, ψ and $F : [0, T] \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ are appropriate functions, where $\mathcal{L}_2^0(\mathbb{V}, \mathbb{H})$ denotes the space of all Hilbert-

Schmidt operators from $\mathbb{V}$ into $\mathbb{H}$. Also, $B^H(t)$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with values on a separable Hilbert space $\mathbb{V}$.

Consider a sequence of time intervals $0 = s_0 = t_0 < t_1 \leq s_1 < t_2 \leq ... < t_N \leq s_N < t_{N+1} = T$, where $T > 0$ is a constant. For each $k = 1, 2, ..., N$, the function $G_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is called non-instantaneous impulsive function. Here, the state $y(\cdot)$ takes values in a real separable Hilbert space $\mathbb{H}$, equipped with an inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. y_0 is an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|y_0\|^2 < \infty$.

Recently, controllability plays a significant role in various fields such as engineering, physics, robotics to economics and social sciences. The most commonly used types of controllability are exact and approximate controllability. This concept was first proposed by Kalman [82] in 1963, and since then, both the theory of stochastic processes and differential equations have greatly benefited from its application (see references [4, 49, 51, 98, 125, 136] and its allusions).

However, the study of the exact controllability of non-instantaneous stochastic integro-differential equations driven by fractional brownian has not been addressed in the standard literature. Motivated by the above consideration, the purpose of this chapter is to investigate the existence of mild solution and the controllability of a class of non-instantaneous stochastic integro-differential equations driven by fractional brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ and nonlocal Conditions. The present chapter is an extension of the work of Melati et al. [125] to the non-instantaneous stochastic integro-differential equations driven by fractional brownian with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. Using certain assumptions, sufficient conditions are derived using an extended version of Darbo's fixed point theorem, resolvent operator theory and the measure of non-compactness technique to analyze the controllability result.

We offer the following summary of the main result of this chapter:

- The paper proposes a new class of stochastic integro-differential equations driven by fractional Brownian motion with non-instantaneous impulsive and nonlocal conditions.
- We investigate the existence of mild solutions and the exact controllability for system

(2.1) using measures of non-compactness and applying the generalized Darbo's fixed point theorem.

- We establish a sufficient condition for the existence of mild solutions and the exact controllability of the system (2.1).

- We reinforce the theoretical results with illustrative examples.

2.2 Existence of mild solution

Let $(\mathbb{V}, (.,.)_{\mathbb{V}}, \|.\|_{\mathbb{V}})$, and $(\mathbb{H}, (.,.)_{\mathbb{H}}, \|.\|_{\mathbb{H}})$, be real separable Hilbert spaces. We denote by $L_b(\mathbb{V}, \mathbb{H})$ the space of all bounded linear operators from $\mathbb{V}$ to $\mathbb{H}$ and $L_b(\mathbb{H})$ whenever $\mathbb{V} = \mathbb{H}$, $\mathcal{C}(\mathbb{R}^+, \mathbb{V})$ indicate the space of all continuous functions from $[0, +\infty)$ into $\mathbb{V}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\{W^H(t)\}_{t\geq 0}$ are the fractional Brownian motion with Hurst parameter $H \in (0,1)$.

Let $L^2\big(\Omega, \mathbb{H}\big)$ be the space of all $\mathcal{F}_t$ -measurable $\mathbb{H}$-valued random variable y such that $\mathbb{E}\|y\|^2 = \int_\Omega \|y\|^2 d\mathcal{P} < \infty$. For $y \in L^2\big(\Omega, \mathbb{H}\big)$,

$$\|y\|_{L^2\left(\Omega, \mathbb{H}\right)} = \left(\int_\Omega \|y\|^2 d\mathcal{P}\right)^{\frac{1}{2}} := \left(\mathbb{E}\|y\|^2\right)^{\frac{1}{2}}.$$

It is clear that $L^2\big(\Omega, \mathbb{H}\big)$ is a Hilbert space with the norm $\|.\|_{L^2\left(\Omega,\mathbb{H}\right)}$. In the sequel, $L^2_0\big(\Omega, \mathbb{H}\big)$ denotes the space of $\mathcal{F}_0$-measurable, $\mathbb{H}$-valued and square integrable stochastic process.

$$L^2_0\big(\Omega, \mathbb{H}\big) = \Big\{ f \in L^2\big(\Omega, \mathbb{H}\big) \ \mid \ f \ \text{ is } \mathcal{F}_0\text{-measurable} \Big\}.$$

Consider the Banach space

$$\begin{aligned}\mathcal{PC}([0,T], \mathbb{H}) = \Big\{ & \mathcal{F}_t\text{-adapted } \mathbb{H}\text{-valued process } y(t) \text{ is continuous every where except} \\ & \text{for some } t \neq t_k \text{ at which } y(t_k^-) \text{ and } y(t_k^+) \text{ exist} \\ & \text{and } y(t_k^-) = y(t_k), \ \ k = 1, 2, \cdots, N \text{ and } \sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2 < \infty \Big\},\end{aligned}$$

with the norm

$$\|y\|_{\mathcal{PC}} = \left(\sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2\right)^{\frac{1}{2}}.$$

In this part, we establish the existence of mild solutions for the system (2.1) by applying the generalized Darbo fixed point theorem. We now introduce the concept of a mild solution for equation (2.1) and present the following definitions.

Definition 2.1. A $\mathcal{F}_t$-adapted stochastic process $y(t) : [0, T] \to \mathbb{H}$ is called a mild solution of(2.1) if $y \in \mathcal{PC}([0,T], \mathbb{H})$, $y(0) + \psi(y) = y_0$ and

$$y(t) = \begin{cases} \mathcal{R}(t)\big(y_0 - \psi(y)\big) + \int_0^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s), & t \in [0, t_1], \\ G_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \mathcal{R}(t-s_k)G_k(s_k, y(t_k^-)) + \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{2.2}$$

The following hypotheses will be required throughout the paper:

(H_3) The resolvent operator $\mathcal{R}(t)$, for $t \geq 0$, is continuous with respect to the operator norm topology, and there exists a constant $M > 0$ such that

$$\|\mathcal{R}(t)\| \leq M.$$

(H_4) The nonlinear function $\varphi : J \times \mathcal{PC}([0,T], \mathbb{H}) \to \mathbb{H}$ satisfies

(1) For each $y \in \mathcal{PC}([0,T], \mathbb{H})$, $\varphi(.,y)$ is measurable and for any $t \in I$, $\varphi(t,.)$ is continuous.

(2) For some positive number $r > 0$, there exists a constant $k_1 > 0$, function $\omega \in L^1(J, \mathbb{R}^+)$ and a continuous non-decreasing function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathbb{E}\|\varphi(t,y)\|^2 \leq \omega(t)\pi_\varphi\big(\mathbb{E}\|y\|^2\big), \quad \liminf_{r\to+\infty} \frac{\pi_\varphi(r)}{r} = k_1 < +\infty.$$

(3) There exists a positive constant k_2 such that for any bounded set $D \subset \mathbb{H}$

$$\alpha\big(\varphi(t, D)\big) \leq k_2 \alpha(D).$$

(H_5) The function $F : [0,T] \to \mathcal{L}_2^0(\mathbb{V},\mathbb{H})$, satisfying the following condition

$$\int_0^t \|F(s)\|^2_{\mathcal{L}_2^0} ds < \infty.$$

(H_6) The nonlocal function $\psi : \mathcal{PC}([0,T],\mathbb{H}) \to \mathbb{H}$ is continuous and compact, and there exists a constant $C_\psi > 0$, such that

$$\mathbb{E}\|\psi(y)\|^2 \leq C_\psi.$$

(H_7) The impulsive function $G_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is continuous and compact, and there exist constants $C_{G_k} > 0$, $k = 1,2,\cdots,N$, such that

$$\mathbb{E}\| G_k(t,y)\|^2 \leq C_{G_k}\mathbb{E}\|y\|^2.$$

where we have used the notation

$$C_G = \max_{k=1,2,\cdots,N} C_{G_k}, \qquad \eta = \max_{k=1,2,\cdots,N} \|\omega\|_{L^1[s_k,t_{k+1}]}.$$

Remark 2.1. The function f is independent of $y(t)$, $t \in [0,T]$. From the functional point of view, we know that

$$\alpha\Big(\int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big) = 0.$$

Theorem 2.1. *Suppose that (H_1)-(H_7) are satisfied, then the problem (2.1) has at least one mild solution provided that*

$$L := \max_{0\leq k\leq N}\Big\{3M^2\big(C_G + (t_{k+1}-s_k)\,\eta k_1\big)\Big\} < 1. \tag{2.3}$$

Proof. Consider the operator $\Pi : \mathcal{PC}([0,T],\mathbb{H}) \to \mathcal{PC}([0,T],\mathbb{H})$ defined by

$$\Psi y(t) = \begin{cases} \mathcal{R}(t)\big(y_0 - \psi(y)\big) + \int_0^t \mathcal{R}(t-s)\varphi\big(s,y(s)\big)ds \\ \qquad + \int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s), & t \in [0,t_1], \\ G_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \mathcal{R}(t-s_k)G_k(s_k, y(t_k^-)) + \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s,y(s)\big)ds \\ \qquad + \int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{2.4}$$

Further, finding the solution of the operator equation $\Psi y(t) = y(t)$ leads us to find a solution to problem (2.1).

Now, we will show that by using the generalized Darbo's fixed point theorem, the operator Ψ has a fixed point. Obviously, the fixed point of $\Psi y(t)$ is the solution of the problem (2.1). For each finite constant $r > 0$, let

$$\Omega_r = \Big\{ y \in \mathcal{PC}([0,T],\mathbb{H}) : \|y\|^2_{\mathcal{PC}} \leq r \Big\},$$

is a closed bounded and convex set in $\mathcal{PC}([0,T],\mathbb{H})$.

The proof falls naturally into four steps.

Step 1. We claim that there exists a positive number r such that $\Psi(\Omega_r) \subset \Omega_r$. If this is not true, then, for each positive integer r, there exists $y_r \in \Omega_r$ such that for $t \in [0,T], t$ may depending upon r. However, on the other hand, we consider three cases.

Case I. For $t \in [0,t_1]$ by (2.4) and assumptions

$$\begin{aligned}\mathbb{E}\Big\|\Psi y_r(t)\Big\|^2 =& \mathbb{E}\Big\|\mathcal{R}(t)\big(y_0 - \psi(y_r)\big) + \int_0^t \mathcal{R}(t-s)\varphi\big(s, y_r(s)\big)ds + \int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big\|^2 \\ \leq& 3\mathbb{E}\Big\|\mathcal{R}(t)\big(y_0 - \psi(y_r)\big)\Big\|^2 + 3\mathbb{E}\Big\|\int_0^t \mathcal{R}(t-s)\varphi\big(s, y_r(s)\big)ds\Big\|^2 \\ &+ 3\mathbb{E}\Big\|\int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big\|^2,\end{aligned} \tag{2.5}$$

by using assumptions (H_3)-(H_6), Lemma 1.15 and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Psi y_r(t)\|^2 \leq& 3M^2\mathbb{E}\Big\|\big(y_0 - \psi(y_r)\big)\Big\|^2 + 3M^2 t_1 \int_0^t \mathbb{E}\Big\|\varphi\big(s, y_r(s)\big)\Big\|^2 ds \\ &+ 6M^2 H t_1^{2H-1} \int_0^t \|F(s)\|^2_{\mathcal{L}_2^0} ds \\ \leq& 3M^2\big(\mathbb{E}\|y_0\|^2 + \mathbb{E}\|\psi(y_r)\|^2\big) + 3M^2 t_1 \int_0^t \omega(s)\pi_\varphi\big(\mathbb{E}\|y_r\|^2\big)ds \\ &+ 6M^2 H t_1^{2H-1} \int_0^t \|F(s)\|^2_{\mathcal{L}_2^0} ds \\ \leq& 3M^2\big(\mathbb{E}\|y_0\|^2 + C_\psi\big) + 3M^2 t_1 \|\omega\|_{L^1[0,t_1]}\pi_\varphi(r) + 6M^2 H t_1^{2H-1} \int_0^t \|F(s)\|^2_{\mathcal{L}_2^0} ds.\end{aligned} \tag{2.6}$$

Case II. For $t \in (t_k, s_k], k = 1 \cdots, N$. By assumption (H_7), we get

$$\begin{aligned}\mathbb{E}\|\Psi y_r(t)\|^2 \leq & \mathbb{E}\|\, G_k(t, y_r(t_k^-))\|^2 \\ \leq & C_{G_k}\mathbb{E}\|y_r(t)\|^2 \\ \leq & C_G r.\end{aligned} \tag{2.7}$$

Case III. For $t \in (s_k, t_{k+1}], k = 1 \cdots, N$. By Lemma 1.15, and using assumptions (H_3)-(H_5), we have

$$\begin{aligned}\mathbb{E}\Big\|\Psi y_r(t)\Big\|^2 \leq & 3\mathbb{E}\Big\|\mathcal{R}(t-s_k)G_k(s_k, y_r(t_k^-))\Big\|^2 + \mathbb{E}\Big\|\int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y_r(s)\big)ds\Big\|^2 \\ & + 3\mathbb{E}\Big\|\int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big\|^2 \\ \leq & 3M^2\mathbb{E}\Big\|G_k(s_k, y_r(t_k^-))\Big\|^2 + 3M^2\left(t_{k+1}-s_k\right)\int_{s_k}^t \mathbb{E}\Big\|\varphi\big(s, y_r(s)\big)\Big\|^2 ds \\ & + 6M^2H\left(t_{k+1}-s_k\right)^{2H-1}\int_{s_k}^t \|F(s)\|^2_{\mathcal{L}_2^0}ds \\ \leq & 3M^2\mathbb{E}\Big\|G_k(s_k, y_r(t_k^-))\Big\|^2 + 3M^2\left(t_{k+1}-s_k\right)\int_{s_k}^t \omega(s)\pi_\varphi\big(\mathbb{E}\|y_r\|^2\big)ds \\ & + 6M^2H\left(t_{k+1}-s_k\right)^{2H-1}\int_{s_k}^t \|F(s)\|^2_{\mathcal{L}_2^0}ds \\ \leq & 3M^2C_G r + 3M^2\left(t_{k+1}-s_k\right)\|\omega\|_{L^1[s_k,t_{k+1}]}\pi_\varphi(r) \\ & + 6M^2H\left(t_{k+1}-s_k\right)^{2H-1}\int_{s_k}^t \|F(s)\|^2_{\mathcal{L}_2^0}ds,\end{aligned} \tag{2.8}$$

from (2.6), (2.7) and (2.8), we divide by r and take the lower bound as $r \to +\infty$, we have

$$1 < \mathbb{E}\Big\|\Psi y_r(t)\Big\|^2 \leq L,$$

with

$$1 < L := \max_{0\leq k\leq N}\Big\{3M^2\big(C_G + \left(t_{k+1}-s_k\right)\eta k_1\big)\Big\}.$$

Which contradict with condition (2.3), hence $\Psi(\Omega_r) \subset \Omega_r$.

Step 2. We prove that the operator Ψ is continuous in Ω_r.

Let us consider a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{PC}([0,T], \mathbb{H})$ such that $\lim_{n\to+\infty} y_n = y \in \mathcal{PC}([0,T], \mathbb{H})$.

By Hölder's inequality, Lemma 1.15 and using (H_3)-(H_7), we have

$$\lim_{n\to+\infty}\varphi(s, y_n(s)) = \varphi(s, y(s)), \tag{2.9}$$

$$\lim_{n\to+\infty}\psi(y_n)=\psi(y), \tag{2.10}$$

$$\lim_{n\to+\infty}G_k(s,y_n(t_k^-))=G_k(s,y(t_k^-)). \tag{2.11}$$

By assumption (H_4), for a.e $s\in[0,T]$, we obtain

$$\begin{aligned}\mathbb{E}\Big\|\varphi(s,y_n(s))-\varphi(s,y(s))\Big\|^2 \leq& 2\mathbb{E}\|\varphi(s,y_n(s))\|^2+2\mathbb{E}\|\varphi(s,y(s))\|^2\\ \leq& 4\omega(s)\pi_\varphi\big(r\big).\end{aligned} \tag{2.12}$$

Case I. For $t\in[0,t_1]$, using the fact that the function $s\mapsto\omega(s)\pi_\varphi\big(r\big)$ is Lebesgue integrable for $s\in[0,T]$ and $t\in[0,t_1]$ so by (2.9), (2.10), (2.12), Lemma 1.15 and the Lebesgue dominated convergence theorem, we see that

$$\begin{aligned}\mathbb{E}\|\Psi y_n(t)-\Psi y(t)\|^2 \leq& 2M^2\mathbb{E}\|\psi(y_n)-\psi(y)\|^2\\ &+2M^2\int_0^t\mathbb{E}\Big\|\varphi(s,y_n(s))-\varphi(s,y(s))\Big\|^2ds\\ &\longrightarrow 0 \quad \text{as} \qquad n\to+\infty.\end{aligned}$$

Case II. For $t\in(t_k,s_k]$, $k=1,2,\cdots,N$, by (2.11), we get

$$\mathbb{E}\|\Pi y_n(t)-\Pi y(t)\|^2 \leq \mathbb{E}\|G_k(s,y_n(t_k^-))-G_k(s,y(t_k^-))\|^2\longrightarrow 0 \quad \text{as} \quad n\to+\infty.$$

Case III. For $t\in(s_k,t_{k+1}]$, $k=1,2,\cdots,N$, by Lemma 1.15,(2.9), (2.11), (2.12) and the Lebesgue dominated convergence theorem, we can deduce that

$$\begin{aligned}\mathbb{E}\|\Psi y_n(t)-\Psi y(t)\|^2 \leq& 2M^2\mathbb{E}\|G_k(s,y_n(t_k^-))-G_k(s,y(t_k^-))\|^2\\ &+2M^2\int_{s_k}^t\mathbb{E}\Big\|\varphi(s,y_n(s))-\varphi(s,y(s))\Big\|^2ds\\ &\longrightarrow 0 \quad \text{as} \qquad n\to+\infty.\end{aligned}$$

Thus

$$\mathbb{E}\|\Psi y_n(t)-\Psi y(t)\|^2\longrightarrow 0 \quad \text{as} \qquad n\to+\infty.$$

Therefore Ψ is continuous in Ω_r.

Step 3. We now establish the equicontinuous of the operator $\Psi:\Omega_r\longrightarrow\Omega_r$. Since the impulsive function g_k is compact, then $\Psi(\Omega_r)$ is equicontinuous on $(t_k,s_k]$, $k=1,2,\cdots,N$.

Case I. Let r_1, $r_2 \in [0,t_1]$, $r_1 < r_2$ and $x \in \Omega_r$, using Lemma 1.15, hypotheses (H_3)-(H_6) and by Hölder's inequality, we got

$$\begin{aligned}
\mathbb{E}\Big\|\Psi y(r_2)-\Psi y(r_1)\Big\|^2 \leq & \mathbb{E}\Big\|\big(\mathcal{R}(r_2)-\mathcal{R}(r_1)\big)\big(y_0-\psi(y)\big) \\
& + \int_0^{r_2}\mathcal{R}(r_2-s)\varphi(s,y(s))ds - \int_0^{r_1}\mathcal{R}(r_1-s)\varphi(s,y(s))ds \\
& + \int_0^{r_2}\mathcal{R}(r_2-s)F\big(s\big)dW^H(s) - \int_0^{r_1}\mathcal{R}(r_1-s))F\big(s\big)dW^H(s)\Big\|^2 \\
\leq & 5\mathbb{E}\Big\|\big(\mathcal{R}(r_2)-\mathcal{R}(r_1)\big)\big(y_0-\psi(y)\big)\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{r_1}^{r_2}\mathcal{R}(r_2-s)\varphi(s,y(s))ds\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_0^{r_1}\Big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big)\varphi(s,y(s))ds\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{r_1}^{r_2}\mathcal{R}(r_2-s)F\big(s\big)dW^H(s)\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_0^{r_1}\Big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big)F\big(s\big)dW^H(s)\Big\|^2
\end{aligned}$$

$$\begin{aligned}
\leq & 5\Big\|\mathcal{R}(r_2)-\mathcal{R}(r_2)\Big\|^2\big(\mathbb{E}\|y_0\|^2+\mathbb{E}\|\psi(y)\|^2\big) \\
& + 5M^2\left(r_2-r_1\right)\int_{r_1}^{r_2}\mathbb{E}\|\varphi(s,y(s))\|^2ds \\
& + 5M^2\int_0^{r_1}\Big\|\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big\|^2ds\int_0^{r_1}\mathbb{E}\|\varphi(s,y(s))\|^2ds \\
& + 10M^2Ht_1^{2H-1}\left(r_2-r_1\right)\int_{r_1}^{r_2}\|F(s)\|^2_{\mathcal{L}_2^0}ds \\
& + 10Ht_1^{2H-1}\int_0^{r_1}\Big\|\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big\|^2ds\int_0^{r_1}\|F(s)\|^2_{\mathcal{L}_2^0}ds \\
\leq & \sum_{i=1}^{5}p_i.
\end{aligned}$$

For the purpose of proving that $\mathbb{E}\Big\|\Psi y(r_2)-\Psi y(r_1)\Big\|^2 \longrightarrow 0 \quad \text{as} \quad r_2-r_1\to 0$ we only need to check independently of $y\in\Omega_r$ when $r_2-r_1\to 0$.

For $p_1,\cdots p_5$, since the resolvent operator is continuous in operator norm topology for $t\geq 0$, the nonlocal function ψ is compact and taking use of the function $s\to\Phi(s)\pi_\varphi(r)$ is

Lebesgue integrable, we can easily see that

$$p_1 := \left\|\mathcal{R}(r_2) - \mathcal{R}(r_1)\right\|^2 \left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0,$$
$$p_2 := 5M^2 \left(r_2 - r_1\right) \pi_\varphi\left(r\right) \int_{r_1}^{r_2} \omega(s)ds \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0,$$
$$p_3 := 5M^2 \pi_\varphi\left(r\right) \int_0^{r_1} \left\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\right\|^2 ds \int_0^{r_1} \omega(s)ds \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0,$$
$$p_4 := 10M^2 H t_1^{2H-1} \left(r_2 - r_1\right) \int_{r_1}^{r_2} \|F(s)\|^2_{\mathcal{L}_2^0} ds \quad \text{as} \quad r_2 - r_1 \to 0,$$

and

$$p_5 := 10 H t_1^{2H-1} \int_0^{r_1} \left\|\mathcal{R}(r_2 - s) - \mathcal{R}(r_1 - s)\right\|^2 ds \int_0^{r_1} \|F(s)\|^2_{\mathcal{L}_2^0} ds \quad \text{as} \quad r_2 - r_1 \to 0.$$

Consequently, $\mathbb{E}\left\|\Psi y(r_2) - \Psi y(r_1)\right\|^2 \longrightarrow 0 \quad \text{as} \quad r_2 - r_1 \to 0$ independently of $y \in \Omega_r$ when $r_2 - r_1 \to 0$, it follows that $\Psi(\Omega_r)$ is equicontinuous on $[0, t_1]$.

Case II. For any $x \in \Omega_r$ and r_1, $r_2 \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, $r_1 < r_2$ by Lemma

1.15, and hypotheses (H_3)-(H_6) , we have

$$\begin{aligned}
\mathbb{E}\Big\|\varPsi y(r_2)-\varPsi y(r_1)\Big\|^2 \leq & \mathbb{E}\Big\|\big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\big)\,G_k(s_k,y(t_k^-)) \\
& + \int_{s_k}^{r_2}\mathcal{R}(r_2-s)\varphi(s,y(s))ds-\int_{s_k}^{r_1}\mathcal{R}(r_1-s)\varphi(s,y(s))ds \\
& + \int_{s_k}^{r_2}\mathcal{R}(r_2-s)F\big(s\big)dW^H(s)-\int_{s_k}^{r_1}\mathcal{R}(r_1-s)F\big(s\big)dW^H(s)\Big\|^2 \\
\leq & 5\mathbb{E}\Big\|\big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\big)\,G_k(s_k,y(t_k^-))\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{r_1}^{r_2}\mathcal{R}(r_2-s)\varphi(s,y(s))ds\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{s_k}^{r_1}\Big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big)\varphi(s,y(s))ds\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{r_1}^{r_2}\mathcal{R}(r_2-s)F\big(s\big)dW^H(s)\Big\|^2 \\
& + 5\mathbb{E}\Big\|\int_{s_k}^{r_1}\Big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big)F\big(s\big)dW^H(s)\Big\|^2 \\
\leq & 5\mathbb{E}\Big\|\big(\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\big)C_G r\Big\|^2 \\
& + 5M^2\,(r_2-r_1)\int_{r_1}^{r_2}\mathbb{E}\|\varphi(s,y(s))\|^2ds \\
& + 5M^2\int_{s_k}^{r_1}\Big\|\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big\|^2ds\int_{s_k}^{r_1}\mathbb{E}\|\varphi(s,y(s))\|^2ds \\
& + 10M^2H\,(t_{k+1}-s_k)^{2H-1}\,(r_2-r_1)\int_{r_1}^{r_2}\|F(s)\|^2_{\mathcal{L}_2^0}ds \\
& + 10H\,(t_{k+1}-s_k)^{2H-1}\int_{s_k}^{r_1}\Big\|\mathcal{R}(r_2-s)-\mathcal{R}(r_1-s)\Big\|^2ds\int_{s_k}^{r_1}\|F(s)\|^2_{\mathcal{L}_2^0}ds.
\end{aligned}$$

We observe that $\mathbb{E}\Big\|\varPsi y(r_2)-\varPsi y(r_1)\Big\|^2 \longrightarrow 0$ independently of $y\in\Omega_r$ when $r_2-r_1\to 0$, under the same reasoning as in **Case I** and the fact that g_k is compact. Which implies that $\varPsi(\Omega_r)$ is equicontinuous on $(s_k,t_{k+1}]$ for $k=1,2,\cdots,N$.

Thus, $\mathbb{E}\Big\|\varPsi y(r_2)-\varPsi y(r_1)\Big\|^2 \longrightarrow 0$ at every interval on $[0,T]$. Thus, we determine $\varPi(\Omega_r)$ is equicontinuous on each $[0,T]$.

Step 4. Denote $I=\overline{co}\,\varPi(\Omega_r)$. Where $\overline{co}$ is the closure of convex hull, it can be shown that the map $\varPsi: I\to I$ is equicontinuous on each interval, and $I\subset\Omega_r$ is also equicontinuous.

In what follows we will prove that there exists a constant $0\leq\lambda<1$ and a positive integer n_0 such that for any bounded and nonprecompact subset $C \subset I$

$$\alpha_{\mathcal{PC}}\big(\Psi^{n_0}(C)\big) \leq \lambda \alpha_{\mathcal{PC}}(C). \tag{2.13}$$

For any $C \subset I$ by the definition of operator Ψ^n and the equicontinuity of I, we get that $\Psi^n \subset \Omega_r$ is also equicontinuous. It follows by Lemma 1.9, that

$$\alpha_{\mathcal{PC}}\big(\Psi^n(C)\big) = \max_{t\in[0,T]} \alpha(\Psi^n(C)\,(t)\,), \quad n = 1, 2, \cdots, N. \tag{2.14}$$

And there exists a finite sequence $C_1 = (x_N^1) \subset C$ such that

$$\alpha(\Psi(C)\,(t)\,) \leq 2\alpha(\Psi(C_1)\,(t)\,). \tag{2.15}$$

Furthermore, for any bounded set $C_1, C_2 \subset C$ by Lemma 1.15 and (H_4) we can deduce that

$$\begin{aligned}
&\left\| \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s,C_1(s)\big)ds - \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s,C_2(s)\big)ds \right\| \\
&\qquad = \left(\int_{s_k}^t \left\| \Big(\mathcal{R}(t-s)\Big[\varphi\big(s,C_1(s)\big) - \varphi\big(s,C_2(s)\big)\Big] ds \Big) \right\|^2 \right)^{\frac{1}{2}} \\
&\qquad \leq M\Big(\int_{s_k}^t \|\varphi\big(s,C_1(s)\big) - \varphi\big(s,C_2(s)\big)\|^2 ds \Big)^{\frac{1}{2}}.
\end{aligned}$$

Then, by Theorem 1.9 - **8.**, we get

$$\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s,C(s)\big)ds \Big) \leq M\Big(\int_{s_k}^t \alpha\big(\varphi\big(s,C(s)\big)\big)^2 ds \Big)^{\frac{1}{2}}. \tag{2.16}$$

Therefore, by Lemma 1.15, Theorem 1.9, (2.14), (2.15), (2.16), conditions (H_3)-(H_7), we get for $t \in [0, t_1]$ that

$$\begin{aligned}
\alpha\Big(\Psi^1\,(C)\,(t)\Big) =& \alpha\Big(\Psi\,(C)\,(t)\Big) \leq 2\alpha\Big(\Psi\,(C_1)\,(t)\Big) \\
\leq& 2\alpha\Big(\mathcal{R}(t)\big(y_0 - \psi(y_N^1)\big) + \int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds \\
&\quad + \int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big) \\
\leq& 2\alpha\Big(\mathcal{R}(t)\big(y_0 - \psi(y_N^1)\big)\Big) + 2\alpha\Big(\int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\Big) \\
&\quad + 2\alpha\Big(\int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\alpha\Big(\int_0^t \mathcal{R}(t-s)\varphi\big(s,y_N^1(s)\big)ds\Big)\\
&\leq 2M\Big(\int_0^t\Big[k_2\alpha\big(C_1(s)\big)\Big]^2ds\Big)^{\frac{1}{2}}\\
&\leq 2Mk_2\sqrt{t_1}\,\alpha_{\mathcal{PC}}\big(C\big).
\end{aligned}
$$

For every $t\in(t_k,s_k]$, $k=1,2,\cdots N$, and since $G_k\big(s_k,x(t_k^-)\big)$ is compact according to assumption (H_7), we have

$$
\begin{aligned}
\alpha\Big(\Psi^1C(t)\Big)=&\alpha\Big(\Psi C(t)\Big)\leq 2\alpha\Big(\Psi C_1(t)\Big)\\
\leq&2\alpha\Big(G_k\big(s_k,y_N^1(t_k^-)\big)\Big),
\end{aligned}
$$

at the moment, we obtain

$$
\alpha\Big(\Psi^1C(t)\Big)=0.
$$

And similarly, for $t\in(s_k,t_{k+1}]$, $k=1,2,\cdots,N$, we have

$$
\begin{aligned}
\alpha\Big(\Psi^1C(t)\Big)=&\alpha\Big(\Psi C(t)\Big)\leq 2\alpha\Big(\Psi C_1(t)\Big)\\
\leq&2\alpha\Big(\mathcal{R}(t-s)G_k\big(s_k,y_N^1(t_k^-)\big)+\int_{s_k}^t\mathcal{R}(t-s)\varphi\big(s,y_N^1(s)\big)ds\\
&+\int_{s_k}^t\mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big)\\
\leq&2\alpha\Big(\mathcal{R}(t-s)G_k\big(s_k,y_N^1(t_k^-)\big)\Big)+2\alpha\Big(\int_{s_k}^t\mathcal{R}(t-s)\varphi\big(s,y_N^1(s)\big)ds\Big)\\
&+2\alpha\Big(\int_{s_k}^t\mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big)\\
\leq&2\alpha\Big(\int_{s_k}^t\mathcal{R}(t-s)\varphi\big(s,y_N^1(s)\big)ds\Big)\\
\leq&2M\Big(\int_{s_k}^t\Big[k_2\alpha\big(C_1(s)\big)\Big]^2ds\Big)^{\frac{1}{2}}\\
\leq&2Mk_2\sqrt{(t_{k+1}-s_k)}\,\alpha_{\mathcal{PC}}\big(C\big).
\end{aligned}
$$

Moreover, there exists a countable set $C_2=\{x_N^2\}\subset\overline{co}\,\Pi^1(C)$ such that

$$
\alpha\Big(\Psi\big(\overline{co}\,\Pi^1(C)\big)(t)\Big)\leq 2\alpha\Big(\Psi C_2(t)\Big). \tag{2.17}
$$

Hence, by Lemma 1.15, (2.17) and (H_4), (H_5), (H_7), for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, we obtain

$$\begin{aligned}
\alpha\Big(\Psi^2(C)(t)\Big) =& \alpha\Big(\Psi\big(\overline{co}\,\Psi^1(C)\big)(t)\Big) \leq 2\alpha\Big(\Pi C_2(t)\Big) \\
\leq & 2\alpha\Big(\mathcal{R}(t-s)G_k\big(s_k, y_N^2(t_k^-)\big) \\
& \quad + \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y_N^2(s)\big)ds \int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big) \\
\leq & 2\alpha\Big(\mathcal{R}(t-s)G_k\big(s_k, y_N^2(t_k^-)\big)\Big) + 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y_N^2(s)\big)ds\Big) \\
& \quad + 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big) \\
\leq & 2M\Big(\int_{s_k}^t \alpha\big(\varphi\big(s, C_N^2(s)\big)\big)^2 ds\Big)^{\frac{1}{2}} \\
\leq & 2M\Big(\int_{s_k}^t \Big[\alpha\big(C_2(s)\big)\Big]^2 ds\Big)^{\frac{1}{2}} \\
\leq & 2M\Big(\int_{s_k}^t \Big[\alpha\big(\overline{co}\,\Pi^1(C)\big)\Big]^2 ds\Big)^{\frac{1}{2}} \\
\leq & 2M\Big(\int_{s_k}^t \Big(2Mk_2(t_{k+1}-s_k)^{\frac{1}{2}}\alpha_{\mathcal{PC}}\big(C\big)\Big)^2 ds\Big)^{\frac{1}{2}} \alpha_{\mathcal{PC}}\big(C\big) \\
\leq & \big(2Mk_2\big)^2 \sqrt{\frac{(t_{k+1}-s_k)^2}{2}}\,\alpha_{\mathcal{PC}}\big(C\big).
\end{aligned}$$

By using an iterative process for all $t \in [0, T]$, we obtain

$$\alpha\Big(\Psi^n(C)(t)\Big) \leq \big(2Mk_2\big)^n \sqrt{\frac{(t_{k+1}-s_k)^n}{n!}}\,\alpha_{\mathcal{PC}}\big(C\big).$$

Therefore

$$\alpha\Big(\Psi^n(C)\Big) \leq \big(2Mk_2\big)^n \sqrt{\frac{T^n}{n!}}\,\alpha_{\mathcal{PC}}\big(C\big).$$

It has been found that

$$\big(2Mk_2\big)^n \sqrt{\frac{T^n}{n!}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Then, there exists a large enough positive integer n_0 such that

$$\big(2Mk_2\big)^{n_0} \sqrt{\frac{T^{n_0}}{n_0!}} = \lambda < 1.$$

As a result, we demonstrated that (2.13) is met when $0 \leq \lambda < 1$ and a positive integer n_0 exist. The operator has at least one fixed point, which is a mild solution of (2.1), according to Theorem 2.1.

□

Example 2.1. The following example is given to illustrate the the proposed theory. Let us consider the non-instantaneous impulsive stochastic integro-differential equation driven by a fractional Brownian motion as follows

$$\begin{cases} \frac{\partial}{\partial t} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + \int_0^t K(t-s) \frac{\partial^2}{\partial x^2} z(t,x) ds \\ \qquad\qquad + \varphi\big(t, z(t,x)\big) + \Phi(t) dW^H(t), & t \in [0,1] \cup (2,3],\ x \in [0,\pi], \\ z(t,x) = G_1\big(t, z(1^-, x)\big) & t \in (1,2], x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \in [0,1] \cup (2,3], \\ z(0,x) + \psi(z) = 0 & x \in [0,\pi]. \end{cases} \tag{2.18}$$

where W^H denotes a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$.

Let $\mathbb{V} = \mathbb{H} := L^2\big([0,\pi], \mathbb{R}\big)$. be the Hilbert space with the scalar product $(u,v) = \int_0^\pi u(x)v(x)dx$. We define the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $Au = \frac{\partial z^2}{\partial x^2}$. with domain

$$D(A) = \left\{ z \in \mathbb{H}, \frac{\partial z}{\partial x}, \frac{\partial z^2}{\partial x^2} \in \mathbb{H} \text{ and } z(0) = z(\pi) = 0 \right\}.$$

Then, it is well known that $Au = \sum_{n=1}^{\infty} n^2 (u, e_n) e_n, u \in \mathbb{H}$, where $e_n(u) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nu),\ n = 1, 2, \cdots$ and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ on $\mathbb{H}$, which is given by $T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} (u, e_n) e_n, u \in \mathbb{H}$ and $e_n(u) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nu)$, $n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A. It is well known that $\{T(t)\}_{t \geq 0}$ is compact, such that $\|T(t)\|^2 \leq 1$. In order to define the operator $Q : \mathbb{H} \to \mathbb{H}$, we choose a sequence $(\sigma_n)_{n \in \mathbb{N}}$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $W_Q^H(s)$ by

$$W_Q^H(s) = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\beta_n^H e_n,$$

where $H \in \left(\frac{1}{2}, 1\right)$, and $(\beta_n^H(t))_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

Assume that K is a bounded function in $\mathcal{C}^1(\mathbb{R}^+, \mathbb{H})$, and that K' is both bounded and uniformly continuous. As a result, assumptions (H_1) and (H_2) hold. Consequently, we can conclude that Equation 2.18, has a resolvent operator $h(t)$ for $t \geq 0$, which is norm continuous for $t > 0$, as established by Theorems 1.14 and 1.6.

Let $t \in [0, 3]$ and $\mathcal{PC} := \mathcal{PC}\big([0, 3], \mathbb{H}\big)$. For z, we refer to the segment solution defined in the standard manner $z(.,.) : [0, 3] \times [0, \pi] \longrightarrow \mathbb{H}$, with

$$y(t)(z) = z(t, x), \;\; t \in [0, T] \;\;\; x \in [0, \pi].$$

By the definition of f, g, φ, ψ one easily verify that assumptions (H_3)-(H_7) hold with the following functions $\varphi : ([0, 1] \cup (2, 3]) \times \mathcal{PC}\big([0, 3], \mathbb{H}\big) \to \mathbb{H}$, $g_1 : (1, 2] \times \mathbb{H} \to \mathbb{H}$ and the nonlocal function $\psi : \mathcal{PC}\big([0, 3], \mathbb{H}\big) \to \mathbb{H}$ and $f : [0, 3] \to L_2^0(\mathbb{V}, \mathbb{H})$, defined by

$$\varphi\big(t, z(t, x)\big) = \frac{t^{\frac{1}{2}} \sin(z(t, x))}{e^7\,(1 + \|z\|_2)}\big(z(t, x)\big) \quad t \in [0, 1] \cup (2, 3], \;\; x \in [0, \pi], \tag{2.19}$$

$$G_1\big(t, z(1^-, x)\big) = \int_0^{\pi}\int_1^t \zeta(s, x)\frac{z(1^-, w)}{3e^4\big(1 + \|z(1^-, w)\|_2\big)} ds\, dw \quad t \in (1, 2], \;\; x \in [0, \pi], \tag{2.20}$$

$$\psi(z) = \int_0^{\pi}\int_0^3 b(s, x)\cos(z(s, w)ds\, dw, \tag{2.21}$$

$$F(t) = \Phi(t) \quad t \in [0, 1] \cup (2, 3], \tag{2.22}$$

where $\zeta, b : [0, T] \times [0, \pi] \to \mathbb{R}^+$ are continuous functions such that $\zeta(t, \pi) = b(t, \pi) = 0$.

We now present Lemma 2.1 to prove the compactness of a class of functions.

Lemma 2.1. *[125] Let $g : \mathcal{PC}\big([0, T], \mathbb{H}\big) \to \mathbb{H}$ be a operator defined by*

$$g(z)(x) = \int_0^{\pi}\int_0^T \varpi(s, x)\vartheta(z(s, w))ds\, dw,$$

where $\varpi : [0, T] \times [0, \pi] \to \mathbb{R}$ and $\vartheta : \mathbb{H} \to \mathbb{H}$ are continuous functions where ϑ satisfies

$$\|\vartheta(z)\|^2 \leq C\big(\|z\|^2 + 1\big), \quad \text{for all } \; z \in \mathcal{PC}\big([0, T], \mathbb{H}\big), \quad \text{for } C > 0.$$

Then, g is a compact.

Proof. Let $\mathbb{B} \subset \mathcal{C}\big([0,T], \mathbb{H}\big)$ a bounded set, then there exists $L > 0$ such that

$$\|z\|_\infty \sup_{t\in[0,T]} \|z(t,.)\|_{L^2(\Omega,\mathbb{H})} \leq l.$$

Let $z \in \mathbb{B}$, applying Hölder inequality and Fubini's theoerem, we have

$$\begin{aligned}
\big|g(z)(x)\big|^2 =& \bigg|\int_0^\pi\int_0^T \varpi(s,x)\vartheta(z(s,w))ds\,dw\bigg|^2 \\
\leq& \big\|\varpi\big\|_\infty^2 \bigg|\int_0^\pi\int_0^T \vartheta(z(s,w))ds\,dw\bigg|^2 \\
\leq& \pi T\big\|\varpi\big\|_\infty^2 \int_0^T \big\|\vartheta(z(s,.))\big\|_{\mathbb{H}}^2 ds \\
\leq& \pi T^2\big\|\varpi\big\|_\infty^2 C\big(l^2+1\big).
\end{aligned}$$

We conclude

$$\big\|g(z)\big\|_{\mathbb{H}}^2 \leq \pi T^2\big\|\varpi\big\|_\infty^2 C\big(l^2+1\big).$$

Consequently, g is bounded on $\mathbb{B}$.

Next, we will show that the operator $g(\mathbb{B})$ satisfied the "integral" equicontinuity condition. Let $x, \xi \in [0,\pi]$, we have

$$\begin{aligned}
\int_0^\pi \big|g(z)(x+\xi) - g(z)(x)\big|^2 dx =& \int_0^\pi \bigg|\int_0^\pi\int_0^T \Big(\varpi(s,x+\xi) - \varpi(s,x)\Big)\vartheta(z(s,w))ds\,dw\bigg|^2 dx \\
\leq& \pi C T^2\big(l^2+1\big)\int_0^\pi\int_0^T \big|\varpi(s,x+\xi) - \varpi(s,x)\big|^2 ds\,dx.
\end{aligned}$$

Thus,

$$\|\tau_\xi\psi(z) - \psi(z)\|_{\mathbb{H}}^2 \to 0 \text{ as } \xi \to 0,$$

where, $\tau_\xi\psi(z) = g(z)(x+\xi)$. We deduce, from Kolmogorov-Riesz-Fréchet theorem [[31], Theorem 4.26], that $\varphi(\mathbb{B})$ is relatively compact in $\mathbb{H}$.

□

Corollary 2.1. *[125] Let $\Upsilon : [0,T]\times\mathbb{H} \to \mathbb{H}$ be a operator defined by*

$$\Upsilon(z)(t,x) = \int_0^\pi\int_0^t \varpi(s,x)\vartheta(z(s,w))ds\,dw,$$

where $\varpi : [0,T]\times[0,\pi] \to \mathbb{R}$ and $\vartheta : \mathbb{H} \to \mathbb{H}$ are continuous functions where ϑ satisfies

$$\|\vartheta(z)\|^2 \leq C\big(\|z\|^2+1\big), \quad \textit{for all } z \in \mathcal{PC}\big([0,T], \mathbb{H}\big), \quad \textit{for } C > 0.$$

Then, for all bounded $\mathbb{B} \subset \mathbb{H}$, $\Upsilon : [0,T]\times\mathbb{B} \to \mathbb{H}$ is a relatively compact.

Proof. In the same way as the proof of Lemma 2.1, We prove this. □

Assumptions (H_3)-(H_7) are readily verified by using the definitions of f, g_1, φ, and ψ, hold with

$$\omega(t) = t, \qquad \eta = 1, \qquad \pi_\varphi(\left\|z\right\|_2^2) = \frac{\|z\|_2^2}{e^{14}}, \qquad k_1 = \frac{1}{e^{14}}.$$

Corollary 2.1 and Lemma 2.1 hold that ψ and g_1 are compact. As a result, (H_6) and (H_7) are satisfied. Thus, the totality of assumptions in Theorem 2.1 is satisfied. This implies that, the system (2.18) on $[0,T]$ has a mild solution.

2.3 Controllability Result

In this section, we formulate sufficient conditions for the exact controllability of SIDEs with NII driven by a FBM of the form:

$$\begin{cases} dy(t) = \left[Ay(t) + \displaystyle\int_0^t h(t-s)y(t)ds\right]dt \\ \qquad\qquad +\varphi(t, y(t))dt + Bu(t)dt + F(t)dW^H(t), \quad t \in \cup_{k=0}^N (s_k, t_{k+1}], \\ y(t) = G_k(t, y(t_k^-)), \qquad\qquad t \in \cup_{k=1}^N (t_k, s_k], \\ y(0) + \psi(y) = y_0 \in \mathbb{H}. \end{cases} \tag{2.23}$$

The functions φ, ψ, f, and g_k are functions previously defined. The control function $u(.)$ takes values in $L^2([0,T], \mathbb{U})$ of admissible control functions for a separable Hilbert space $\mathbb{U}$, B is a linear bounded operator from $\mathbb{U}$ to $\mathbb{H}$.

Definition 2.2. A $\mathcal{F}$-adapted stochastic process $y(t) : [0,T] \to \mathbb{H}$ is called a mild solution of (2.23) if $y(0) + \psi(y) = y_0 \in \mathbb{H}$ and for each $t \in [0, T]$

$$y(t) = \begin{cases} \mathcal{R}(t)\big(y_0 - \psi(y)\big) + \displaystyle\int_0^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \displaystyle\int_0^t \mathcal{R}(t-s)Bu\big(s\big)ds + \displaystyle\int_0^t \mathcal{R}(t-s)f\big(s\big)dW^H(s), & t \in [0, t_1], \\ \\ g_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \\ \mathcal{R}(t-s_k)g_k(s_k, y(t_k^-)) + \displaystyle\int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \displaystyle\int_{s_k}^t \mathcal{R}(t-s)Bu\big(s\big)ds + \displaystyle\int_{s_k}^t \mathcal{R}(t-s)f\big(s\big)dW^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{2.24}$$

Definition 2.3. The stochastic control system (2.23) is said to be exact controllable on $[0, T]$ if for every initial state $y_0, y_1 \in \mathbb{H}$ if the reachable set $\mathfrak{R}(T)$ is dense in the space $u \in L^2\big([0, T], \mathbb{U}\big)$ such that the mild solution of (2.23) satisfies $y(T) + \psi(y) = y_1$, where y_1 is a preassigned terminal state.

To prove the controllability result, the following hypotheses are necessary:

(H_8) The linear operator $\mathcal{G}$ from $L^2([0, T], \mathbb{U})$ into $L^2([0, T], \mathbb{H})$ defined by

$$\mathcal{G}_u = \int_{s_k}^T \mathcal{R}(T-s)Bu\big(s\big)ds,$$

has an inverse operator $\mathcal{G}^{-1}$ that takes values in $L^2([0, T], \mathbb{U}) / \ker \mathcal{G}$, where

$$\ker \mathcal{G} = \left\{ y \in L^2([0, T], \mathbb{U}) : \mathcal{G}_y = 0 \right\}.$$

(1) There exists two positive constants C_B, $C_{\mathcal{G}}$ such that

$$\|B\|^2 \leq C_B, \qquad \|\mathcal{G}^{-1}\|^2 \leq C_{\mathcal{G}}.$$

(2) There exists $K_B \in \mathbb{R}^+$, $K_{\mathcal{G}} \in L^1([0, T], \mathbb{R}^+)$ such that for any bounded set $D_1 \subset U$, $D_2 \subset \mathbb{H}$

$$\alpha\big(B(D_1)\big) \leq K_B \alpha(D_1), \qquad \alpha\big(\mathcal{G}^{-1}(D_2(t))\big) \leq K_{\mathcal{G}}(t)\alpha(D_2(t)).$$

The main result of this paper is given in the next theorem

Theorem 2.2. *Suppose that* (H_1)-(H_8) *hold. Then, the stochastic integro-differential system (2.23) is controllable on* $[0.T]$ *. provided that*

$$4M^2\Big(C_g + T\eta k_1\Big)\big(1 + 5C_{\mathcal{G}}C_BT^2\big) < 1. \tag{2.25}$$

Proof. To prove our result, we transform (2.23) into a fixed point problem. Consider the operator $\Psi \in \mathcal{PC}\big([0,T],\mathbb{H}\big)$ defined by

$$\Psi y(t) = \begin{cases} \mathcal{R}(t)\big(y_0 - \psi(y)\big) + \displaystyle\int_0^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \displaystyle\int_0^t \mathcal{R}(t-s)Bu\big(s\big)ds + \int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s), & t \in [0, t_1], \\ G_k(t, y(t_k^-)), & t \in \cup_{k=1}^N (t_k, s_k], \\ \\ \mathcal{R}(t-s_k)G_k(s_k, y(t_k^-)) + \displaystyle\int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y(s)\big)ds \\ \qquad + \displaystyle\int_{s_k}^t \mathcal{R}(t-s)Bu\big(s\big)ds + \int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s), & t \in \cup_{k=1}^N (s_k, t_{k+1}]. \end{cases} \tag{2.26}$$

Using the hypothesis (H_8) for an arbitrary function $y(.)$, define the stochastic control

$$\begin{aligned} u_y\,(t) = \mathcal{G}^{-1}\bigg(& y_1 - \mathcal{R}(T)\big(y_0 - \psi(y)\big) - \mathcal{R}(T-s_k)G_k(s_k, y(t_k^-)) \\ & - \int_{s_k}^T \mathcal{R}(T-s)\varphi\big(s, y(s)\big)ds - \int_{s_k}^T \mathcal{R}(T-s)F\big(s\big)dW^H(s)\bigg)\,(t)\,, \end{aligned} \tag{2.27}$$

for $u_y \in \Omega_r$, using Lemma 1.15, (H_3)-(H_7) and (H_8), we obtain the following result

$$\begin{aligned}
\mathbb{E}\left\|u_y\right\|^2 \leq & C_{\mathcal{G}}\mathbb{E}\Big\|y_1 - \mathcal{R}(T)\big(y_0 - \psi(y)\big) - \mathcal{R}(T-s_k)G_k(s_k, y(t_k^-)) \\
& - \int_{s_k}^{T} \mathcal{R}(T-s)\varphi\big(s, y(s)\big)ds - \int_{s_k}^{T} \mathcal{R}(T-s)F\big(s\big)dW^H(s)\Big\|^2 \\
\leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\big(\mathbb{E}\|y_0\|^2 + \mathbb{E}\|\psi(y)\|^2\big) + M^2\mathbb{E}\Big\|G_k(s_k, y(t_k^-))\Big\|^2 \\
& + M^2\left(T-s_k\right)\int_{s_k}^{T}\mathbb{E}\Big\|\varphi\big(s, y(s)\big)\Big\|^2 ds + 2M^2H\left(T-s_k\right)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
\leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_{G_k}\mathbb{E}\|(s_k, y(t_k^-))\|^2 \\
& + M^2\left(T-s_k\right)\int_{s_k}^{T}\omega(s)\pi_\varphi\big(\mathbb{E}\|y\|^2\big)ds + 2M^2H\left(T-s_k\right)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
\leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_G r \\
& + M^2\left(T-s_k\right)\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) + 2M^2H\left(T-s_k\right)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\left\|u_y\right\|^2 \leq & 5C_{\mathcal{G}}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) + M^2C_G r \\
& + M^2\left(T-s_k\right)\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) \\
& + 2M^2H\left(T-s_k\right)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big).
\end{aligned} \tag{2.28}$$

step 1 The proof is similar as in problem (2.1). Here, we merely demonstrate the existence of a constant $r > 0$ such that $\Psi(\Omega_r) \subset \Omega_r$. Let's assume that this is untrue. Then for each $r > 0$, there would exist $y_r \in \Omega_r$ and $t_r \in [0, T]$ such that $\mathbb{E}\Big\|\Psi\big(y_r\big)\left(t_r\right)\Big\|^2 > r$.

Case I. For $t \in [0, t_1]$, using Lemma1.15, (2.28), and hypotheses (H_3)- (H_6) and (H_8), we have

$$\begin{aligned}
\mathbb{E}\Big\|\Psi\big(y_r\big)\,(t)\,\Big\|^2 \leq & 4M^2\Big(\mathbb{E}\|y_0+\psi(y)\|^2 + t_1\int_0^{t_r}\mathbb{E}\Big\|\varphi\big(s,y_r(s)\big)\Big\|^2 ds \\
& + C_B t_1\int_0^{t_r}\mathbb{E}\Big\|u_y\,(s)\,\Big\|^2 ds + 2Ht_1^{2H-1}\int_0^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
\leq & 4M^2\big(\mathbb{E}\|y_0\|^2 + C_\psi\big) + 4M^2 t_1\int_0^{t_r}\omega(s)\pi_\varphi\left(\mathbb{E}\|y_r\|^2\right)ds \\
& + 20M^2 C_{\mathcal{G}} C_B t_1\int_0^{t_r}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\
& + M^2 C_G r + M^2\,(T-s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) \\
& + 2M^2 H\,(T-s_k)^{2H-1}\int_{s_k}^T\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big)ds \\
& + 8M^2 Ht_1^{2H-1}\int_0^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0}ds \\
\leq & 4M^2\big(\mathbb{E}\|y_0\|^2 + C_\psi\big) + 4M^2 t_1\|\omega\|_{L^1[0,t_1]}\pi_\varphi\,(r) \\
& + 20M^2 C_{\mathcal{G}} C_B t_1^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\
& + M^2 C_G r + M^2\,(T-s_k)\,\|\omega\|_{L[s_k,T]}\pi_\varphi(r) \\
& + 2M^2 H\,(T-s_k)^{2H-1}\int_{s_k}^T\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
& + 8M^2 Ht_1^{2H-1}\int_0^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0}ds,
\end{aligned}$$

consequently, we have

$$\begin{aligned}
\mathbb{E}\Big\|\Psi\big(y_r\big)\,(t)\,\Big\|^2 \leq & 4M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi + t_1\|\omega\|_{L^1[0,t_1]}\pi_\varphi\,(r)\,\Big) \\
& + 20M^2 C_{\mathcal{G}} C_B t_1^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\
& + M^2 C_G r + M^2\,(T-s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) \\
& + 2M^2 H\,(T-s_k)^{2H-1}\int_{s_k}^T\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
& + 8M^2 Ht_1^{2H-1}\int_0^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0}ds.
\end{aligned} \tag{2.29}$$

Case II. For $t_r \in (t_k, s_k]$, $k = 1, 2, \cdots, N$, using assumption (H_7), we obtain

$$\begin{aligned}\mathbb{E}\|\Psi y_r(t)\|^2 \leq& \mathbb{E}\| G_k(t, y_r(t_k^-))\|^2 \\ \leq& C_{G_k}\mathbb{E}\|y_r\|^2 \\ \leq& C_G r.\end{aligned} \tag{2.30}$$

Case III. For $t_r \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, by Lemma1.15, (2.28), and assumptions (H_3)- (H_5), (H_7) and (H_8), we get the following results

$$\begin{aligned}\mathbb{E}\|\Psi y_r(t)\|^2 \leq& 4M^2\Big(\mathbb{E}\| G_k(s_k, y_r(t_k^-))\|^2 + (t_{k+1} - s_k)\int_{s_k}^{t_r}\mathbb{E}\Big\|\varphi\big(s, y_r(s)\big)\Big\|^2 ds \\ &+ C_B\,(t_{k+1} - s_k)\int_{s_k}^{t_r}\mathbb{E}\Big\|u_y\,(s)\,\Big\|^2 ds \\ &+ 2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0} ds\Big) \\ \leq& 4M^2\Big(\mathbb{E}\| G_k(s_k, y_r(t_k^-))\|^2 + (t_{k+1} - s_k)\int_{s_k}^{t}\mathbb{E}\Big\|\varphi\big(s, y_r(s)\big)\Big\|^2 ds \\ &+ 5C_{\mathcal{G}}C_B\,(t_{k+1} - s_k)\int_{s_k}^{t_r}\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\ &+ M^2C_G r + M^2\,(T - s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) \\ &+ 2M^2H\,(T - s_k)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0} ds\Big) ds \\ &+ 2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0} ds\Big) \\ \leq& 4M^2C_G r + 4M^2\,(t_{k+1} - s_k)\int_{s_k}^{t_r}\omega(s)\pi_\varphi\left(\mathbb{E}\|y_r\|^2\right) ds \\ &+ 20M^2C_{\mathcal{G}}C_B\,(t_{k+1} - s_k)^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\ &+ M^2C_G r + M^2\,(T - s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) \\ &+ 2M^2H\,(T - s_k)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0} ds\Big) \\ &+ 8M^2H\,(t_{k+1} - s_k)^{2H-1}\int_{s_k}^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0} ds,\end{aligned}$$

we obtain

$$\begin{aligned}
\mathbb{E}\|\Psi y_r(t)\|^2 \leq & 4M^2\Big(C_G r + (t_{k+1}-s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r)\Big) \\
& + 20M^2 C_{\mathcal{G}} C_B (t_{k+1}-s_k)^2 \Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\
& \quad + M^2 C_G r + M^2 (T-s_k)\,\|\omega\|_{L^1[s_k,T]}\pi_\varphi(r) \\
& \quad + 2M^2 H (T-s_k)^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{\mathcal{L}_2^0}ds\Big) \\
& + 8M^2 H (t_{k+1}-s_k)^{2H-1}\int_{s_k}^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0}ds.
\end{aligned} \tag{2.31}$$

Combining the three cases (2.29), (2.30), (2.31), we obtain

$$\begin{aligned}
r < \mathbb{E}\|\Psi x_r(t)\|^2 \leq & 4M^2\Big(\mathbb{E}\|y_0\|^2 + C_\psi + C_G r + T\eta\pi_\varphi(r)\Big) \\
& + 20M^2 C_{\mathcal{G}} C_B T^2\Big(\mathbb{E}\|y_1\|^2 + 2M^2\left(\mathbb{E}\|y_0\|^2 + C_\psi\right) \\
& \quad + M^2 C_G r + M^2 T\eta\pi_\varphi(r) \\
& \quad + 2M^2 H T^{2H-1}\int_{s_k}^{T}\|F(s)\|^2_{L_2^0}ds\Big) \\
& + 8M^2 H (t_{k+1}-s_k)^{2H-1}\int_{s_k}^{t_r}\|F(s)\|^2_{\mathcal{L}_2^0}ds.
\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$, we have

$$\begin{aligned}
1 < \mathbb{E}\|\Psi x_r(t)\|^2 \leq & 4M^2\Big(C_G + T\eta k_1\Big) + 20M^4 C_{\mathcal{G}} C_B T^2\big(C_G + T\eta k_1\big) \\
1 \leq & 4M^2\Big(C_G + T\eta k_1\Big)\big(1 + 5M^2 C_{\mathcal{G}} C_B T^2\big),
\end{aligned}$$

which is contradicted with (2.25), hence, there exists a constant $r > 0$ such that $\Psi(\Omega_r) \subset \Omega_r$.

Using the same method as in problem 2.4, we show that the operator Ψ is continuous in Ω_r and equicontinuous for each $t \in [0, T]$.

$$
\begin{aligned}
\alpha\left(u_C(t)\right) \leq & \alpha\Bigg(\mathcal{G}^{-1}\Big(y_1 - \mathcal{R}(T)\big(y_0 - \psi(C)\big) - \mathcal{R}(T-s_k)G_k(s_k, C(t_k^-)) \\
& \qquad - \int_{s_k}^{T} \mathcal{R}(T-s)\varphi\big(s, C(s)\big)ds - \int_{s_k}^{T} \mathcal{R}(T-s)F\big(s\big)dW^H(s)\Big)(t)\Bigg) \\
\leq & K_{\mathcal{G}}(t)\alpha\Big(y_1 - \mathcal{R}(T)\big(y_0 - \psi(C)\big) - \mathcal{R}(T-s_k)G_k(s_k, C(t_k^-)) \\
& \qquad - \int_{s_k}^{T} \mathcal{R}(T-s)\varphi\big(s, C(s)\big)ds - \int_{s_k}^{T} \mathcal{R}(T-s)F\big(s\big)dW^H(s)\Big) \\
\leq & \|K_{\mathcal{G}}\|_{L^1[0,T]}\Big(2\alpha\left(y_1 - \mathcal{R}(T)\big(y_0 - \psi(C)\big)\right) + 2\alpha\left(\mathcal{R}(T-s_k)G_k(s_k, C(t_k^-))\right) \\
& \qquad + 2\alpha\left(\int_{s_k}^{T} \mathcal{R}(T-s)\varphi\big(s, C(s)\big)ds\right) + 2\alpha\left(\int_{s_k}^{T} \mathcal{R}(T-s)F\big(s\big)dW^H(s)\right)\Big) \\
\leq & 2\|K_{\mathcal{G}}\|_{L^1[0,T]}\alpha\left(\int_{s_k}^{T} \mathcal{R}(T-s)\varphi\big(s, C(s)\big)ds\right).
\end{aligned} \tag{2.32}
$$

In the same way as in step 4 of the proof of the previous theorem, by Lemma 1.15, Theorem 1.9, conditions (H_3)-(H_7), (2.14), (2.15), (2.16) and (2.32) for $t \in [0, t_1]$, we obtain

$$
\begin{aligned}
\alpha\Big(\Psi^1(C)(t)\Big) =& \alpha\Big(\Psi(C)(t)\Big) \leq 2\alpha\Big(\Psi(C_1)(t)\Big) \\
\leq & 2\alpha\Big(\mathcal{R}(t)\big(y_0 - \psi(y_N^1)\big) + \int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds \\
& \qquad + \int_0^t \mathcal{R}(t-s)Bu\big(s\big)ds + \int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big) \\
\leq & 2\alpha\Big(\mathcal{R}(t)\big(y_0 - \psi(y_N^1)\big)\Big) + 2\alpha\Big(\int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\Big) \\
& \qquad + 2\alpha\Big(\int_0^t \mathcal{R}(t-s)Bu_N^1\big(s\big)ds\Big) + 2\alpha\Big(\int_0^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big) \\
\leq & 2\alpha\Big(\int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\Big) + 2\Big(2MK_B\|K_{\mathcal{G}}\|_{L^1[0,T]}\int_0^t u_N^1\big(s\big)ds\Big) \\
\leq & 2\alpha\Big(\int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\Big) \\
& + 2\Big(2MK_BT\|K_{\mathcal{G}}\|_{L^1[0,T]}\alpha\Big(\int_0^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\Big)ds\Big) \\
\leq & 2Mk_2\sqrt{T}\big(1 + MK_BT\|K_{\mathcal{G}}\|_{L^1[0,T]}\big)\alpha_{\mathcal{PC}}\big(C\big).
\end{aligned}
$$

For all $t \in (t_k, s_k]$, $k = 1, 2, \cdots N$, By (H_7), since $G_k\big(s_k, x(t_k^-)\big)$ is compact, we get

$$\begin{aligned}\alpha\Big(\Psi^1 C(t)\Big) =&\alpha\Big(\Psi C(t)\Big) \leq 2\alpha\Big(\Psi C_1(t)\Big)\\ \leq& 2\alpha\Big(G_k\big(s_k, y_N^1(t_k^-)\big)\Big),\end{aligned}$$

at the moment, we obtain

$$\alpha\Big(\Psi^1 C(t)\Big) = 0.$$

Similarly, for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$, we have

$$\begin{aligned}\alpha\Big(\Psi^1 C(t)\Big) =&\alpha\Big(\Psi C(t)\Big) \leq 2\alpha\Big(\Psi C_1(t)\Big)\\ \leq& 2\alpha\Big(\mathcal{R}(t-s)G_k\big(s_k, y_N^1(t_k^-)\big) + \int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\\ &+ \int_{s_k}^t \mathcal{R}(t-s)Bu\big(s\big)ds + \int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big)\\ \leq& 2\alpha\Big(\mathcal{R}(t-s)G_k\big(s_k, y_N^1(t_k^-)\big)\Big) + 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)\varphi\big(s, y_N^1(s)\big)ds\Big)\\ &+ 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)Bu_N^1\big(s\big)ds\Big) + 2\alpha\Big(\int_{s_k}^t \mathcal{R}(t-s)F\big(s\big)dW^H(s)\Big)\\ \leq& 2Mk_2.\Big(1 + MK_BT\|K_{\mathcal{G}}\|_{L^1[0,T]}\Big)\sqrt{(t_{k+1}-s_k)}\alpha_{\mathcal{PC}}\big(C\big).\end{aligned}$$

Also that, there exists a large enough positive integer n_0 such that

$$\big(2Mk_2\big)^{n_0}.\Big(1 + MK_BT\|K_{\mathcal{G}}\|_{L^1[0,T]}\Big)^{n_0}\sqrt{\frac{T^{n_0}}{n_0!}} = \lambda < 1,$$

where $0 \leq \lambda < 1$. Thus, condition (1.2) is satisfied. According to Theorem 1.6, the operator 2.2 has at least one fixed point. Hence, the system is exactly controllable on the interval $[0, T]$.

□

Example 2.2. Let us consider the following the non-instantaneous impulsive stochastic integro-differential equation driven by a fractional Brownian motion:

$$\begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_0^t K(t-s)\dfrac{\partial^2}{\partial x^2}z(t,x)ds \\ \qquad\qquad +\varphi\big(t,z(t,x)\big) + v(t,x) + \varPhi(t)dW^H(t), & t \in [0,1]\cup(2,3],\ x\in[0,\pi]\,, \\ z(t,x) = G_1\big(t, z(1^-,x)\big) & t\in(1,2], x\in[0,\pi]\,, \\ z(t,0) = z(t,\pi) = 0, & t\in[0,1]\cup(2,3], \\ z(0,x) + \psi(z) = 0 & x\in[0,\pi]\,. \end{cases} \tag{2.33}$$

Let $\mathbb{U} = \mathbb{V} = \mathbb{H} := L^2\big(\,[0,\pi]\,,\mathbb{R}\big)$. Our presumptions are as follows:

We define $\varphi : ([0,1]\cup(2,3])\times\mathcal{PC}\big(\,[0,3]\,,\mathbb{H}\big) \to \mathbb{H}$, $g_1 : (1,2]\times\mathbb{H}\to\mathbb{H}$, $\psi : \mathcal{PC}\big(\,[0,3]\,,\mathbb{H}\big) \to \mathbb{H}$ and $f : [0,3] \to L_2^0(\mathbb{V},\mathbb{H})$, defined by

$$\begin{aligned} \varphi\big(t,z(t,x)\big) =& \frac{t^{\frac{1}{2}}\sin\big(z(t,x)\big)}{e^7\big(1+\|z\|_2\big)}\big(z(t,x)\big) \quad t\in[0,1]\cup(2,3],\ \ x\in[0,\pi]\,, \\ G_1\big(t,z(1^-,x)\big) =& \int_0^\pi\int_1^t \zeta(s,x)\frac{z(1^-,x)}{3e^4\big(1+\|z(1^-,x)\|_2\big)}ds\,dx \quad t\in(1,2],\ \ x\in[0,\pi]\,, \\ \psi(z) =& \int_0^\pi\int_0^3 b(t,x)\cos(z(s,x)ds\,dx, \\ F(t) =& \varPhi(t) \quad t\in[0,1]\cup(2,3], \end{aligned}$$

where $\zeta, b : [0,T]\times[0,\pi] \to \mathbb{R}^+$ are continuous functions such that $\zeta(t,\pi) = b(t,\pi) = 0$.

Define the bounded linear operator $B : \mathbb{U}\to\mathbb{H}$ by

$$Bu(t)(x) = v(t,x),\ \ x\in[0,\pi]\,,\ \ u\in L^2\left([0,T]\,,\mathbb{U}\right).$$

The operator : $L^2\left([0,T]\,,\mathbb{U}\right) \to L^2\left([0,T]\,,\mathbb{H}\right)$ defined by

$$\mathcal{G}_u(x) = \int_{s_k}^T \mathcal{R}(T-s)v(s,x)ds,$$

has an inverse $\mathcal{G}_u^{-1}$ and satisfies condition (H_8). Lemma 2.1 and Corollary 2.1 establish that ψ and g_1 are compact.

Next, it shows us that all requirements of Theorem 2.2 are satisfied in the above example. Therefore, the system corresponding to (2.33) is exact controllable.

Chapter 3

Existence, Uniqueness and UHR Stability of FSIDEs with NII

In this chapter, we present the main results obtained in our article [114]. We propose a new class of Caputo fractional stochastic integro-differential equations with non-instantaneous impulsive and nonlocal conditions. The existence, uniqueness, and Ulam-Hyers-Rassias stability are established. This is achieved by applying fixed point theory, stochastic analysis, and fractional calculus.

3.1 Introduction

Ulam introduced the concept of functional equation stability in 1940, posing the question: under what conditions does an additive mapping approach an approximately additive mapping? Hyers [78] provided the first answer to this question, focusing on Banach spaces. This form of stability is known as Ulam-Hyers stability (UH stability). In 1978, Rassias [137] extended Ulam-Hyers stability by incorporating variables into the theory. The concept of stability arises when an inequality replaces a functional equation, and the solutions to this inequality remain close to the original equation's solutions. Essentially, for every solution of the perturbed equation, there exists a solution to the original equation that is close to it. Since then, a number of researchers have become interested in the theory of Ulam-Hyers stability. Many interesting results on HU stability have been gained (see [12, 21, 80, 92, 147, 152]), and some research on Ulam-Hyers-Rassias stability (UHR stability) has been provided (see [19, 67, 92, 184] and the references therein). In recent times, the UH and UHR stability of stochastic differential equations has been examined by many authors. Anguraj et al. [12], Selvam et al. [147], Lang et al. [92], Ben Makhlouf et al. [19] studied the existence and UHR stability of solutions for stochastic differential equations Wang et al. [172], Song and Luo [152], Ben Makhlouf et al [21], Kahouli et al. [80], Rhaim et al [140] addressed the UH stability of fractional stochastic differential equations.

Inspired by the above works and to the best of our knowledge, there is currently no literature exploring the UH and UHR stability of Caputo fractional stochastic integro-differential equations with non-instantaneous impulsive and nonlocal conditions. The purpose of this chapter is studies the existence and uniqueness of mild solutions for the following class of fractional stochastic integro-differential equations with non-instantaneous impulses and nonlocal conditions:

$$\begin{cases} {}_0^cD_t^\beta y(t) = Ay(t) + \hbar\Big(t, y(t), K_1y(t)\Big) + \aleph\Big(t, y(t), K_2y(t)\Big)\dfrac{dW(t)}{dt}, & t \in \bigcup\limits_{k=0}^{m}(s_k, t_{k+1}], \\ y(t) = H_k(t, y(t_k^-)), \qquad t \in \bigcup\limits_{k=1}^{m}(t_k, s_k], \\ y(0) + \gamma(y) = y_0 \in \mathbb{H}, \end{cases} \tag{3.1}$$

where ${}_0^cD_t^\beta$ is the Caputo fractional derivative of order β, $\beta \in (\frac{1}{2}, 1)$, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of bounded linear operators on a separable Hilbert space $\mathbb{H}$. Let $J = [0, T]$, $T > 0$ is a constant, $\hbar : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, $\aleph : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ and $\gamma : \mathcal{PC}\big(J, \mathbb{H}\big) \to \mathbb{H}$ are given functions satisfying certain assumptions $0 = t_0 = s_0 < t_1 < s_1 < t_2 \cdots < t_m < s_m < t_{m+1} = T$, $H_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ are non-instantaneous impulsive functions for all $k = 0, 1, 2, \cdots, m$, the terms $K_iy(t)$ is given by $K_iy(t) = \displaystyle\int_0^t \wp_i(t, s)y(s)ds$ where $\wp_i \in \mathcal{C}\big(D, \mathbb{R}^+\big)$, for $i = 1, 2$ are the set of all positive continuous functions on $D := \Big\{(t, s) : 0 \leq s < t \leq T\Big\}$, and y_0 is an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|y_0\| < \infty$.

The chapter is organized as follows: **??** introduces some fundamental notations and preliminaries. 3.2 provides some sufficient conditions for the existence and uniqueness of a mild solution for the system (3.1) and includes an example to demonstrate the obtained result. 3.3 Provides the UH and UHR stability of a mild solution for the system (3.1) using fundamental Definition 3.6.

3.2 Existence and Uniqueness of the Mild solution

In this section, we establish the existence and uniqueness of the mild solution for a class of FSIDEs by applying the Banach contraction principle and Krasnoselskii's fixed point theorem.

Let $(\mathbb{H}, (.,.)_{\mathbb{H}}, \|.\|_{\mathbb{H}})$, and $(\mathbb{K}, (.,.)_{\mathbb{K}}, \|.\|_{\mathbb{K}})$, two real separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\{W(t)\}_{t\geq 0}$ be a $\mathcal{Q}$-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with the covariance operator $\mathcal{Q}$ such that $tr(\mathcal{Q}) < \infty$. It is assumed that there exists a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}$ and positive real numbers λ_k such that $\mathcal{Q}e_k = \lambda_k e_k$, $k = 1, 2, \cdots$, and a sequence of independent Brownian motions such that

$$(W(t), e) = \sum_{n=1}^{\infty} \sqrt{\lambda_k}\beta_n(e_k, e)\beta_k(t), \quad e \in \mathbb{H}, \quad t \geq 0.$$

Let $L^2(J, \mathbb{H})$ be the space of all $\mathbb{H}$-valued square integrable operators defined on $[0, T]$, with norm $\|y\|_{L^2(J,\mathbb{H})} = \left(\int_0^T \|y(t)\|^2 dt \right)^{\frac{1}{2}}$.

In the sequel, $L_0^2(J, \mathbb{H})$ denotes the space of $\mathcal{F}_0$- measurable, $\mathbb{H}$-valued and square integrable stochastic process.

$$L_0^2(J, \mathbb{H}) = \left\{ f \in L^2(J, \mathbb{H}) \mid f \text{ is } \mathcal{F}_0\text{-measurable} \right\}.$$

Consider the space

$$\begin{aligned} \mathcal{PC}([0,T], \mathbb{H}) = \Big\{ & \mathcal{F}_t\text{-adapted } \mathbb{H}\text{-valued process } y(t) \text{ is continuous every where} \\ & \text{except for some } t \neq t_k \text{ at which } y(t_k^-) \text{ and } y(t_k^+) \text{ exist} \\ & \text{and } y(t_k^-) = y(t_k), \quad k = 1, 2, \cdots, m \text{ and } \sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2 < \infty \Big\}, \end{aligned}$$

which is a Banach endowed with the norm

$$\|y\|_{\mathcal{PC}} = \left(\sup_{0\leq t\leq T} \mathbb{E}\|y(t)\|^2 \right)^{\frac{1}{2}}.$$

Lemma 3.1. *[39] Let the space*

$$\mathcal{M}(\mathbb{H},\mathbb{K}) = \Big\{\phi \text{ is an } L_b\big(\mathbb{H},\mathbb{K}\big)\text{-valued stochastic process on } J \text{ such that } \Phi(t) \text{ is } \mathcal{F}_t\text{-measurable for all } t\in[0,T], \int_0^T \mathbb{E}\big\|\phi(t)\big\|^2 dt < \infty\Big\}.$$

If $\Phi \in \mathcal{M}\big(\mathbb{H},\mathbb{K}\big)$, *then*

$$\mathbb{E}\left\|\int_0^T \phi(t)dW(t)\right\|^2 \le tr\big(\mathcal{Q}\big)\int_0^T \mathbb{E}\big\|\phi(t)\big\|^2 dt.$$

Now, we present the fundamental concept of mild solutions for fractional stochastic integro-differential equations (3.1).

Definition 3.1. A stochastic process $y \in \mathcal{PC}\big([0,T],\mathbb{H}\big)$ is a mild solution of the system (3.1) if y satisfies the following stochastic integral:

$$y(t) = \begin{cases} U(t)\big(y_0-\gamma(y)\big) \\ \quad +\int_0^t (t-s)^{\beta-1}V(t-s)\hbar\Big(s,y(s),K_1y(s)\Big)ds \\ \quad +\int_0^t (t-s)^{\beta-1}V(t-s)\aleph\Big(s,y(s),K_2y(s)\Big)dW(s) & t\in[0,t_1], \\ H_k(t,y(t_k^-)), & t\in \bigcup_{k=1}^m (t_k,s_k], \\ U(t-s_k)H_k(s_k,y(s_k)) \\ \quad +\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\hbar\Big(s,y(s),K_1y(s)\Big)ds \\ \quad +\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\aleph\Big(s,y(s),K_2y(s)\Big)dW(s), & t\in \bigcup_{k=1}^m (s_k,t_{k+1}], \end{cases} \tag{3.2}$$

where

$$U(t) := \int_0^\infty \sigma_\beta(r)T(t^\beta r)dr, \qquad V(t) := \beta\int_0^\infty r\sigma_\beta(r)T(t^\beta r)dr,$$

$$\sigma_\beta(r) := \frac{1}{\beta}r^{-1-\frac{1}{\beta}}\varrho_\beta\big(r^{-\frac{1}{\beta}}\big),$$

and

$$\varrho_\beta(r) := \frac{1}{\pi}\sum_{n=1}^\infty\big(-1\big)^{n-1}r^{-\beta n-1}\frac{\Gamma\big(\beta n+1\big)}{n!}\sin(n\pi\beta), \qquad r\in(0,+\infty).$$

σ_β is a probability density function defined on $(0,+\infty)$.

We will now present the set of hypotheses that will be utilized in our main theorem.

$(\hat{H}_1)$ $\mathcal{T}(t)$ is a compact operator for every $t > 0$.

$(\hat{H}_2)$ The function $\hbar : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ satisfies

(1) For each $t \in [0, T]$, $\hbar(., u, v)$ is measurable and for any $u, v \in \mathbb{H}$, $\hbar(t, ., .)$ is continuous.

(2) There exist a continuous, non-decreasing function $\psi_\hbar : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $L_\hbar > 0$, the function $\varphi_\hbar \in L^{\frac{1}{p_1}}(J, \mathbb{R}^+)$ such that

$$\mathbb{E}\|\hbar(t, u, v)\|^2 \leq \varphi_\hbar(t)\psi_\hbar\Big(\Big\|u\Big\|^2_{\mathcal{PC}}\Big), \quad \liminf_{u \to +\infty} \frac{\psi_\hbar(u)}{u} = L_\hbar < +\infty.$$

(3) There exist two positive constant $C_\hbar, C'_\hbar$ such that

$$\mathbb{E}\|\hbar(t, u, v) - \hbar(t, u', v')\|^2 \leq C_\hbar \mathbb{E}\Big\|u - u'\Big\|^2 + C'_\hbar \mathbb{E}\Big\|v - v'\Big\|^2,$$

for each $(t, u, v), (t, u', v') \in J \times \mathbb{H} \times \mathbb{H}$.

$(\hat{H}_3)$ The function $\aleph : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ satisfying

(1) For all $t \in [0, T]$, $\aleph(., u, v)$ is measurable and for any $u, v \in \mathbb{H}$, $\aleph(t, ., .)$ is continuous.

(2) There exist a continuous non-decreasing function $\psi_\aleph : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $L_\aleph > 0$, the function $\varphi_\aleph \in L^{\frac{1}{p}}(J, \mathbb{R}^+)$ such that

$$\mathbb{E}\|\aleph(t, u, v)\|^2 \leq \varphi_\aleph(t)\psi_\aleph\Big(\Big\|u\Big\|^2_{\mathcal{PC}}\Big), \quad \liminf_{u \to +\infty} \frac{\psi_\aleph(u)}{u} = L_\aleph < +\infty.$$

(3) There exist two positive constant $C_\aleph, C'_\aleph$ such that

$$\mathbb{E}\|\aleph(t, u, v) - \aleph(t, u', v')\|^2 \leq C_\aleph \mathbb{E}\Big\|u - u'\Big\|^2 + C'_\aleph \mathbb{E}\Big\|v - v'\Big\|^2,$$

for every $(t, u, v), (t, u', v') \in J \times \mathbb{H} \times \mathbb{H}$.

$(\hat{H}_4)$ The nonlocal function $\gamma : \mathcal{PC}([0, T], \mathbb{H}) \to \mathbb{H}$ is continuous, there exists a constant $C_\gamma > 0$, satisfying the conditions

(1)
$$\mathbb{E}\|\gamma(y)\|^2 \le C_\gamma, \quad \forall y \in \mathcal{PC}([0,T], \mathbb{H}).$$

(2)
$$\mathbb{E}\|\gamma(y)-\gamma(x)\|^2 \le C_\gamma \left\|y-x\right\|^2_{\mathcal{PC}}, \quad \forall x,y \in \mathcal{PC}([0,T], \mathbb{H}).$$

$(\hat{H}_5)$ $H_k : (t_k, s_k] \times \mathbb{H} \to \mathbb{H}$ is continuous function, for $k = 1, 2, \cdots, m$, there exists constants $C_{H_k} \in (0, \frac{1}{2})$, satisfying the conditions

(1)
$$\mathbb{E}\|H_k(t, y(t_k^-))\|^2 \le C_{H_k}\left\|y\right\|^2_{\mathcal{PC}}, \quad \forall y \in \mathbb{H}, \ t \in (t_k, s_k].$$

(2)
$$\mathbb{E}\left\|H_k(t,x(t_k^-)) - H_k(t,y(t_k^-))\right\|^2 \le C_{H_k}\mathbb{E}\left\|y(t)-x(t)\right\|^2, \quad \forall x,y \in \mathbb{H}, \ t \in (t_k, s_k].$$

The following should be noted:

$$N_\hbar = \max_{k=0,1,\cdots m}\{\|\varphi_\hbar\|\}_{L^{\frac{1}{p_1}}[s_k,t_{k+1}]}, \qquad N_\aleph = \max_{k=0,1,\cdots m}\{\|\varphi_\aleph\|\}_{L^{\frac{1}{p}}[s_k,t_{k+1}]}.$$

$$C_{K_1} = \sup_{t\in[0,T]}\int_0^t \wp_1(t,s)ds < \infty, \qquad C_{K_2} = \sup_{t\in[0,T]}\int_0^t \wp_2(t,s)ds < \infty.$$

Theorem 3.1. *Assume that the assumptions* $(\hat{H}_1)$-$(\hat{H}_5)$ *are satisfied. Then the system* (3.1) *has a mild solution on* $[0,T]$, *provided that*

$$C := 3M^2C_{H_k} + \frac{3M^2L_\hbar T^{(a+1)(1-p_1)}N_\hbar}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}} + \frac{3M^2Tr\big(\mathcal{Q}\big)L_\aleph T^{(b+1)(1-p)}N_\aleph}{\Gamma^2(\beta)\,(b+1)^{(1-p)}} < 1. \tag{3.3}$$

And

$$M^2C_\gamma + \left(M^2+1\right)C_{H_k} < 1.$$

Proof. Defined the operator $\mathit{\Gamma} : \mathcal{PC}([0,T],\mathbb{H}) \to \mathcal{PC}([0,T],\mathbb{H})$ by

$$
\big(\mathit{\Gamma} y\big)(t) = \begin{cases} U(t)\big(y_0 - \gamma(y)\big) \\ \qquad + \int_0^t (t-s)^{\beta-1} V(t-s) \hbar\Big(s, y(s), K_1 y(s)\Big) ds \\ \qquad + \int_0^t (t-s)^{\beta-1} V(t-s) \aleph\Big(s, y(s), K_2 y(s)\Big) dW(s) & t \in [0, t_1], \\ \\ H_k(t, y(t_k^-)), & t \in \bigcup_{k=0}^{m} (t_k, s_k], \\ \\ U(t-s_k) H_k(s_k^-, y(s_k)) \\ \qquad + \int_{s_k}^t (t-s)^{\beta-1} V(t-s) \hbar\Big(s, y(s), K_1 y(s)\Big) ds \\ \qquad + \int_{s_k}^t (t-s)^{\beta-1} V(t-s) \aleph\Big(s, y(s), K_2 y(s)\Big) dW(s), t \in \bigcup_{k=1}^{m} (s_k, t_{k+1}]. \end{cases} \tag{3.4}
$$

The existence of mild solution of the system (3.1) is equivalent to the existence of a fixed point of the operator $\mathit{\Gamma}$ defined by (3.4). Obviously, the fixed point of $\mathit{\Gamma} y(t)$ is the solution of the problem (3.1). For each finite constant $r > 0$, let

$$
\Omega_r = \Big\{ y \in \mathcal{PC}([0,T],\mathbb{H}) : \|y\|^2_{\mathcal{PC}} \leq r \Big\}.
$$

It is evident that Ω_r is a bounded closed and convex set in $\mathcal{PC}([0,T],\mathbb{H})$.

The proof falls naturally into four steps.

Step 1 Now we can show that the operator $\mathit{\Gamma}$ is bounded set to a bounded set in Ω_r

We prove that there exists a constant $r > 0$ such that $\mathit{\Gamma}(\Omega_r) \subset \Omega_r$. If this is not true, then for all r, there will exist $y_r \in \Omega_r$ and $t \in [0,T]$ such that

$$
\mathbb{E}\Big\| \mathit{\Gamma} y_r(t) \Big\|^2 > r.
$$

We consider (3.4) and $p, p_1 \in (0,1), a = \frac{2(\beta-1)}{1-p_1}, b = \frac{2(\beta-1)}{1-p}, a, b \in (-1,0)$, by using $(\hat{H}_2)$-$(\hat{H}_5)$, Hölder's inequality Lemma3.1 and Lemma 1.9, we have

Case 1: For $t \in [0, t_1]$ then by $(\hat{H}_2)$-$(\hat{H}_4)$, we have

$$\begin{aligned} r <& \mathbb{E}\Big\| \mathit{\Gamma} y_r(t) \Big\|^2 \\ \leq& 3\mathbb{E}\Big\| U(t)\big(y_0 - \gamma(y_r)\big) \Big\|^2 \\ &+ 3\mathbb{E}\Bigg\| \int_0^t \big(t-s\big)^{\beta-1} V\big(t-s\big)\hbar\Big(s, y_r(s), K_1 y_r(s)\Big) ds \Bigg\|^2 \\ &+ 3\mathbb{E}\Bigg\| \int_0^t \big(t-s\big)^{\beta-1} V\big(t-s\big)\aleph\Big(s, y_r(s), K_2 y_r(s)\Big) dW(s) \Bigg\|^2 \\ =& 3J_1 + 3J_2 + 3J_3. \end{aligned} \tag{3.5}$$

Using Lemma 1.9, $(\hat{H}_4)$, are shown

$$\begin{aligned} J_1 :=& \mathbb{E}\Big\| U(t)\big(y_0 - \gamma(y_r)\big) \Big\|^2 \\ \leq& 2M^2\big(\mathbb{E}\|y_0\|^2 + \mathbb{E}\|\gamma(y_r)\|^2\big) \\ \leq& 2M^2\big(\mathbb{E}\|y_0\|^2 + C_\gamma\big). \end{aligned} \tag{3.6}$$

Using Hölder's inequality, Lemma 1.9 and $(\hat{H}_2)$, we get

$$\begin{aligned} J_2 :=& \mathbb{E}\Bigg\| \int_0^t \big(t-s\big)^{\beta-1} V\big(t-s\big)\hbar\Big(s, y_r(s), K_1 y_r(s)\Big) ds \Bigg\|^2 \\ \leq& \frac{M^2 T}{\Gamma^2(\beta)} \int_0^t \big(t-s\big)^{2(\beta-1)} \mathbb{E}\Big\| \hbar\big(s, y_r(s), K_1 y_r(s)\big) \Big\|^2 ds \\ \leq& \frac{M^2 T}{\Gamma^2(\beta)} \Big[\int_0^t \big(t-s\big)^{\frac{2(\beta-1)}{1-p_1}} ds \Big]^{1-p_1} \Bigg[\Big(\int_0^t \mathbb{E}\Big\| \hbar\big(s, y_r(s), K_1 y_r(s)\big) \Big\|^2 ds \Big)^{\frac{1}{p_1}} \Bigg]^{p_1} \\ \leq& \frac{M^2 T}{\Gamma^2(\beta)} \Big[\int_0^t \big(t-s\big)^{a} ds \Big]^{1-p_1} \Big[\int_0^t \big(\varphi_\hbar(s)\psi_\hbar\big(\big\|y\big\|_{\mathcal{PC}}^2\big)\big)^{\frac{1}{p_1}} ds \Big]^{p_1} \\ \leq& \frac{M^2 T t_1^{(a+1)(1-p_1)}}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}} \psi_\hbar\big(\big\|y\big\|_{\mathcal{PC}}^2\big) \Big[\int_0^t \big(\varphi_\hbar(s)\big)^{\frac{1}{p_1}} ds \Big]^{p_1} \\ \leq& \frac{M^2 T t_1^{(a+1)(1-p_1)}}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}} \psi_\hbar\big(\big\|y\big\|_{\mathcal{PC}}^2\big) \|\varphi_\hbar\|_{L^{\frac{1}{p_1}}[0,t_1]}. \end{aligned} \tag{3.7}$$

By hypothesis $(\hat{H}_3)$, Lemma 1.9, 3.1 and Hölder's inequality, we have

$$\begin{aligned}
J_3 :=& \mathbb{E}\left\| \int_0^t (t-s)^{\beta-1} V(t-s) \aleph\Big(s, y_r(s), K_2 y_r(s)\Big) dW(s) \right\|^2 \\
\leq& Tr(\mathcal{Q}) \int_0^t \mathbb{E}\left\| (t-s)^{\beta-1} V(t-s) \aleph\big(s, y_r(s), K_2 y_r(s)\big) \right\|^2 ds \\
\leq& Tr(\mathcal{Q}) \int_0^t (t-s)^{2(\beta-1)} \mathbb{E}\left\| V(t-s) \aleph\big(s, y_r(s), K_2 y_r(s)\big) \right\|^2 ds \\
\leq& \frac{M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)} \int_0^t (t-s)^{2(\beta-1)} \mathbb{E}\left\| \aleph\big(s, y_r(s), K_2 y_r(s)\big) \right\|^2 ds \\
\leq& \frac{M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)} \left[\int_0^t (t-s)^{\frac{2(\beta-1)}{1-p}} ds \right]^{1-p} \left[\int_0^t \big(\varphi_\aleph(s) \psi_\aleph(\|y\|^2_{\mathcal{PC}})\big)^{\frac{1}{p}} ds \right]^p \\
\leq& \frac{M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)} \psi_\aleph(\|y\|^2_{\mathcal{PC}})^{\frac{1}{p}} \left[\int_0^t (t-s)^b ds \right]^{1-p} \left[\int_0^t (\varphi_\aleph(s) ds \right]^p \\
\leq& \frac{M^2 Tr(\mathcal{Q}) t_1^{(b+1)(1-p)}}{\Gamma^2(\beta)\,(b+1)^{(1-p)}} \psi_\aleph(\|y\|^2_{\mathcal{PC}}) \|\varphi_\aleph\|_{L^{\frac{1}{p}}[0,t_1]}.
\end{aligned} \tag{3.8}$$

Inserting (3.6), (3.7) and (3.8) in (3.5), we obtain

$$\begin{aligned}
\mathbb{E}\left\| \mathit{\Gamma} y_r(t) \right\|^2 \leq& 6M^2\big(\mathbb{E}\|y_0\|^2 + C_\gamma\big) + \frac{3M^2 T t_1^{(a+1)(1-p_1)}}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}} \psi_\hbar(\|y\|^2_{\mathcal{PC}}) \|\varphi_\hbar\|_{L^{\frac{1}{p_1}}[0,t_1]} \\
&+ \frac{3M^2 Tr(\mathcal{Q}) t_1^{(b+1)(1-p)}}{\Gamma^2(\beta)\,(b+1)^{(1-p)}} \psi_\aleph(\|y\|^2_{\mathcal{PC}}) \|\varphi_\aleph\|_{L^{\frac{1}{p}}[0,t_1]}.
\end{aligned} \tag{3.9}$$

Case 2: Using Lemma 1.9 for $t \in (t_k, s_k]$, $k = 0, 1, 2, \cdots, m$, we obtain

$$\begin{aligned}
\mathbb{E}\left\| \mathit{\Gamma} y_r(t) \right\|^2 =& \mathbb{E}\|H_k(t, y_r(t_k^-))\|^2 \\
\leq& C_{H_k} \|y\|^2_{\mathcal{PC}}.
\end{aligned} \tag{3.10}$$

Case 3: By $(\hat{H}_2)$, $(\hat{H}_3)$, $(\hat{H}_5)$, Hölder's inequality and Lemma 1.9, 3.1, for $t \in (s_k, t_{k+1}]$, $k =$

$0, 1, 2, \cdots, m$, we obtain

$$
\begin{aligned}
r <&\mathbb{E}\Big\|\varGamma y_r(t)\Big\|^2 \\
\leq&3\mathbb{E}\Big\|U(t-s_k)H_k(s_k^-, y_r(s_k))\Big\|^2 \\
&+3\mathbb{E}\Big\|\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\hbar\Big(s, y_r(s), K_1y_r(s)\Big)ds\Big\|^2 \\
&+3\mathbb{E}\Big\|\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\aleph\Big(s, y_r(s), K_2y_r(s)\Big)dW(s)\Big\|^2 \\
=&3J_1' + 3J_2' + 3J_3'.
\end{aligned}
\tag{3.11}
$$

By Lemma 1.9 and $(\hat{H}_5)$, we have

$$
\begin{aligned}
J_1' :=&\mathbb{E}\Big\|U(t-s_k^-)H_k(s_k, y_r(s_k))\Big\|^2 \\
\leq&M^2\mathbb{E}\|H_k(s_k, y_r(s_k))\|^2 \\
\leq&M^2C_{H_k}\Big\|y\Big\|_{\mathcal{PC}}^2.
\end{aligned}
\tag{3.12}
$$

Using Hölder's inequality, Lemma 1.9 and $(\hat{H}_2)$, we get

$$
\begin{aligned}
J_2' :=& \mathbb{E}\Big\|\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\hbar\Big(s, y_r(s), K_1y_r(s)\Big)ds\Big\|^2 \\
\leq&\frac{M^2T}{\Gamma^2(\beta)}\int_{s_k}^t (t-s)^{2(\beta-1)}\mathbb{E}\Big\|V(t-s)\hbar(s, y_r(s), K_1y_r(s))\Big\|^2 ds \\
\leq&\frac{M^2T}{\Gamma^2(\beta)}\Big[\int_{s_k}^t (t-s)^a ds\Big]^{1-p_1}\Big[\int_{s_k}^t \Big(\mathbb{E}\Big\|\hbar(s, y_r(s), K_1y_r(s))\Big\|^2\Big)^{\frac{1}{p_1}} ds\Big]^{p_1} \\
\leq&\frac{M^2T(t-s_k)^{(a+1)(1-p_1)}}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}}\Big[\int_{s_k}^t \Big(\varphi_\hbar(s)\psi_\hbar(\Big\|y\Big\|_{\mathcal{PC}}^2)\Big)^{\frac{1}{p_1}} ds\Big]^{p_1} \\
\leq&\frac{M^2T(t-s_k)^{(a+1)(1-p_1)}}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}}\psi_\hbar(\Big\|y\Big\|_{\mathcal{PC}}^2)\Big[\int_{s_k}^t (\varphi_\hbar(s))^{\frac{1}{p_1}} ds\Big]^{p_1} \\
\leq&\frac{M^2T(t-s_k)^{(a+1)(1-p_1)}}{\Gamma^2(\beta)\,(a+1)^{(1-p_1)}}\psi_\hbar(\Big\|y\Big\|_{\mathcal{PC}}^2)\|\varphi_\hbar\|_{L^{\frac{1}{p_1}}[s_k, t_{k+1}]}.
\end{aligned}
\tag{3.13}
$$

By hypothesis $(\hat{H}_3)$, Lemma 1.9, 3.1 and Hölder's inequality, we have

$$\begin{aligned} J_3' :=& \mathbb{E}\left\| \int_{s_k}^{t} (t-s)^{\beta-1} V(t-s) \aleph\Big(s, y_r(s), K_2 y_r(s)\Big) dW(s) \right\|^2 \\ \leq & Tr(\mathcal{Q}) \int_{s_k}^{t} \mathbb{E}\left\| (t-s)^{\beta-1} V(t-s) \aleph(s, y_r(s), K_2 y_r(s)) \right\|^2 ds \\ \leq & Tr(\mathcal{Q}) \int_{s_k}^{t} (t-s)^{2(\beta-1)} \mathbb{E}\left\| V(t-s) \aleph(s, y_r(s), K_2 y_r(s)) \right\|^2 ds \\ \leq & \frac{M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)} \int_{s_k}^{t} (t-s)^{2(\beta-1)} \mathbb{E}\left\| \aleph(s, y_r(s), K_2 y_r(s)) \right\|^2 ds \\ \leq & \frac{M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)} \left[\int_{s_k}^{t} (t-s)^{\frac{2(\beta-1)}{1-p}} ds \right]^{1-p} \left[\int_{s_k}^{t} \left(\varphi_{\aleph}(s) \psi_{\aleph}(\|y\|_{\mathcal{PC}}^2) \right)^{\frac{1}{p}} ds \right]^{p} \\ \leq & \frac{M^2 Tr(\mathcal{Q}) (t-s_k)^{(b+1)(1-p)}}{\Gamma^2(\beta) (b+1)^{(1-p)}} \psi_{\aleph}(\|y\|_{\mathcal{PC}}^2) \|\varphi_{\aleph}\|_{L^{\frac{1}{p}}[s_k, t_{k+1}]}. \end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) and (3.14) in (3.11), we obtain

$$\begin{aligned} \mathbb{E}\left\| \Gamma y_r(t) \right\|^2 \leq & 3M^2 C_{H_k} \|y\|_{\mathcal{PC}}^2 + \frac{M^2 T (t-s_k)^{(a+1)(1-p_1)}}{\Gamma^2(\beta) (a+1)^{(1-p_1)}} \psi_{\aleph}(\|y\|_{\mathcal{PC}}^2) \|\varphi_{\hbar}\|_{L^{\frac{1}{p_1}}[s_k, t_{k+1}]} \\ & + \frac{3M^2 Tr(\mathcal{Q}) (t-s_k)^{(b+1)(1-p)}}{\Gamma^2(\beta) (b+1)^{(1-p)}} \psi_{\aleph}(\|y\|_{\mathcal{PC}}^2) \|\varphi_{\aleph}\|_{L^{\frac{1}{p}}[s_k, t_{k+1}]}. \end{aligned} \tag{3.15}$$

Dividing both sides of (3.9), (3.10), (3.15) by r and taking limit as $r \to \infty$, hence, we have

$$\mathbb{E}\left\| \Gamma y_r(t) \right\|^2 \leq C,$$

with,

$$C := 3M^2 C_{H_k} + \frac{3M^2 L_{\hbar} T^{(a+1)(1-p_1)} N_{\hbar}}{\Gamma^2(\beta) (a+1)^{(1-p_1)}} + \frac{M^2 Tr(\mathcal{Q}) L_{\aleph} T^{(b+1)(1-p)} N_{\aleph}}{\Gamma^2(\beta) (b+1)^{(1-p)}} < 1.$$

Now we decompose Γ as $\Gamma = \Gamma_1 + \Gamma_2$, we can define Γ_1 and Γ_2 as follows:

$$(\Gamma_1 y)(t) = \begin{cases} U(t)\big(y_0 - \gamma(y)\big), & t \in [0, t_1], \\ H_k(t, y(t_k^-)), & t \in \bigcup_{k=1}^{m} (t_k, s_k], \\ U(t-s_k) H_k(s_k, y(s_k^-)), & t \in \bigcup_{k=1}^{m} (s_k, t_{k+1}]. \end{cases}$$

And

$$
\left(\mathit{\Gamma}_2 y\right)(t)=\begin{cases}\displaystyle\int_0^t (t-s)^{\beta-1} V(t-s)\hbar\Big(s,y(s),K_1y(s)\Big)ds \\ \qquad\qquad\displaystyle+\int_0^t (t-s)^{\beta-1} V(t-s)\aleph\Big(s,y(s),K_2y(s)\Big)dW(s) & t\in[0,t_1], \\ 0 & t\in\bigcup_{k=1}^{m}(t_k,s_k], \\ \displaystyle\int_{s_k}^t (t-s)^{\beta-1} V(t-s)\hbar\Big(s,y(s),K_1y(s)\Big)ds \\ \qquad\qquad\displaystyle+\int_{s_k}^t (t-s)^{\beta-1} V(t-s)\aleph\Big(s,y(s),K_2y(s)\Big)dW(s), & t\in\bigcup_{k=1}^{m}(s_k,t_{k+1}].\end{cases}
$$

Verifying that $\mathit{\Gamma}_1$ is a contraction and $\mathit{\Gamma}_2$ is a completely continuous operator will allow us to apply Theorem 1.4.

Step 2 In order to demonstrate that $\mathit{\Gamma}_1$ is a contraction, let $x,y\in\Omega_r$

Case 1 For any $t\in[0,t_1]$, using condition $(\hat{H}_4)$ and Lemma 1.9, we obtain

$$
\begin{aligned}
\mathbb{E}\Big\|\left(\mathit{\Gamma}_1 y\right)(t)-\left(\mathit{\Gamma}_1 x\right)(t)\Big\|^2 =&\mathbb{E}\Big\|U(t)\left(\gamma(y)-\gamma(x)\right)\Big\|^2\\
\leq& M^2C_\gamma\mathbb{E}\|y-x\|^2.
\end{aligned}
$$

Case 2 For each $t\in(t_k,s_k]$, $k=0,1,2,\cdots,m,$, by condition $(\hat{H}_5)$, we therefore obtain

$$
\begin{aligned}
\mathbb{E}\Big\|\left(\mathit{\Gamma}_1 y\right)(t)-\left(\mathit{\Gamma}_1 x\right)(t)\Big\|^2 =&\mathbb{E}\Big\|H_k(t,y(t_k^-))-H_k(t,x(t_k^-))\Big\|^2\\
\leq& C_{H_k}\mathbb{E}\|y(t)-x(t)\|^2.
\end{aligned}
$$

Case 3 For each $t\in(s_k,t_{k+1}]$, $k=0,1,2,\cdots,m$, by Lemma 1.9 and condition $(\hat{H}_5)$, we get

$$
\begin{aligned}
\mathbb{E}\Big\|\left(\mathit{\Gamma}_1 y\right)(t)-\left(\mathit{\Gamma}_1 x\right)(t)\Big\|^2 =&\mathbb{E}\Big\|U(t-s_k)\left(H_k(t,y(t_k^-))-H_k(t,x(t_k^-))\right)\Big\|^2\\
\leq& M^2C_{H_k}\mathbb{E}\|y(t)-x(t)\|^2.
\end{aligned}
$$

Then, we get

$$
\mathbb{E}\Big\|\left(\mathit{\Gamma}_1 y\right)(t)-\left(\mathit{\Gamma}_1 x\right)(t)\Big\|^2\leq\Big(M^2C_\gamma+\left(M^2+1\right)C_{H_k}\Big)\mathbb{E}\Big\|y-x\Big\|^2.
$$

We establish that $\mathit{\Gamma}_1$ is a contraction.

We now prove that $\mathit{\Gamma}_2$ is a completely continuous operator.

Step 3 We show that $\mathit{\Gamma}_2$ is continuous on $\mathcal{PC}([0,T],\mathbb{H})$. Let (y_n) be a sequence in $\mathcal{PC}([0,T],\mathbb{H})$ such that $\lim_{n\to\infty} y_n(t) = y(t)$ in $\mathcal{PC}([0,T],\mathbb{H})$. For every t in $[0,T]$. Thus, by , we obtain

$$\hbar\big(s, y_n(s), K_1 y_n(s)\big) \ \to \hbar\big(s, y(s), K_1 y(s)\big) \text{ as } n \to \infty,$$

and

$$\aleph\big(s, y_n(s), K_2 y_n(s)\big) \ \to \aleph\big(s, y(s), K_2 y(s)\big) \text{ as } n \to \infty.$$

Therefore by dominated convergence theorem, we can conclude that

$$\mathbb{E}\Big\|\hbar\left(s, y_n(s), K_1 y_n(s)\right) - \hbar\left(s, y(s), K_1 y(s)\right)\Big\|^2 \to 0 \text{ as } n \to \infty, \tag{3.16}$$

also

$$\mathbb{E}\Big\|\aleph\left(s, y_n(s), K_2 y_n(s)\right) - \aleph\left(s, y(s), K_2 y(s)\right)\Big\|^2 \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Case 1 For $t \in [0, t_1]$, by (3.16), (3.17), Lemma 1.9, Lemma 3.1, Hölder's inequality, and hypotheses $(\hat{H}_2)$-$(\hat{H}_3)$, we get the following

$$\begin{aligned}
&\mathbb{E}\Big\|\left(\mathit{\Gamma}_2 y_n\right)(t) - \left(\mathit{\Gamma}_2 y\right)(t)\Big\|^2 \\
\leq& 2\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1} V(t-s)\Big(\hbar\big(s, y_n(s), K_1 y_n(s)\big) - \hbar\big(s, y(s), K_1 y(s)\big)\Big)ds\Big\|^2 \\
&\quad + 2\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1} V(t-s)\Big(g\big(s, y_n(s), K_2 y_n(s)\big) - \aleph\big(s, y(s), K_2 y(s)\big)\Big)dW(s)\Big\|^2 \\
\leq& \frac{2M^2}{\Gamma^2(\beta)}\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1}\Big(\hbar\big(s, y_n(s), K_1 y_n(s)\big) - \hbar\big(s, y(s), K_1 y(s)\big)\Big)ds\Big\|^2 \\
&\quad + \frac{2M^2}{\Gamma^2(\beta)}\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1}\Big(\aleph\big(s, y_n(s), K_2 y_n(s)\big) - \aleph\big(s, y(s), K_2 y(s)\big)\Big)dW(s)\Big\|^2 \\
\leq& \frac{2M^2 t_1}{\Gamma^2(\beta)}\int_0^t (t-s)^{2(\beta-1)}\mathbb{E}\Big\|\hbar\big(s, y_n(s), K_1 y_n(s)\big) - \hbar\big(s, y(s), K_1 y(s)\big)\Big\|^2 ds \\
&\quad + \frac{2M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)}\int_0^t (t-s)^{2(\beta-1)}\mathbb{E}\Big\|\aleph\big(s, y_n(s), K_2 y_n(s)\big) - \aleph\big(s, y(s), K_2 y(s)\big)\Big\|^2 ds.
\end{aligned}$$

Case 2 For $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m$, using (3.16), (3.17), Lemma 1.9, Lemma 3.1, Hölder's inequality, and hypotheses $(\hat{H}_2)$-$(\hat{H}_3)$, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big\| \left(\mathit{\Gamma}_2 y_n\right)(t) - \left(\mathit{\Gamma}_2 y\right)(t)\Big\|^2 \\
\leq & 2\mathbb{E}\Big\| \int_{s_k}^{t} (t-s)^{\beta-1} V(t-s)\Big(\hbar\big(s, y_n(s), K_1 y_n(s)\big) - \hbar\big(s, y(s), K_1 y(s)\big)\Big) ds\Big\|^2 \\
&\quad + 2\mathbb{E}\Big\| \int_{s_k}^{t} (t-s)^{\beta-1} V(t-s)\Big(\aleph\big(s, y_n(s), K_2 y_n(s)\big) - \aleph\big(s, y(s), K_2 y(s)\big)\Big) dW(s)\Big\|^2 \\
\leq & \frac{2M^2}{\Gamma^2(\beta)}\mathbb{E}\Big\| \int_{s_k}^{t} (t-s)^{\beta-1} \Big(\hbar\big(s, y_n(s), K_1 y_n(s)\big) - \hbar\big(s, y(s), K_1 y(s)\big)\Big) ds\Big\|^2 \\
&\quad + \frac{2M^2}{\Gamma^2(\beta)}\mathbb{E}\Big\| \int_{s_k}^{t} (t-s)^{\beta-1} \Big(\aleph\big(s, y_n(s), G_\beta y_n(s)\big) - \aleph\big(s, y(s), K_2 y(s)\big)\Big) dW(s)\Big\|^2 \\
\leq & \frac{2M^2 (t-s_k)}{\Gamma^2(\beta)} \int_{s_k}^{t} (t-s)^{2(\beta-1)} \mathbb{E}\Big\|\hbar\big(s, y_n(s), K_1 y_n(s)\big) - \hbar\big(s, y(s), K_1 y(s)\big)\Big\|^2 ds \\
&\quad + \frac{2M^2 Tr(\mathcal{Q})}{\Gamma^2(\beta)} \int_{s_k}^{t} (t-s)^{2(\beta-1)} \mathbb{E}\Big\|\aleph\big(s, y_n(s), K_2 y_n(s)\big) - \aleph\big(s, y(s), K_2 y(s)\big)\Big\|^2 ds.
\end{aligned}
$$

Thus, we see that $\mathbb{E}\Big\|\mathit{\Gamma}_2 y_n - \mathit{\Gamma}_2 y\Big\|^2 \to 0$ as $n \to 0$.

As a result, which implies that the mapping $\mathit{\Gamma}_2$ is continuous on Ω_r.

Step 4 In this step, our aim is to prove that $\mathit{\Gamma}_2(\Omega_r)$ is an equicontinuous

Case 1 Let $\tau_1, \tau_2 \in [0, t_1]$, $\tau_1 < \tau_2$, for any $y \in \Omega_r$, we have

$$\begin{aligned}
\mathbb{E}\Big\| (\mathit{\Gamma}_2 y)(\tau_2) - (\mathit{\Gamma}_2 y)(\tau_1)\Big\|^2 \\
\leq & 2\mathbb{E}\Big\| \int_0^{\tau_2} \big(\tau_2 - s\big)^{\beta-1} V\big(\tau_2 - s\big) \hbar\big(s, y(s), K_1 y(s)\big) ds \\
& - \int_0^{\tau_1} \big(\tau_1 - s\big)^{\beta-1} V\big(\tau_1 - s\big) \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
& + 2\mathbb{E}\Big\| \int_0^{\tau_2} \big(\tau_2 - s\big)^{\beta-1} V\big(\tau_2 - s\big) \aleph\big(s, y(s), K_2 y(s)\big) dW(s) \\
& - \int_0^{\tau_1} \big(\tau_1 - s\big)^{\beta-1} V\big(\tau_1 - s\big) \aleph\big(s, y(s), GK_2 y(s)\big) dW(s) \Big\|^2 \\
\leq & \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \big(\tau_2 - s\big)^{\beta-1} \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
& + \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_0^{\tau_1} \Big(\big(\tau_2 - s\big)^{\beta-1} - \big(\tau_1 - s\big)^{\beta-1}\Big) \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
& + \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \big(\tau_2 - s\big)^{\beta-1} g\big(s, y(s), K_2 y(s)\big) dW(s) \Big\|^2 \\
& + \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_0^{\tau_1} \Big(\big(\tau_2 - s\big)^{\beta-1} - \big(\tau_1 - s\big)^{\beta-1}\Big) \aleph\big(s, y(s), K_2 y(s)\big) dW(s) \Big\|^2 \\
\leq & \frac{4M^2}{\Gamma^2(\beta)} \sum_{i=1}^{4} S_i.
\end{aligned} \tag{3.18}$$

Now, we only need to make sure that S_1, S_2, S_3 and S_4 tend to 0 independently of $y \in \Omega_r$ as $\tau_2 - \tau_1 \to 0$.

We calculate S_1 and S_2, by $(\hat{H}_2)$, Lemma 1.9 and Hölder's inequality, we get

$$\begin{aligned}
S_1 := & \mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \big(\tau_2 - s\big)^{\beta-1} \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
\leq & T\Big[\int_{\tau_1}^{\tau_2} \big(\tau_2 - s\big)^{\frac{2(\beta-1)}{1-p_1}} ds \Big]^{1-p_1} \Big[\int_{\tau_1}^{\tau_2} \Big(\mathbb{E}\Big\| \hbar\big(s, y(s), K_1 y(s)\big) \Big\|^2 \Big)^{\frac{1}{p_1}} ds \Big]^{p_1} \\
\leq & \frac{T\big(\tau_2 - \tau_2\big)^{(a+1)(1-p_1)}}{(a+1)^{1-p_1}} \psi_\hbar\big(\big\|y\big\|_{\mathcal{PC}}^2\big) \|\varphi_\hbar\|_{L^{\frac{1}{p_1}}[0,t_1]} \to 0 \text{ as } \tau_2 - \tau_1 \to 0,
\end{aligned}$$

and, we have

$$\begin{aligned}
S_2 :=&\mathbb{E}\Big\|\int_0^{\tau_1}\Big(\big(\tau_2-s\big)^{q-1}-\big(\tau_1-s\big)^{\beta-1}\Big)\hbar\big(s,y(s),K_1y(s)\big)ds\Big\|^2\\
\leq&\int_0^{\tau_1}\Big(\big(\tau_1-s\big)^{2(\beta-1)}-\big(\tau_2-s\big)^{2(\beta-1)}\Big)\mathbb{E}\Big\|\hbar\big(s,y(s),K_1y(s)\big)\Big\|^2ds\\
\leq&\Big[\int_0^{\tau_1}\Big(\big(\tau_1-s\big)^{2(\beta-1)}-\big(\tau_2-s\big)^{2(\beta-1)}\Big)^{\frac{1}{1-p_1}}ds\Big]^{1-p_1}\\
&\qquad\cdot\Big[\int_0^{\tau_1}\big(\mathbb{E}\Big\|\hbar\big(s,y(s),K_1y(s)\big)\Big\|^2\big)^{\frac{1}{p_1}}ds\Big]^{p_1}\\
\leq&\frac{\psi_\hbar\big(\|y\|^2_{\mathcal{PC}}\big)}{(a+1)}\Big(\big(\tau_2-\tau_1\big)^{a+1}-\big(\tau_2\big)^{a+1}+\big(\tau_1\big)^{a+1}\Big)^{(1-p_1)}t\Big[\int_0^{\tau_1}\big(\varphi_\hbar(s)\big)^{\frac{1}{p_1}}ds\Big]^{p_1}\\
\leq&\frac{\big(\tau_2-\tau_1\big)^{(a+1)(1-p_1)}}{(a+1)^{1-p_1}}\psi_\hbar\big(\|y\|^2_{\mathcal{PC}}\big)\|\varphi_\hbar\|_{L^{\frac{1}{p_1}}[0,t_1]}\to 0\ \text{ as }\ \tau_2-\tau_1\to 0.
\end{aligned}$$

We calculate S_3 and S_4 using Lemma 1.9, Lemma 3.1, $(\hat{H}_3)$ and Hölder's inequality, we get

$$\begin{aligned}
K_3 :=&\mathbb{E}\Big\|\int_{\tau_1}^{\tau_2}\big(\tau_2-s\big)^{\beta-1}\aleph\big(s,y(s),K_2y(s)\big)dW(s)\Big\|^2\\
\leq&Tr\big(\mathcal{Q}\big)\int_{\tau_1}^{\tau_2}\big(\tau_2-s\big)^{2(\beta-1)}\mathbb{E}\Big\|\aleph\big(s,y(s),K_2y(s)\big)\Big\|^2ds\\
\leq&Tr\big(\mathcal{Q}\big)\Big(\int_{\tau_1}^{\tau_2}\big(\tau_2-s\big)^{b}\Big)^{1-p}\Big(\int_{\tau_1}^{\tau_2}\big(\varphi_\aleph(s)\psi_\aleph\big(\|y\|^2_{\mathcal{PC}}\big)\big)^{\frac{1}{p}}ds\Big)^{p}\\
\leq&\frac{Tr\big(\mathcal{Q}\big)\big(\tau_2-\tau_1\big)^{(b+1)(1-p)}}{(b+1)^{1-p}}\psi_\aleph\big(\|y\|^2_{\mathcal{PC}}\big)\|\varphi_\aleph\|_{L^{\frac{1}{p}}[0,t_1]}\to 0\ \text{ as }\ \tau_2-\tau_1\to 0,
\end{aligned}$$

and

$$\begin{aligned}
S_4 :=&\mathbb{E}\Big\|\int_0^{\tau_1}\Big(\big(\tau_2-s\big)^{\beta-1}-\big(\tau_1-s\big)^{\beta-1}\Big)g\big(s,y(s),K_2y(s)\big)dW(s)\Big\|^2\\
\leq&Tr\big(\mathcal{Q}\big)\int_0^{\tau_1}\Big(\big(\tau_1-s\big)^{2(\beta-1)}-\big(\tau_2-s\big)^{2(\beta-1)}\Big)\varphi_\aleph(s)\psi_\aleph\big(\|y\|^2_{\mathcal{PC}}\big)ds\\
\leq&Tr\big(\mathcal{Q}\big)\Bigg(\int_0^{\tau_1}\Big(\big(\tau_1-s\big)^{2(\beta-1)}-\big(\tau_2-s\big)^{2(\beta-1)}\Big)^{\frac{1}{1-p}}ds\Bigg)^{1-p}\psi_\aleph\big(\|y\|^2_{\mathcal{PC}}\big)\|\varphi_g\|_{L^{\frac{1}{p}}[0,t_1]}\\
\leq&\frac{Tr\big(\mathcal{Q}\big)}{(b+1)}\Big(\big(\tau_2-\tau_1\big)^{b+1}-\big(\tau_2\big)^{b+1}+\big(\tau_1\big)^{b+1}\Big)^{(1-p)}\psi_\aleph\big(\|y\|^2_{\mathcal{PC}}\big)\|\varphi_\aleph\|_{L^{\frac{1}{p}}[0,t_1]}\\
\leq&\frac{Tr\big(\mathcal{Q}\big)\big(\tau_2-\tau_1\big)^{(b+1)(1-p)}}{(b+1)^{(1-p)}}\psi_\aleph\big(\|y\|^2_{\mathcal{PC}}\big)\|\varphi_\aleph\|_{L^{\frac{1}{p}}[0,t_1]}\to 0\ \text{ as }\ \tau_2-\tau_1\to 0.
\end{aligned}$$

Substituting (S_1)-(S_4) into (3.18), thus, for each $y \in \Omega_r$, can be easily seen that

$$\mathbb{E}\Big\| \left(\mathit{\Gamma}_2 y\right)\left(\tau_2\right) - \left(\mathit{\Gamma}_2 y\right)\left(\tau_1\right)\Big\|^2 \to 0 \text{ as } \tau_2 - \tau_1 \to 0.$$

Which means that $\mathit{\Gamma}_2\left(\Omega_r\right)$ is equicontinuous.

Case 2 Let $\tau_1, \tau_2 \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m$, $\tau_1 < \tau_2$, for any $y \in \Omega_r$, we have

$$\begin{aligned}
&\mathbb{E}\Big\| \left(\mathit{\Gamma}_2 y\right)\left(\tau_2\right) - \left(\mathit{\Gamma}_2 y\right)\left(\tau_1\right)\Big\|^2 \\
&\quad \leq 2\mathbb{E}\Big\| \int_{s_k}^{\tau_2} \left(\tau_2 - s\right)^{\beta-1} V\left(\tau_2 - s\right) \hbar\big(s, y(s), K_1 y(s)\big) ds \\
&\qquad\quad - \int_{s_k}^{\tau_1} \left(\tau_1 - s\right)^{\beta-1} V\left(\tau_1 - s\right) \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
&\qquad + 2\mathbb{E}\Big\| \int_{s_k}^{\tau_2} \left(\tau_2 - s\right)^{\beta-1} V\left(\tau_2 - s\right) \aleph\big(s, y(s), K_2 y(s)\big) dW(s) \\
&\qquad\quad - \int_{s_k}^{\tau_1} \left(\tau_1 - s\right)^{\beta-1} V\left(\tau_1 - s\right) \aleph\big(s, y(s), K_2 y(s)\big) dW(s) \Big\|^2 \\
&\quad \leq \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \left(\tau_2 - s\right)^{\beta-1} \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
&\qquad + \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_{s_k}^{\tau_1} \Big(\left(\tau_2 - s\right)^{\beta-1} - \left(\tau_1 - s\right)^{\beta-1} \Big) \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
&\qquad + \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \left(\tau_2 - s\right)^{\beta-1} \aleph\big(s, y(s), K_2 y(s)\big) dW(s) \Big\|^2 \\
&\qquad + \frac{4M^2}{\Gamma^2(\beta)} \mathbb{E}\Big\| \int_{s_k}^{\tau_1} \Big(\left(\tau_2 - s\right)^{\beta-1} - \left(\tau_1 - s\right)^{\beta-1} \Big) \aleph\big(s, y(s), K_2 y(s)\big) dW(s) \Big\|^2 \\
&\quad \leq \frac{4M^2}{\Gamma^2(\beta)} \sum_{i=1}^{4} S_i'.
\end{aligned}$$

In the same manner as S_1 and S_3, we display S_1' and S_3'. It suffices to show S_2' and S_4'. Using assumptions $(\hat{H}_2)$-$(\hat{H}_3)$, Lemma 1.9, Lemma 3.1 and Hölder's inequality, we obtain

$$\begin{aligned}
S_1' :=& \mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \left(\tau_2 - s\right)^{\beta-1} \hbar\big(s, y(s), K_1 y(s)\big) ds \Big\|^2 \\
\leq& \frac{T\big(\tau_2 - \tau_2\big)^{(a+1)(1-p_1)}}{(a+1)^{1-p_1}} \psi_\hbar\Big(\big\|y\big\|_{\mathcal{PC}}^2\Big) \|\varphi_\hbar\|_{L^{\frac{1}{p_1}}[0,t_1]} \to 0 \text{ as } \tau_2 - \tau_1 \to 0.
\end{aligned}$$

Also, we get

$$\begin{aligned}
S_2' :=&\mathbb{E}\Big\| \int_{s_k}^{\tau_1} \Big(\big(\tau_2 - s\big)^{\beta-1} - \big(\tau_1 - s\big)^{\beta-1}\Big)\hbar\big(s, y(s), K_1 y(s)\big)ds\Big\|^2 \\
\leq&(\tau_1 - s_k) \int_{s_k}^{\tau_1} \Big(\big(\tau_1 - s\big)^{2(q-1)} - \big(\tau_2 - s\big)^{2(\beta-1)}\Big)\mathbb{E}\Big\|\hbar\big(s, y(s), K_1 y(s)\big)\Big\|^2 ds \\
\leq&(\tau_1 - s_k)\Big[\int_{s_k}^{\tau_1} \Big(\big(\tau_1 - s\big)^{2(\beta-1)} - \big(\tau_2 - s\big)^{2(\beta-1)}\Big)^{\frac{1}{1-p^1}} ds\Big]^{1-p_1} \\
&\qquad\qquad .\Big[\int_{s_k}^{\tau_1} \big(\mathbb{E}\Big\|\hbar\big(s, y(s), K_1 y(s)\big)\Big\|^2\big)^{\frac{1}{p_1}} ds\Big]^{p_1} \\
\leq&\frac{(\tau_1 - s_k)}{(a+1)^{1-p_1}}\Big((\tau_2 - \tau_1)^{a+1} - (\tau_2 - s_k)^{a+1} + (\tau_1 - s_k)^{a+1} \Big)^{(1-p_1)} \\
&\qquad\qquad .\Big[\int_{s_k}^{\tau_1} \big(\varphi_\hbar(s)\psi_\hbar\big(\big\|y\big\|_{\mathcal{PC}}^2\big)\big)^{\frac{1}{p_1}} ds\Big]^{p_1} \\
\leq&\frac{(\tau_1 - s_k)\big(\tau_2 - \tau_1\big)^{(a+1)(1-p_1)}}{(a+1)^{(1-p_1)}}\psi_\hbar\big(\big\|y\big\|_{\mathcal{PC}}^2\big)\left\|\varphi_\hbar\right\|_{L^{\frac{1}{p_1}}[s_k, t_{k+1}]} \to 0 \text{ as } \tau_2 - \tau_1 \to 0.
\end{aligned}$$

And we also have

$$\begin{aligned}
S_3' :=&\mathbb{E}\Big\| \int_{\tau_1}^{\tau_2} \big(\tau_2 - s\big)^{\beta-1}\aleph\big(s, y(s), K_2 y(s)\big)dW(s)\Big\|^2 \\
\leq&\frac{Tr\big(\mathcal{Q}\big)\big(\tau_2 - \tau_1\big)^{(b+1)(1-p)}}{(b+1)^{1-p}}\psi_\aleph\big(\big\|y\big\|_{\mathcal{PC}}^2\big)\left\|\varphi_\aleph\right\|_{L^{\frac{1}{p}}[s_k, t_{k+1}]} \to 0 \quad \text{as} \quad \tau_2 - \tau_1 \to 0.
\end{aligned}$$

Also

$$\begin{aligned}
S_4' :=&\mathbb{E}\Big\| \int_{s_k}^{\tau_1} \Big(\big(\tau_2 - s\big)^{\beta-1} - \big(\tau_1 - s\big)^{\beta-1}\Big)\aleph\big(s, y(s), G_2 y(s)\big)dW(s)\Big\|^2 \\
\leq&Tr(\mathcal{Q}) \int_{s_k}^{\tau_1} \Big(\big(\tau_1 - s\big)^{2(\beta-1)} - \big(\tau_2 - s\big)^{2(\beta-1)}\Big)\mathbb{E}\Big\|\aleph\big(s, y(s), K_2 y(s)\big)\Big\|^2 ds \\
\leq&\frac{Tr\big(\mathcal{Q}\big)\big(\tau_2 - \tau_1\big)^{(b+1)(1-p)}}{(b+1)^{(1-p)}}\psi_\aleph\big(\big\|y\big\|_{\mathcal{PC}}^2\big)\left\|\varphi_\aleph\right\|_{L^{\frac{1}{p}}[s_k, t_{k+1}]} \to 0 \quad \text{as} \quad \tau_2 - \tau_1 \to 0.
\end{aligned}$$

Therefore, for $y \in \Omega_r$, $\mathbb{E}\Big\|\left(\mathit{\Gamma}_2 y\right)(\tau_2) - \left(\mathit{\Gamma}_2 y\right)(\tau_1)\Big\|^2 \to 0$ as $\tau_2 - \tau_1 \to 0$.

As a result $\mathit{\Gamma}_2$ is equicontinuous.

Consequently, the Kolmogorov-Riesz-Fréchet theorem (see [30]) indicates that $\mathit{\Gamma}_2(\Omega_r) \subset \Omega_r$ is relatively compact. $\mathit{\Gamma}_2$ is therefore implied to be a completely continuous operator by the continuity of $\mathit{\Gamma}_2$ and the relative compactness of $\mathit{\Gamma}_2(\Omega_r) \subset \Omega_r$.

Hence, $\varGamma = \varGamma_1 + \varGamma_2$ has a fixed point on Ω_r, as demonstrated by Krasnoselskii's fixed point theorem 1.4. As a result, a mild solution to the problem (3.1) exists. The proof is complete.

□

Then we will prove the uniqueness of the mild solution of FSIDE 3.1 based on contraction mapping theory.

Theorem 3.2. *Assume $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_4)$-$(\hat{H}_5)$-(2) holds, then the system (3.1) has a unique mild solution on $[0,T]$ provided that*

$$\Lambda := \max\Big\{\Lambda_1, \Lambda_2\Big\} < 1. \tag{3.19}$$

With

$$\Lambda_1 := 3M^2 C_\gamma + \frac{3M^2\big(C_\hbar + C'_\hbar C_{K_1} T\big) t_1^{2\mathbb{E}}}{\Gamma^2(\beta)\,(2\beta-1)} + \frac{3M^2 tr(\mathcal{Q})\big(C_\aleph + C'_\aleph C_{K_2} T\big) t_1^{2\beta-1}}{\Gamma^2(\beta)\,(2\beta-1)},$$

and

$$\Lambda_2 := 3M^2 C_H + \frac{3M^2\big(C_\hbar + C'_\hbar C_{K_1} T\big) T^{2\beta}}{\Gamma^2(\beta)\,(2\beta-1)} + \frac{3M^2 tr(\mathcal{Q})\big(C_\aleph + C'_\aleph C_{K_2} T\big) T^{2\beta-1}}{\Gamma^2(\aleph)\,(2\beta-1)}.$$

Proof. Let $x, y \in \Omega_r$, we consider Lemma 3.1, Lemma 1.9, $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_4)$-$(\hat{H}_5)$-(2) we obtain

Case 1 For any $t \in [0, t_1]$, by Lemma 1.9, we get

$$\begin{aligned}
&\mathbb{E}\Big\|\,(\varGamma y)\,(t) - (\varGamma x)\,(t)\Big\|^2 \\
&\qquad = \mathbb{E}\Bigg\| U(t)\big(y_0 - \gamma(y)\big) + \int_0^t (t-s)^{\beta-1} V(t-s)\hbar\big(s, y(s), K_1 y(s)\big)ds \\
&\qquad\qquad + \int_0^t (t-s)^{\beta-1} V(t-s)\aleph\big(s, y(s), K_2 y(s)\big)dW(s)\Bigg\|^2 \\
&\qquad\quad - U(t)\big(y_0 - \gamma(x)\big) - \int_0^t (t-s)^{\beta-1} V(t-s)\hbar\big(s, x(s), K_1 x(s)\big)ds \\
&\qquad\qquad - \int_0^t (t-s)^{\beta-1} V(t-s)\aleph\big(s, x(s), K_2 x(s)\big)dW(s)\Bigg\|^2 \\
&\qquad \le 3M^2\mathbb{E}\Big\|\gamma(y) - \gamma(x)\Big\|^2 \\
&\qquad\quad + \frac{3M^2}{\Gamma^2(\beta)}\mathbb{E}\Bigg\|\int_0^t (t-s)^{\beta-1}\Big(\hbar\big(s, y(s), K_1 y(s)\big) - \hbar\big(s, x(s), K_1 x(s)\big)\Big)ds\Bigg\|^2 \\
&\qquad\quad + \frac{3M^2}{\Gamma^2(\beta)}\mathbb{E}\Bigg\|\int_0^t (t-s)^{\beta-1}\Big(\aleph\big(s, y(s), K_2 y(s)\big) - \aleph\big(s, x(s), K_2 x(s)\big)\Big)dW(s)\Bigg\|^2.
\end{aligned}$$

By applying Lemma 3.1, $(\hat{H}_2)$-$(\hat{H}_4)$, we have

$$
\begin{aligned}
\mathbb{E}\Big\| (\mathit{\Gamma} y)(t) - (\mathit{\Gamma} x)(t)\Big\|^2 \leq & 3M^2 C_\gamma \mathbb{E}\Big\|y-x\Big\|^2 \\
& + \frac{3M^2 t_1}{\Gamma^2(\beta)} \int_0^t (t-s)^{\beta-1}\Big(C_\hbar \mathbb{E}\Big\|y(s)-x(s)\Big\|^2 \\
& \qquad + C'_\hbar \mathbb{E}\Big\|K_1 y(s) - K_1 x(s)\Big\|^2\Big) ds \\
& + \frac{3M^2 tr(\mathcal{Q})}{\Gamma^2(\beta)} \int_0^t (t-s)^{\beta-1}\Big(C_\aleph \mathbb{E}\Big\|y(s)-x(s)\Big\|^2 \\
& \qquad + C'_\aleph \mathbb{E}\Big\|K_2 y(s) - K_2 x(s)\Big\|^2\Big) ds \\
\leq & 3M^2 C_\gamma \mathbb{E}\Big\|y-x\Big\|^2 \\
& + \frac{3M^2 t_1 \big(C_\hbar + C'_\hbar C_{K_1} T\big)}{\Gamma^2(\beta)} \int_0^t (t-s)^{\beta-1} \mathbb{E}\Big\|y(s)-x(s)\Big\|^2 ds \\
& + \frac{3M^2 tr(\mathcal{Q}) \big(C_\aleph + C'_\aleph C_{K_2} T\big)}{\Gamma^2(\beta)} \int_0^t (t-s)^{\beta-1} \mathbb{E}\Big\|y(s)-x(s)\Big\|^2 ds.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathbb{E}\Big\| (\mathit{\Gamma} y)(t) - (\mathit{\Gamma} x)(t)\Big\|^2 \leq \Big(3M^2 C_\gamma + & \frac{3M^2 \big(C_\hbar + C'_\hbar C_{K_1} T\big) t_1^{2q}}{\Gamma^2(\beta)(2\beta-1)} \\
& + \frac{3M^2 tr(\mathcal{Q}) \big(C_\aleph + C'_\aleph C_{K_2} T\big) t_1^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\Big) \mathbb{E}\Big\|y-x\Big\|^2.
\end{aligned}
$$

Case 2 For each $t \in (t_k, s_k]$, $k = 0, 1, 2, \cdots, m$, by assumption $(\hat{H}_5)$, we obtain

$$
\begin{aligned}
\mathbb{E}\Big\| (\mathit{\Gamma} y)(t) - (\mathit{\Gamma} x)(t)\Big\|^2 = & \mathbb{E}\Big\|H_k(t, y(t_k^-)) - H_k(t, x(t_k^-))\Big\|^2 \\
\leq & C_{H_k} \mathbb{E}\|y(t) - x(t)\|^2.
\end{aligned}
$$

Hence, we have

$$
\mathbb{E}\Big\| (\mathit{\Gamma} y)(t) - (\mathit{\Gamma} x)(t)\Big\|^2 \leq C_{H_k} \mathbb{E}\Big\|y-x\Big\|^2.
$$

Case 3 For every $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m,$, by Lemma 1.9, $(\hat{H}_2)$, $(\hat{H}_3)$, $(\hat{H}_5)$ and Hölder's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\Big\| \left(\mathit{\Gamma} y\right)(t) - \left(\mathit{\Gamma} x\right)(t)\Big\|^2 \\
&\qquad =\mathbb{E}\Big\| U(t-s_k)\big(H_k(t, y(t_k^-))\big) + \int_{s_k}^{t} \big(t-s\big)^{\beta-1} V\big(t-s\big)\hbar\big(s, y(s), K_1 y(s)\big)ds \\
&\qquad\quad + \int_{s_k}^{t} \big(t-s\big)^{\beta-1} V\big(t-s\big)\aleph\big(s, y(s), K_2 y(s)\big)dW(s) \\
&\qquad\quad - U(t-s_k)\big(H_k(t, x(t_k^-))\big) - \int_{s_k}^{t} \big(t-s\big)^{\beta-1} V\big(t-s\big)\hbar\big(s, x(s), K_1 x(s)\big)ds \\
&\qquad\quad - \int_{s_k}^{t} \big(t-s\big)^{\beta-1} V\big(t-s\big)\aleph\big(s, x(s), K_2 x(s)\big)dW(s)\Big\|^2 \\
&\qquad \leq 3M^2 \mathbb{E}\Big\| H_k(t, y(t_k^-)) - H_k(t, x(t_k^-))\Big\|^2 \\
&\qquad\quad - \frac{3M^2}{\Gamma^2(B)}\mathbb{E}\Big\| \int_{s_k}^{t} \big(t-s\big)^{\beta-1}\Big(\hbar\big(s, y(s), K_1 y(s)\big) - \hbar\big(s, x(s), K_1 x(s)\big)\Big)ds\Big\|^2 \\
&\qquad\quad + \frac{3M^2}{\Gamma^2(\beta)}\mathbb{E}\Big\| \int_{s_k}^{t} \big(t-s\big)^{\beta-1}\Big(\aleph\big(s, y(s), K_2 y(s)\big) - \aleph\big(s, x(s), K_2 x(s)\big)\Big)dW(s)\Big\|^2.
\end{aligned}
$$

Therefore, by Lemmas 3.1 , $(\hat{H}_2)$, $(\hat{H}_3)$ and $(\hat{H}_5)$, we have

$$
\begin{aligned}
&\mathbb{E}\Big\| \left(\mathit{\Gamma} y\right)(t) - \left(\mathit{\Gamma} x\right)(t)\Big\|^2 \\
&\qquad \leq 3M^2 C_H \mathbb{E}\Big\| y(t) - x(t)\Big\|^2 \\
&\qquad\quad + \frac{3M^2\big(C_\hbar + C'_\hbar C_{K_1} T\big)(t-s_k)}{\Gamma^2(\beta)} \int_{s_k}^{t} \big(t-s\big)^{2(\beta-1)}\mathbb{E}\Big\| y(s) - x(s)\Big\|^2 ds \\
&\qquad\quad + \frac{3M^2 tr(\mathcal{Q})\big(C_\aleph + C'_\aleph C_{K_2} T\big)}{\Gamma^2(\beta)} \int_{s_k}^{t} \big(t-s\big)^{2(\beta-1)}\mathbb{E}\Big\| y(s) - x(s)\Big\|^2 ds.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathbb{E}\Big\| \left(\mathit{\Gamma} y\right)(t) - \left(\mathit{\Gamma} x\right)(t)\Big\|^2 \leq & \Bigg(3M^2 C_H + \frac{3M^2\big(C_\hbar + C'_\hbar C_{K_1} T\big)(t-s_k)^{2\beta}}{\Gamma^2(\beta)\,(2\beta-1)} \\
& \qquad + \frac{3M^2 tr(\mathcal{Q})\big(C_\aleph + C'_\aleph C_{K_2} T\big)(t-s_k)^{2\beta-1}}{\Gamma^2(\beta)\,(2\beta-1)}\Bigg)\mathbb{E}\Big\| y - x\Big\|^2.
\end{aligned}
$$

Hence, it follows from (3.2) that

$$
\mathbb{E}\Big\| \left(\mathit{\Gamma} y\right)(t) - \left(\mathit{\Gamma} x\right)(t)\Big\|^2 \leq \Lambda \mathbb{E}\Big\| y - x\Big\|^2.
$$

The above inequalities imply that $\mathit{\Gamma}$ is contraction on Ω_r.

Clearly, we can conclude that the operator Γ has a unique fixed point $y(t) \in \Omega_r$ based on the Banach contraction principle. Thus, the Theorem is proved.

$\square$

We now present the second result of this chapter.

3.3 Ulam-Hyers-Rassias Stability results

In this section , we examine the UH Stability of the solution for the system (3.1)

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a Banach space, $\mathcal{C}^1\big([0,T], \mathbb{X}\big)$ is defined as a space of continuously differentiable functions. Let $f : [0,T] \times \mathbb{X} \to \mathbb{X}$ be a continuous operator, $\xi\,[0,T] \to \mathbb{R}^+$ be a continuous nondecreasing function and $\varepsilon > 0$ be a positive real number. We take the following

$$y'(t) = f(t, y(t)), \quad \text{for each } t \in [0,T], \tag{3.20}$$

and the following differential inequations for each $t \in [0,T]$

$$\|y'(t) - f(t, y(t))\|_{\mathbb{X}} \leq \varepsilon, \tag{3.21}$$

$$\|y'(t) - f(t, y(t))\|_{\mathbb{X}} \leq \upsilon_f(t), \tag{3.22}$$

$$\|y'(t) - f(t, y(t))\|_{\mathbb{X}} \leq \varepsilon\xi(t). \tag{3.23}$$

According to [142], the stability of (3.20) in terms of Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias are as follows.

Definition 3.2. [142] The equation (3.20) is Ulam-Hyers stable if there exists a positive constant c_f such that for every $\varepsilon > 0$ and for every solution $y \in \mathcal{C}^1\big([0,T], \mathbb{X}\big)$ of (3.21) there exists a solution $x \in \mathcal{C}^1\big([0,T], \mathbb{X}\big)$ equation (3.20) with

$$\|y(t) - x(t)\|_{\mathbb{X}} \leq c_f\varepsilon, \quad \text{for every } t \in [0,T].$$

Definition 3.3. [142] The equation (3.20) is generalized Ulam-Hyers stable if there exists $\upsilon(\varepsilon) \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\upsilon(0) = 0$ such that for all $\varepsilon > 0$ and for each solution $y \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{X})$ of the inequality (3.22) there exists a solution $x \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{X})$ problem (3.20) with

$$\|y(t) - x(t)\|_{\mathbb{X}} \leq \upsilon(\varepsilon), \quad \text{for every } t \in [0, T]\,.$$

Definition 3.4. [142] The equation (3.20) is Ulam-Hyers-Rassias stable with respect to ξ if there exists $c_{f,\xi} > 0$ so that for all $\varepsilon > 0$ and for every solution $y \in \mathcal{C}^1\big([0,T], \mathbb{X}\big)$ of (3.23) there exists a solution $x \in \mathcal{C}^1\big([0,T], \mathbb{X}\big)$ of (3.20) with

$$\|y(t) - x(t)\|_{\mathbb{X}} \leq c_{f,\xi}\varepsilon\xi(t), \quad \text{for every } t \in [0, T]\,.$$

Definition 3.5. [142] The equation(3.20) is generalized Ulam-Hyers-Rassias stable with respect to ξ if there exists a real number $c_{f,\xi}$ such that for each solution $y \in \mathcal{C}^1\big([0,T], \mathbb{X}\big)$ of the inequality (3.22) there exists a solution $x \in \mathcal{C}^1\big([0,T], \mathbb{X}\big)$ Problem (3.20) with

$$\|y(t) - x(t)\|_{\mathbb{X}} \leq c_{f,\xi}\xi(t), \quad \text{for every } t \in [0, T]\,.$$

Remark 3.1. It is clear that

(i) Definition 3.2 $\Longrightarrow$ Definition 3.3.

(ii) Definition 3.4 $\Longrightarrow$ Definition 3.5.

(iii) Definition 3.4 for $\xi(t) = 1 \Longrightarrow$ Definition 3.2.

Similar to [3, 142] Ulam-Hyers stability and Ulam-Hyers-Rassias stability of (3.1) are given as follows:

Definition 3.6. The equation (3.1) is Ulam-Hyers stable with respect to ε if there exists a contant $C_1 > 0$, such that for each $\varepsilon > 0$ and a solution $y \in \mathcal{PC}\big([0,T], \mathbb{H}\big)$ of the following inequality:

$$\begin{cases} \mathbb{E}\Big\| y(t) - U(t)\big(y_0 - \gamma(y)\big) \\ \qquad - \int_0^t \big(t-s\big)^{\beta-1} V\big(t-s\big) \hbar\Big(s, y(s), K_1 y(s)\Big) ds \\ \qquad - \int_0^t \big(t-s\big)^{\beta-1} V\big(t-s\big) \aleph\Big(s, y(s), K_2 y(s)\Big) dW(s) \Big\|^2 \leq \varepsilon, & t \in [0, t_1]\,, \\ \mathbb{E}\Big\| y(t) - H_k(t, y(t_k^-)) \Big\|^2 \leq \varepsilon, & t \in \bigcup_{k=1}^{m} (t_k, s_k], \\ \\ \mathbb{E}\Big\| y(t) - U(t-s_k) H_k(s_k, y(s_k)) \\ \qquad - \int_{s_k}^t \big(t-s\big)^{\beta-1} V\big(t-s\big) - \hbar\Big(s, y(s), K_1 y(s)\Big) ds \\ \qquad - \int_{s_k}^t \big(t-s\big)^{\beta-1} V\big(t-s\big) \aleph\Big(s, y(s), K_2 y(s)\Big) dW(s) \Big\|^2 \leq \varepsilon, & t \in \bigcup_{k=1}^{m} (s_k, t_{k+1}], \end{cases} \tag{3.24}$$

and there is a solution $x \in \mathcal{PC}\big([0,T]\,, \mathbb{H}\big)$ of system (3.1), satisfies

$$\mathbb{E}\Big\| y(t) - x(t) \Big\|^2 \leq C_1.\varepsilon, \quad \forall t \in [0, T]\,.$$

With the initial value $y(0) = x(0)$.

Definition 3.7. Equation (3.1) is Ulam-Hyers-Rassias stable in the mean square with respect to $\xi : [0,T] \to \mathbb{R}^+$ is a continuous nondecreasing function and if there exists a contant $C_2 > 0$, such that for each solution $y \in \mathcal{PC}\big([0,T],\mathbb{H}\big)$ of the following inequality:

$$\begin{cases} \mathbb{E}\Big\| y(t) - U(t)\big(y_0 - \gamma(y)\big) \\ \qquad - \int_0^t (t-s)^{\beta-1} V(t-s)\hbar\Big(s, y(s), K_1 y(s)\Big)ds \\ \qquad - \int_0^t (t-s)^{\beta-1} V(t-s)\aleph\Big(s, y(s), K_2 y(s)\Big)dW(s)\Big\|^2 \le \varepsilon\xi(t), \quad t \in [0, t_1], \\ \mathbb{E}\Big\| y(t) - H_k(t, y(t_k^-))\Big\|^2 \le \varepsilon\xi(t), \qquad t \in \bigcup_{k=1}^m (t_k, s_k], \\ \\ \mathbb{E}\Big\| y(t) - U(t-s_k)H_k(s_k, y(s_k)) \\ \qquad - \int_{s_k}^t (t-s)^{\beta-1} V(t-s) - \hbar\Big(s, y(s), K_1 y(s)\Big)ds \\ \qquad - \int_{s_k}^t (t-s)^{\beta-1} V(t-s)\aleph\Big(s, y(s), K_2 y(s)\Big)dW(s)\Big\|^2 \le \varepsilon\xi(t), \quad t \in \bigcup_{k=1}^m (s_k, t_{k+1}], \end{cases} \tag{3.25}$$

there is a solution $x \in \mathcal{PC}\big([0,T],\mathbb{H}\big)$ of system (3.1), with the initial value $y(0) = x(0)$., satisfies

$$\mathbb{E}\Big\| y(t) - x(t)\Big\|^2 \le C_2\varepsilon\xi(t), \quad \forall t \in [0,T]. \tag{3.26}$$

where C_2 is a constant that does not depend on $x(t)$.

Remark 3.2. It is clear that Definition 3.7 for $\xi(t) = 1 \implies$ Definition 3.6.

Theorem 3.3. *[50](generalized Grönwall inequality) Let $x(t)$ be real, continuous, and nonnegative on $[0,T]$ such that*

$$x(t) \le \omega(t) + \upsilon(t)\int_0^t k(t,s)x(s)ds,$$

where $\omega(t) \ge 0, \upsilon(t) \ge 0$ and $k(t,s) \ge 0$ are continuous functions for $0 \le s \le t \le T$. Then

$$x(t) \le \delta(t)\exp\Big(\nu(t)\int_0^t K(t,s)ds\Big),$$

where

$$\delta(t) = \sup_{0\le s\le t}\omega(s),\ \nu(t) = \sup_{0\le s\le t}\upsilon(s),\ K(t,s) = \sup_{s\le r\le t}k(r,t).$$

Theorem 3.4. *Assume that* $(\hat{H}_2)$-$(\hat{H}_3)$-(3) *and* $(\hat{H}_4)$-$(\hat{H}_5)$- (2) *are valid. If*

$$1 - 4M^2 C_{H_k} > 0.$$

The system (3.1) is Ulam-Hyers stable and also generalized Ulam-Hyers stable with respect to ε.

Proof. Consider $y(t)$ to be the solution to FSIDEs (3.1) and $x(t)$ to be the solution to inequality (3.24).

Case 1 For each $t \in [0, t_1]$, by Lemma 1.9, Lemma 3.1 and Theorem 3.3, we have

$$\begin{aligned}
&\mathbb{E}\Big\|y(t) - x(t)\Big\|^2 \\
&\quad \leq \mathbb{E}\Big\|y(t) - U(t)\big(x_0 - \gamma(x)\big) - \int_0^t (t-s)^{\beta-1} V(t-s)\hbar\Big(s, x(s), K_1 x(s)\Big)ds \\
&\qquad - \int_0^t (t-s)^{\beta-1} V(t-s)\aleph\Big(s, x(s), K_2 x(s)\Big)dW(s)\Big\|^2 \\
&\quad \leq 3\mathbb{E}\Big\|y(t) - U(t)\big(y_0 - \gamma(y)\big) - \int_0^t (t-s)^{q-1} V(t-s)\hbar\Big(s, y(s), K_1 y(s)\Big)ds \\
&\qquad - \int_0^t (t-s)^{\beta-1} V(t-s)\aleph\Big(s, y(s), K_2 y(s)\Big)dW(s)\Big\|^2 \\
&\qquad + 3\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1} V(t-s)\Big(\hbar\big(s, y(s), K_1 y(s)\big) - \hbar\big(s, x(s), K_1 x(s)\big)\Big)ds\Big\|^2 \\
&\qquad + 3\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1} V(t-s)\Big(\hbar\big(s, y(s), K_2 y(s)\big) - \aleph\big(s, x(s), K_2 x(s)\big)\Big)dW(s)\Big\|^2 \\
&\quad \leq 3\varepsilon + \frac{3M^2 T}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} \\
&\qquad\qquad \cdot \mathbb{E}\Big\|\hbar\big(s, y(s), K_1 y(s)\big) - \hbar\big(s, x(s), K_1 x(s)\big)\Big\|^2 ds \\
&\qquad + \frac{3M^2 Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} \\
&\qquad\qquad \cdot \mathbb{E}\Big\|\aleph\big(s, y(s), K_2 y(s)\big) - \aleph\big(s, x(s), K_2 x(s)\big)\Big\|^2 ds.
\end{aligned}$$

Then, using hypotheses $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_4)$- (2), we get

$$\begin{aligned}
\mathbb{E}\Big\|y(t) - x(t)\Big\|^2 \leq & 3\varepsilon + \frac{3M^2\big(C_\hbar + C'_\hbar C_{K_1} T\big)T}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} \mathbb{E}\Big\|y(s) - x(s)\Big\|^2 ds \\
& + \frac{3M^2\big(C_\aleph + C'_\aleph K_{K_2} T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} \mathbb{E}\Big\|y(s) - x(s)\Big\|^2 ds.
\end{aligned}$$

Hence, using Grönwall inequality (Theorem 3.3), we get

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq c_1\epsilon, \tag{3.27}$$

with

$$c_1 = 3\exp\left(\frac{3M^2\big(C_\hbar + C_f' C_{K_1} T\big)T^{2\beta} + 3M^2 Tr\big(\mathcal{Q}\big)\big(C_\aleph + C_\aleph' C_{K_2} T\big)T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\right).$$

Case 2 For each $t \in (t_k, s_k]$, $k = 0, 1, 2, \cdots, m$, by $(\hat{H}_5)$-(2), we have

$$\begin{aligned}\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq & 2\mathbb{E}\Big\|y(t)-H_k(t, y(t_k^-))\Big\|^2 \\ & + 2\mathbb{E}\Big\|H_k(t, y(t_k^-)) - H_k(t, x(t_k^-))\Big\|^2 \\ \leq & 2\epsilon + 2C_{H_k}\mathbb{E}\Big\|y(t)-x(t)\Big\|^2.\end{aligned}$$

Therefore

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq c_2\epsilon, \tag{3.28}$$

with $c_2 = \frac{2}{1-2C_{H_k}}$.

Case 3 For each $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m$, by $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_5)$-(2), Lemma

1.9 and Theorem 3.3, we obtain

$$
\begin{aligned}
&\mathbb{E}\Big\|y(t)-x(t)\Big\|^2\\
\leq&\mathbb{E}\Big\|y(t)-U(t-s_k)H_k(s_k,x(s_k))\\
&\quad-\int_{s_k}^{t}\big(t-s\big)^{\beta-1}V\big(t-s\big)-\hbar\Big(s,x(s),K_1x(s)\Big)ds\\
&\quad-\int_{s_k}^{t}\big(t-s\big)^{\beta-1}V\big(t-s\big)\aleph\Big(s,x(s),K_2x(s)\Big)dW(s)\Big\|^2\\
\leq&4\mathbb{E}\Big\|y(t)-U(t-s_k)H_k(s_k,y(s_k))\\
&\qquad-\int_{s_k}^{t}\big(t-s\big)^{\beta-1}V\big(t-s\big)-\hbar\Big(s,y(s),K_1y(s)\Big)ds\\
&\qquad-\int_{s_k}^{t}\big(t-s\big)^{\beta-1}V\big(t-s\big)\aleph\Big(s,y(s),K_2y(s)\Big)dW(s)\Big\|^2\\
&\quad+4\mathbb{E}\Big\|U(t-s_k)\big(H_k(s_k,y(s_k))-H_k(s_k,x(s_k))\big)\Big\|^2\\
&\quad+4\mathbb{E}\Big\|\int_{s_k}^{t}\big(t-s\big)^{\beta-1}V\big(t-s\big)\\
&\qquad\qquad\cdot\Big(\hbar\big(s,y(s),K_1y(s)\big)-\hbar\big(s,x(s),K_1x(s)\big)\Big)ds\Big\|^2\\
&\quad+4\mathbb{E}\Big\|\int_{s_k}^{t}\big(t-s\big)^{\beta-1}V\big(t-s\big)\\
&\qquad\qquad\cdot\Big(\aleph\big(s,y(s),K_2y(s)\big)-g\big(s,x(s),K_2x(s)\big)\Big)dW(s)\Big\|^2\\
\leq&4\varepsilon+4M^2\mathbb{E}\Big\|H_k(s_k,y(s_k))-H_k(s_k,x(s_k))\Big\|^2\\
&\quad+\frac{4M^2T}{\Gamma^2(\beta)}\int_{s_k}^{t}\big(t-s\big)^{2\beta-2}\mathbb{E}\Big\|\hbar\big(s,y(s),K_1y(s)\big)-\hbar\big(s,x(s),K_1x(s)\big)\Big\|^2ds\\
&\quad+\frac{4M^2Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)}\int_{s_k}^{t}\big(t-s\big)^{2\beta-2}\mathbb{E}\Big\|\aleph\big(s,y(s),K_2y(s)\big)-\aleph\big(s,x(s),K_2x(s)\big)\Big\|^2ds\\
\leq&\frac{4\varepsilon}{1-4M^2C_{H_k}}+\frac{4M^2\big(C_\hbar+C'_\hbar C_{K_1}T\big)T}{\Gamma^2(\beta)(1-4M^2C_{H_k})}\int_{s_k}^{t}\big(t-s\big)^{2\beta-2}\mathbb{E}\Big\|y(s)-x(s)\Big\|^2ds\\
&\quad+\frac{4M^2\big(C_\aleph+C'_\aleph C_{K_2}T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)(1-4M^2C_{H_k})}\int_{s_k}^{t}\big(t-s\big)^{2\beta-2}\mathbb{E}\Big\|y(s)-x(s)\Big\|^2ds.
\end{aligned}
$$

Then applying Theorem 3.3 (Generalized Grönwall inequality), we have

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq \frac{4}{1-4M^2C_{H_k}}.\varepsilon$$
$$.\exp\Bigg(\Big(\frac{4M^2\big(C_\hbar+C'_\hbar C_{K_1}T\big)T}{\Gamma^2(\beta)(1-4M^2C_{H_k})}+\frac{4M^2\big(C_\aleph+C'_\aleph C_{K_2}T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)(1-4M^2C_{H_k})}\Big)\int_{s_k}^t\big(t-s\big)^{2\beta-2}ds\Bigg).$$

As a result, we have

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq c_3\varepsilon, \tag{3.29}$$

with

$$c_3=\frac{4}{1-4M^2C_{H_k}}\exp\Bigg(\frac{4M^2\big(C_\hbar+C'_\hbar C_{K_1}T\big)T^{2\beta}+4M^2\big(C_\aleph+C'_\aleph C_{K_2}T\big)Tr\big(\mathcal{Q}\big)T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)(1-4M^2C_{H_k})}\Bigg).$$

Thus,

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq C_1\varepsilon. \tag{3.30}$$

As shown in (3.27), (3.28) and (3.29), C_1 exists, with $C_1=\max\{c_1,c_2,c_3\}$ that satisfies Definition 3.6. The solution to (3.1) is Ulam-Hyers stable in reference to ε.

In addition, assume that $\upsilon\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$, $\upsilon(0)=0$ is a non-decreasing function. We can then write from (3.30)

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq \upsilon(\varepsilon), \quad \text{with } \upsilon(0)=0. \tag{3.31}$$

Hence, (3.31) shows that the solution is also generalized Ulam-Hyers stable.

□

Remark 3.3. Assuming the definition 3.4, we examine FSIDEs (3.1) and inequalities (3.26). Ulam-Hyers-Rassias stability of FSIDEs (3.1) can be verified by repeating the same procedure.

Theorem 3.5. *Suppose that the hypothese* $(\hat{H}_2)$-$(\hat{H}_3)$-(3) *and* $(\hat{H}_4)$-$(\hat{H}_5)$- (2) *are satisfied. If*

$$1-4M^2C_{H_k}>0$$

Then FSIDEs (3.1) is Ulam-Hyers-Rassias stable.

Proof. Let $y(t)$ to be the solution to FSIDEs (3.1) and $x(t)$ to be the solution to inequality (3.24).

Case 1 For any $t \in [0, t_1]$, by Lemma 1.9, Lemma 3.1, Theorem 3.3 and assumptions $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_4)$-(2), we find

$$
\begin{aligned}
&\mathbb{E}\Big\|y(t)-x(t)\Big\|^2\\
\leq & 3\mathbb{E}\Big\|y(t)-U(t)\big(y_0-\gamma(y)\big)\\
&\quad -\int_0^t (t-s)^{\beta-1}V(t-s)\hbar\Big(s,y(s),K_1y(s)\Big)ds\\
&\quad -\int_0^t (t-s)^{\beta-1}V(t-s)\aleph\Big(s,y(s),K_2y(s)\Big)dW(s)\Big\|^2\\
&+3\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1}V(t-s)\\
&\qquad\qquad \cdot\Big(\hbar\big(s,y(s),K_1y(s)\big)-\hbar\big(s,x(s),K_1x(s)\big)\Big)ds\Big\|^2\\
&+3\mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1}V(t-s)\\
&\qquad\qquad \cdot\Big(\aleph\big(s,y(s),K_2y(s)\big)-\aleph\big(s,x(s),K_2x(s)\big)\Big)dW(s)\Big\|^2\\
\leq & 3\varepsilon\xi(t)+\frac{3M^2\big(C_\hbar+C'_\hbar C_{K_1}T\big)T}{\Gamma^2(\beta)}\int_0^t (t-s)^{2\beta-2}\mathbb{E}\Big\|y(s)-x(s)\Big\|^2ds\\
&+\frac{3M^2\big(C_\aleph+C'_\aleph C_{K_2}T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)}\int_0^t (t-s)^{2\beta-2}\mathbb{E}\Big\|y(s)-x(s)\Big\|^2ds.
\end{aligned}
$$

Hence, employing Grönwall inequality (Theorem 3.3), we get

$$
\begin{aligned}
\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq & 3\varepsilon\xi(t)\\
&.\exp\bigg(\Big(\frac{3M^2\big(C_\hbar+C'_\hbar C_{K_1}T\big)T}{\Gamma^2(\beta)}+\frac{3M^2\big(C_\aleph+C'_\aleph C_{K_2}T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)}\Big)\int_0^t (t-s)^{2\beta-2}ds\bigg),
\end{aligned}
$$

therefore, we get

$$
\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq c'_1\varepsilon\xi(t), \tag{3.32}
$$

with

$$
c'_1 = 3\exp\Big(\frac{3M^2\big(C_\hbar+C'_\hbar C_{K_1}T\big)T^{2\beta}+3M^2\big(C_\aleph+C'_\aleph C_{K_2}T\big)Tr\big(\mathcal{Q}\big)T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)}\Big).
$$

Case 2 For each $t \in (t_k, s_k]$, $k = 0, 1, 2, \cdots, m$, by $(\hat{H}_5)$-(2), we have

$$\begin{aligned}\mathbb{E}\Big\|y(t) - x(t)\Big\|^2 \leq & 2\mathbb{E}\Big\|y(t) - H_k(t, y(t_k^-))\Big\|^2 + 2\mathbb{E}\Big\|H_k(t, y(t_k^-)) - H_k(t, x(t_k^-))\Big\|^2 \\ \leq & 2\varepsilon\xi(t) + 2C_{H_k}\mathbb{E}\Big\|y(t) - x(t)\Big\|^2.\end{aligned}$$

Thus

$$\mathbb{E}\Big\|y(t) - x(t)\Big\|^2 \leq c_2'\varepsilon\xi(t), \tag{3.33}$$

with $c_2' = \frac{2}{1-2C_{H_k}}$, for all $C_{H_k} < 1/2$.

Case 3 For each $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \cdots, m$, by $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_5)$-(2) Lemma 1.9, Lemma 3.1 and Theorem 3.3 , we obtain

$$\begin{aligned}\mathbb{E}\Big\|y(t) - x(t)\Big\|^2 \leq & 4\mathbb{E}\Big\|y(t) - U(t - s_k)H_k(s_k, y(s_k)) \\ & - \int_{s_k}^t (t-s)^{\beta-1}V(t-s) - \hbar\Big(s, y(s), K_1y(s)\Big)ds \\ & - \int_{s_k}^t (t-s)^{\beta-1}V(t-s)g\Big(s, y(s), K_2y(s)\Big)dW(s)\Big\|^2 \\ & + 4\mathbb{E}\Big\|U(t-s_k)\big(H_k(s_k, y(s_k)) - H_k(s_k, x(s_k))\big)\Big\|^2 \\ & + 3\mathbb{E}\Big\|\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\Big(\hbar\big(s, y(s), K_1y(s)\big) \\ & - \hbar\big(s, x(s), K_1x(s)\big)\Big)ds\Big\|^2 \\ & + 4\mathbb{E}\Big\|\int_{s_k}^t (t-s)^{\beta-1}V(t-s)\Big(\aleph\big(s, y(s), K_2y(s)\big) \\ & - \aleph\big(s, x(s), K_2x(s)\big)\Big)dW(s)\Big\|^2 \\ \leq & 4\varepsilon\xi(t) + 4M^2\mathbb{E}\Big\|H_k(s_k, y(s_k)) - H_k(s_k, x(s_k))\Big\|^2 \\ & + \frac{4M^2T}{\Gamma^2(\beta)}\int_{s_k}^t (t-s)^{2\beta-2}\mathbb{E}\Big\|\hbar\big(s, y(s), K_1y(s)\big) - \hbar\big(s, x(s), K_1x(s)\big)\Big\|^2 ds \\ & + \frac{4M^2Tr(\mathcal{Q})}{\Gamma^2(\beta)}\int_{s_k}^t (t-s)^{2\beta-2}\mathbb{E}\Big\|\aleph\big(s, y(s), K_2y(s)\big) - \aleph\big(s, x(s), K_2x(s)\big)\Big\|^2 ds.\end{aligned}$$

Then, using assumptions $(\hat{H}_2)$-$(\hat{H}_3)$-(3) and $(\hat{H}_5)$-(2), we obtain

$$\begin{aligned}\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq & \frac{4\varepsilon}{1-4M^2C_{H_k}}\xi(t)\\ &+\frac{4M^2\big(C_\hbar+C_f'C_{K_1}T\big)T}{\Gamma^2(\beta)(1-4M^2C_{H_k})}\int_{s_k}^t(t-s)^{2\beta-2}\mathbb{E}\Big\|y(s)-x(s)\Big\|^2ds\\ &+\frac{4M^2\big(C_\aleph+C_\aleph'C_{K_2}T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)(1-4M^2C_{H_k})}\int_{s_k}^t(t-s)^{2\beta-2}\mathbb{E}\Big\|y(s)-x(s)\Big\|^2ds.\end{aligned}$$

Then applying Theorem 3.3 (Generalized Grönwall inequality), we have

$$\begin{aligned}\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq & \frac{4\varepsilon}{1-4M^2C_{H_k}}.\xi(t)\\ &.\exp\left(\frac{4M^2\big(C_\hbar+C_\hbar'C_{K_1}T\big)T+4M^2\big(C_\aleph+C_\aleph'K_{G_2}T\big)Tr\big(\mathcal{Q}\big)}{\Gamma^2(\beta)(1-4M^2C_{H_k})}\int_{s_k}^t(t-s)^{2\beta-2}ds\right).\end{aligned}$$

Thus, we can get

$$\mathbb{E}\Big\|y(t)-x(t)\Big\|^2 \leq c_3'\varepsilon\xi(t), \tag{3.34}$$

with

$$c_3' = \frac{4}{1-4M^2C_{H_k}}\exp\left(\frac{4M^2\big(C_\hbar+C_\hbar'C_{K_1}T\big)T^{2\beta}+4M^2\big(C_\aleph+C_\aleph'C_{K_2}\big)Tr\big(\mathcal{Q}\big)T^{2\beta-1}}{\Gamma^2(\beta)(2\beta-1)(1-4M^2C_{H_k})}\right).$$

As a result, $C_2=\max\{c_1',c_2',c_3'\}$ exists and satisfies Definition 3.26. The solution (3.1) is Ulam-Hyers-Rassias stable.

□

3.4 Example

In this section, an example is presented to verify our preliminary results in Sections 3.2 and 3.3. Consider the following fractional stochastic integro-differential equations with non-instantaneous impulsive and nonlocal conditions:

$$\begin{cases} {}^cD^{\frac{3}{4}}y(t,z) = \frac{\partial^2}{\partial^2 z}y(t,z) + \frac{1}{\sqrt{50}}\Big(\frac{e^{-t}}{\sqrt{1+e^{-t}}}y(t,z) + \int_0^t e^{-s}y(s,z)ds\Big) \\ \qquad\qquad +\Big(\frac{e^{-2t}}{\sqrt{100(1+e^{-2t})}}y(t,z) + \frac{1}{50}\int_0^t e^{-s}y(s,z)ds\Big)dW(t), & z\in[0,1], t\in\big[0,\frac{1}{3}\big]\cup\Big(\frac{2}{3},1\Big], \\ y(t,z) = \frac{e^{-(t-\frac{1}{3})}y(t,z)}{4\big(1+|y(t,z)|\big)}, & z\in[0,1], t\in(\frac{1}{3},\frac{2}{3}], \\ y(t,0)=y(t,1)=0, & t\in[0,1], \\ y(0,z)+\sum_{i=1}^{2}\frac{1}{3^i}y\big(t_i,z\big) = y_0(z), & z\in[0,1]. \end{cases} \tag{3.35}$$

Let $\mathbb{H} = L^2\big([0,1],\mathbb{R}\big)$ be the Hilbert space with the scalar product. We define the operator $A : D(A)\subset\mathbb{H}\to\mathbb{H}$ by $Ay = \frac{\partial y^2}{\partial^2 z}$, with $D(A) = \big\{y\in\mathbb{H}, \frac{\partial y}{\partial z}, \frac{\partial y^2}{\partial^2 z}\in\mathbb{H}$ and $y(0)=y(1)=0\big\}$, and the semigroup generated by A is defined by $\mathcal{T}(t)v = \sum_{n=1}^{\infty} e^{-n^2 t}(v,v_n)\,v_n$ and $\|\mathcal{T}(t)\|^2 \le e^{-t} < 1, t>0$ where $v_n = \sqrt{2}\sin(ns)$, $n=1,2,\cdots$ is the orthogonal set of eigenvectors in A.

Let $t_0 = s_0 = 0, t_1 = \frac{1}{3}, s_1 = \frac{2}{3}$ and $t_2 = 1$ for $m=1$

$$\begin{aligned} y(t) &= y(t,.), \\ \hbar\big(t,y(t),K_1y(t)\big) &= \frac{1}{\sqrt{50}}\Big(\frac{e^{-t}}{\sqrt{1+e^{-t}}}y(t,z) + \int_0^t e^{-s}y(s,z)ds\Big), \\ K_1y(t) &= \frac{1}{\sqrt{50}}\int_0^t e^{-s}y(s,z)ds, \\ \aleph\big(t,y(t),K_2y(t)\big) &= \Big(\frac{e^{-2t}}{\sqrt{100\,(1+e^{-2t})}}y(t,z) + \frac{1}{50}\int_0^t e^{-s}y(s,z)ds\Big), \\ K_2y(t) &= \frac{1}{50}\int_0^t e^{-s}y(s,z)ds, \\ H_1(t,y(t)) &= \frac{e^{-(t-\frac{1}{3})}y(t,z)}{4\big(1+|y(t,z)|\big)}, \end{aligned}$$

and

$$\gamma(y) = \sum_{i=1}^{2}\frac{1}{3^i}y\big(t_i,.\big).$$

Let $\beta = \frac{3}{4}$, $p=\frac{2}{5}$ and $p_1 = \frac{2}{3}$. It is possible to rewrite the equation (3.35) into the abstract form of (3.1).

It is possible to rewrite the equation (3.35) into the abstract form of (3.1). The assumptions $(\hat{H}_1)$-$(\hat{H}_5)$ and condition (3.3) are easily verified with

$$\varphi_{\hbar}(t) = \frac{1}{50}\left(\frac{e^{-2t}}{1+e^{-t}} + 1\right), \quad \psi_{\hbar}(r) = r, \qquad \varphi_{\aleph}(t) = \frac{1}{100}\left(\frac{e^{-4t}}{1+e^{-2t}} + 1\right), \quad \psi_{\aleph}(r) = r,$$

$$L = I = 1, \qquad C_{\gamma} = \frac{2}{9}, \qquad C_{H_k} = \frac{e^{-\frac{4}{3}}}{16}, \qquad M = 1,$$

$$N_{\hbar} = 0.01432, \qquad N_{\aleph} = 0.05001, \qquad C_{\hbar} = 0,00021, \qquad C_{\aleph} = 0,000026,$$

$$C_{K_1} = C_{K_2} = 1 \qquad C'_{\hbar} = \frac{1}{50}, \qquad C'_{\aleph} = \frac{1}{100}.$$

So, we obtain

$$C = 0.23299 < 1, \qquad \Lambda = 0.66767 \leq 1.$$

Consequently, the system (3.35) has at least one mild solution since Theorem 3.1, 3.2 is valid.

In addition to the above, $\xi(t) = \frac{1}{1+e^{-2t}}$. Using Theorem 3.5, such that

$$C_1 = C_2 = 5,08873 > 0.$$

This leads us to conclude that system (3.35) is Ulam-Hyers-Rassias stable in accordance with Theorem 3.5, also is Ulam-Hyers stable in accordance with Theorem 3.4.

Chapter 4

Convergence of the θ-EM method for a class of SVIDEs

This chapter of the thesis outlines the key findings presented in our scientific article [112]. We examine the convergence analysis of the θ-Euler-Maruyama method for a class of stochastic Volterra integro-differential equations (SVIDEs). At first, we discuss the existence, uniqueness, boundedness and Hölder continuity of the theoretical solution. Subsequently, the strong convergence order of the θ-Euler-Maruyama approach for SVIDEs is established. Finally, we provided numerical examples to illustrate the theoretical results.

4.1 Introduction

Recently, Deng *et al.* [44] examined the semi-implicit Euler method for non-linear time changed stochastic diferential equations. Wang *et al.* [171] investigated the stochastic theta methods (STMs) for stochastic differential equations (SDEs) with non-global Lipschitz drift and diffusion coefficients. Zhang *et al.* [183] examined the Euler-Maruyama (EM) method's numerical analysis of the following generalized SVIDEs:

$$\begin{aligned} dY(t) =& f\Big(Y(t), \int_0^t K_1(t,s)Y(s)ds, \int_0^t \sigma_1(t,s)Y(s)dW(s)\Big)dt \\ &+ g\Big(Y(t), \int_0^t K_2(t,s)Y(s)ds, \int_0^t \sigma_2(t,s)Y(s)dW(s)\Big)dW(t). \end{aligned}$$

Lan *et al.* [91] presented the θ-EM method corresponding to the following SVIDEs:

$$dX(t) = f\Big(X(t), \int_0^t G(t-s)X(s)ds\Big)dt + g\Big(X(t), \int_0^t H(t-s)X(s)ds\Big)dW(t).$$

Inspired and motivated by the above works [44, 91, 171, 183], in this chapter, we study the strong convergence of the θ-Euler-Maruyama method for a class of stochastic Volterra integro-differential equations (SVIDEs) as follows:

$$\begin{aligned} dY(t) = f\Big(Y(t), \int_0^t \sigma_1(t,s)Y(s)dW(s)\Big)dt \\ +g\Big(Y(t), \int_0^t \sigma_2(t,s)Y(s)ds\Big)dW(t), \quad t \in [0,T], \end{aligned} \tag{4.1}$$

with initial condition $Y(0) = Y_0 \in \mathbb{R}$, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions. The kernels $\sigma_i : D \to \mathbb{R}$ are continuous on $D := \{(t,s) : 0 \leq s \leq t \leq T\}$ with the norm $\|\sigma_i\|_\infty = \max_{(t,s)\in D} |\sigma_i(t,s)|$ for $i = 1, 2$.

Let Y be a stochastic process defined by $Y(t,\omega)$ for each $\omega \in \Omega$, where t represents time. To simplify the notation, we refer to $Y(t)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $W(t)$ is a standard Brownian motion (1-dimensional Brownian motion) defined on the same probability space. And $\mathbb{E}\|Y_0\|^2 < \infty$.

The structure of this chapter is as follows: We introduce some fundamental notations and preliminaries in 4.2. We then present the definition of the solution of equation (4.1) and investigate the existence, uniqueness, boundedness and Hölder continuity of the analytic solution in 4.3. The θ-EM method for equation (4.1) is presented, and its order of convergence is taken into account in 4.4. Finally, we provide numerical examples in 4.5 to illustrate the theoretical results.

4.2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions , and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. A 1-dimensional Brownian motion defined on the probability space is denoted by $B(t)$. Let $L^2([0,T],\mathbb{R})$ by the family of Borel measurable functions $\Phi : [0,T] \to \mathbb{R}$ such that for every $T > 0$, $\int_0^T |\Phi(t)|^2 dt < \infty$. We denote $\mathcal{L}^2([0,T], \mathbb{R})$ the family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{\Phi(t)\}_{t\in[0,T]}$ such that $\int_0^T |\Phi(t)|^2 dt < \infty$. $\mathcal{M}^2([0,T], \mathbb{R})$ be the family of processes $\mathcal{F}_t$–adapted $\{\Phi(t)\}_{t\in[0,T]} \in \mathcal{L}^2([0,T], \mathbb{R})$ such that $\mathbb{E}\left[\int_0^T |\Phi(t)|^2 dt\right] < \infty$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$, and $a \vee b := \max\{a, b\}$. If $\mathbb{D}$ is a subset of Ω, the indicator function is denoted by $1_{\mathbb{D}}$. We can write the integral of equation (4.1) as follows:

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big)du \\ &+ \int_0^t g\Big(Y(u), \int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u). \end{aligned} \tag{4.2}$$

We will introduce the definition of the solution, we assume that $Y(.) \in \mathcal{L}^2([0,T], \mathbb{R})$ where

$$\begin{aligned} F(t) :=& f\Big(Y(t), \int_0^t \sigma_1(t,s)Y(s)dW(s)\Big), \\ G(t) :=& g\Big(Y(t), \int_0^t \sigma_2(t,s)Y(s)ds\Big). \end{aligned}$$

Definition 4.1. A solution of (4.2) is a continuous stochastic process $\{Y(t)\}_{t\in[0,T]}$ with values in $\mathbb{R}$ satisfies the following conditions:

i) $Y(.) \in \mathcal{L}^2([0,T], \mathbb{R})$, $F(.) \in \mathcal{L}^1([0,T], \mathbb{R})$ and $G(.) \in \mathcal{L}^2([0,T], \mathbb{R})$,

ii) (4.2) holds for all $t \in [0,T]$ with probability 1. The solution $\{Y(t)\}$ is said to be unique if there is exists other solution $\{\overline{Y}(t)\}$ such that $\{Y(t)\} = \{\overline{Y}(t)\}$ for all $t \in [0,T]$, *i.e.*,
$$\mathbb{P}\Big\{Y(t) = \overline{Y}(t) \ \text{ for all } \ t \in [0,T]\Big\} = 1.$$

Definition 4.2. Let $0 < \delta \leq 1$. A stochastic process $Y(t,\omega) : [0,T] \times \Omega \to \mathbb{R}$ is referred to as Hölder continuous with exponent $\delta > 0$ if a constant M exists such that

$$\mathbb{E}|Y(t) - Y(r)|^2 \leq M|t-r|^{2\delta}, \quad \forall t, r \in [0,T].$$

In this article, we propose this hypotheses.

(A_1) **(Lipschiz condition).** Assume that there exist a positive constant K such that

$$\left|f(x,y)-f(x',y')\right|^2 \vee \left|g(x,y)-g(x',y')\right|^2 \leq K\Big(\left|x-x'\right|^2+\left|y-y'\right|^2\Big),$$

for $x,y,x',y' \in \mathbb{R}$.

(A_2) **(Linear growth condition).** For $x,y \in \mathbb{R}$

$$\left|f(x,y)\right|^2 \vee \left|g(x,y)\right|^2 \leq K'\Big(1+\left|x\right|^2+\left|y\right|^2\Big),$$

where $K' = 2\Big(K \vee |f(0,0)|^2 \vee |g(0,0)|^2\Big)$.

(A_3) **(Mean value theorem).** Assuming that the coefficients $\sigma_i \in C^1(D)$, for $i=1,2$. of (4.1) satisfy

$$\left|\sigma_i'(t,s)\right|^2 \leq K'',$$

with $K''>0$, for all $(t,s)\in D$, and

$$\left|\sigma_i(u,s)-\sigma_i(u_h,s)\right|^2 = \left|\sigma_i(\xi_i,s)(u,u_h)\right|^2 \leq K''h^2,$$

where $\xi_i \in (u,u_h)$.

The existence and uniqueness of the solution to (4.1) under hypothesis (A_2) are demonstrated by the following theorem. The proof of this theorem is similar to the proof in [183], The proof of this theorem proceeds similarly to that of [183], which pertains to the case where $\Theta = 0$.

4.3 Theoretical analysis of the class of SVIDE

In this section, we present the theoretical results. The existence and uniqueness of the solution to (4.2) have been established. Additionally, we verified the Hölder continuity condition for the analytical solutions.

4.3.1 The existence and uniqueness of the analytical solution

We now discuss the theory of existence and uniqueness of the solution of the equation (4.2). We first present the following lemma.

Lemma 4.1. *Assume that* (A_2) *is satisfied. If* $Y(t) \in \mathcal{L}^2([0,T], \mathbb{R})$ *is a solution of SVIDEs* (4.1), *and such that*

$$\mathbb{E}\left[Y(t)\right]^2 \leq M_0, \qquad \forall t \in [0,T], \tag{4.3}$$

where M_0 *depends on* $\sigma_1, \sigma_2, T, K'$ *and* Y_0.

Proof. For every integer $n \geq 1$, τ_n be a stopping time such that

$$\tau_n = T \wedge \inf \Big\{ t \in [0,T] : |Y(t)| \geq n \Big\}.$$

Evidently, $\tau_n \to T$ a.s. by letting $n \to \infty$. Define $Y_n(t) := Y(t \wedge \tau_n)$ for $t \in [0,T]$. It can be verified that $Y_n(t)$ satisfies.

$$\begin{aligned} Y_n(t) = Y(0) &+ \int_0^t f\Big(Y_n(u), \int_0^u \sigma_1(u,s) Y_n(s) \mathbf{1}_{[0,\tau_n]}(s) dW(s)\Big) \mathbf{1}_{[0,\tau_n]}(u) du \\ &+ \int_0^t g\Big(Y_n(u), \int_0^u \sigma_2(u,s) Y_n(s) \mathbf{1}_{[0,\tau_n]}(s) ds\Big) \mathbf{1}_{[0,\tau_n]}(u) dW(u), \qquad t \in [0,T]. \end{aligned}$$

By Cauchy's inequality, the Itô isometry, and the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$, we obtain that

$$\begin{aligned} |Y_n(t)|^2 \leq 3|Y_0|^2 &+ 3t \int_0^t \left| f\Big(Y_n(u), \int_0^u \sigma_1(u,s) Y_n(s) \mathbf{1}_{[0,\tau_n]}(s) dW(s)\Big) \mathbf{1}_{[0,\tau_n]}(u) \right|^2 du \\ &+ 3 \int_0^t \left| g\Big(Y_n(u), \int_0^u \sigma_2(u,s) Y_n(s) \mathbf{1}_{[0,\tau_n]}(s) ds, \Big) \mathbf{1}_{[0,\tau_n]}(u) \right|^2 du. \end{aligned}$$

Using Cauchy's inequality and the Itô isometry, we show that

$$\mathbb{E}|Y_n(t)|^2 \leq 3\mathbb{E}|Y_0| + 3TF + 3G, \tag{4.4}$$

where

$$F := \mathbb{E}\Big[\int_0^t \left| f\Big(Y_n(u), \int_0^u \sigma_1(u,s) Y_n(s) \mathbf{1}_{[0,\tau_n]}(s) dW(s)\Big) \mathbf{1}_{[0,\tau_n]}(u) \right|^2 du \Big],$$

and

$$G := \mathbb{E}\Big[\int_0^t \left| g\Big(Y_n(u), \int_0^u \sigma_2(u,s) Y_n(s) \mathbf{1}_{[0,\tau_n]}(s) ds, \Big) \mathbf{1}_{[0,\tau_n]}(u) \right|^2 du \Big].$$

First, we calculate F, using Cauchy's inequality and assumption (A_2), we get

$$
\begin{aligned}
F \leq & K' \int_0^t \left[1 + \mathbb{E}|Y_n(u)|^2 + \mathbb{E}\left| \int_0^u \sigma_1(u,s) Y_n(s) dW(s) \right|^2 \right] du \\
\leq & K' \int_0^t \left[1 + \mathbb{E}|Y_n(u)|^2 + \|\sigma_1\|_\infty^2 \int_0^u \mathbb{E}|Y_n(s) ds|^2 \right] du \\
\leq & K'T + K'(1 + \|\sigma_1\|_\infty^2 T) \int_0^t \mathbb{E}|Y_n(s)|^2 ds,
\end{aligned} \tag{4.5}
$$

Second, we calculate G, by (A_2) and Itô isometry, we get

$$
\begin{aligned}
g \leq & K' \int_0^t \left[1 + \mathbb{E}|Y_n(u)|^2 + \mathbb{E}\left| \int_0^u \sigma_2(u,s) Y_n(s) ds \right|^2 \right] du \\
\leq & K' \int_0^t \left[1 + \mathbb{E}|Y_n(u)|^2 + \|\sigma_2\|_\infty^2 T \int_0^u \mathbb{E}|Y_n(s) ds|^2 \right] du \\
\leq & K'T + K'(1 + \|\sigma_2\|_\infty^2 T^2) \int_0^t \mathbb{E}|Y_n(s)|^2 ds.
\end{aligned} \tag{4.6}
$$

By substituting (4.5) and (4.6) into (4.4), we have

$$
\begin{aligned}
\mathbb{E}|Y_n(t)|^2 \leq & \, 3\mathbb{E}|Y_0|^2 + 3T\Big(K'T + K'(1 + \|\sigma_1\|_\infty^2 T) \int_0^t \mathbb{E}|Y_n(s)|^2 ds\Big) \\
& + 3\Big(K'T + K'(1 + \|\sigma_2\|_\infty^2 T^2) \int_0^t \mathbb{E}|Y_n(s)|^2 ds\Big) \\
\leq & \, 3\mathbb{E}|Y_0|^2 + 3K'T(T+1) + 3K'T(1 + \|\sigma_1\|_\infty^2 T) \int_0^t \mathbb{E}|Y_n(s)|^2 ds \\
& + 3K'(1 + \|\sigma_2\|_\infty^2 T^2) \int_0^t \mathbb{E}|Y_n(s)|^2 ds \\
= & \, 3\mathbb{E}|Y_0|^2 + 3K'T(T+1) + 3K'(C_1 T + C_2) \int_0^t \mathbb{E}|Y_n(s)|^2 ds,
\end{aligned}
$$

where $C_1 := 1 + \|\sigma_1\|_\infty^2 T$ and $C_2 := 1 + \|\sigma_2\|_\infty^2 T^2$.

By using Grönwall's inequality, we have

$$
\mathbb{E}|Y_n(t)|^2 \leq M_0,
$$

where

$$
M_0 := 3\big(\mathbb{E}|Y_0|^2 + C_3\big) \exp\big(3TC\big),
$$

and

$$
C := K'(C_1 T + C_2), C_3 = K'T(T+1).
$$

Since $\mathbb{E}|Y(t \wedge \tau_n)|^2 \leq M_0$, for $t \geq 0$, letting $t \longrightarrow \infty$, we conclude that $\tau_\infty = \infty$, that is,

$$
\mathbb{E}\left[Y(t)\right]^2 \leq M_0.
$$

□

Theorem 4.1. *Suppose (A_1) holds. Then, there exists a unique solution $Y(t) \in \mathcal{M}^2([0,T],\mathbb{R})$ to 4.1 and*

$$\mathbb{E}\left[Y(t)\right]^2 \leq M_1 \quad \text{for } t \in [0,T],$$

where

$$M_1 := 3\big((1+CT)\mathbb{E}|Y_0|^2 + C_3\big)\exp(3CT).$$

Proof. We will divide the proof into two fundamental steps.

Step I. Uniqueness. Let $Y_1(t)$ and $Y_2(t)$ be two solutions of (4.1). From Lemma 4.1 we have, $Y_1(t), Y_2(t) \in \mathcal{M}^2([0,T];\mathbb{R})$. By (A_1), Cauchy's inequality and Itô isometry, we show that

$$\begin{aligned}
\mathbb{E}|Y_1(t)-Y_2(t)|^2 &= 2\mathbb{E}\bigg|\int_0^t f\Big(Y_1(u),\int_0^u \sigma_1(u,s)Y_1(s)dW(s)\Big)du \\
&\qquad - \int_0^t f\Big(Y_2(u),\int_0^u \sigma_1(u,s)Y_2(s)dW(s)\Big)du\bigg|^2 \\
&+ 2\mathbb{E}\bigg|\int_0^t g\Big(Y_1(u),\int_0^u \sigma_2(u,s)Y_1(s)ds\Big)dW(u) \\
&\qquad - \int_0^t g\Big(Y_2(u),\int_0^u \sigma_2(u,s)Y_2(s)ds\Big)dW(u)\bigg|^2 \\
&\leq 2T\int_0^t \mathbb{E}\bigg|f\Big(Y_1(u)\int_0^u \sigma_1(u,s)Y_1(s)dW(s)\Big) - f\Big(Y_2(u),\int_0^u \sigma_1(u,s)Y_2(s)dW(s)\Big)\bigg|^2 du \\
&+ 2\int_0^t \mathbb{E}\bigg|g\Big(Y_1(u),\int_0^u \sigma_2(u,s)Y_1(s)ds\Big) - g\Big(Y_2(u),\int_0^u \sigma_2(u,s)Y_2(s)ds\Big)\bigg|^2 du \\
&\leq 2TK\int_0^t \Big(\mathbb{E}|Y_1(u)-Y_2(u)|^2 + \int_0^u \mathbb{E}\Big|\sigma_1(u,s)\Big(Y_1(s)-Y_2(s)\Big)\Big|^2 ds\Big)du \\
&+ 2K\int_0^t \Big(\mathbb{E}|Y_1(u)-Y_2(u)|^2 + u\int_0^u \mathbb{E}\Big|\sigma_2(u,s)\Big(Y_1(s)-Y_2(s)\Big)\Big|^2 ds\Big)du \\
&\leq 2TK\int_0^t \Big(\mathbb{E}|Y_1(u)-Y_2(u)|^2 + \|\sigma_1\|_\infty^2\int_0^u \mathbb{E}|Y_1(s)-Y_2(s)|^2 ds\Big)du \\
&+ 2K\int_0^t \Big(\mathbb{E}|Y_1(u)-Y_2(u)|^2 + T\|\sigma_2\|_\infty^2\int_0^u \mathbb{E}|Y_1(s)-Y_2(s)|^2 ds\Big)du \\
&\leq 2K\big(TC_1 + C_2\big)\int_0^t \mathbb{E}|Y_1(s)-Y_2(s)|^2 ds \\
&\leq C\int_0^t \mathbb{E}|Y_1(s)-Y_2(s)|^2 ds.
\end{aligned}$$

Finally, from Grönwall's inequality we conclude that

$$\mathbb{E}|Y_1(t) - Y_2(t)|^2 = 0,$$

which proves that $Y_1(t) = Y_2(t)$ for every $t \in [0, T]$.

Step II. Existence. Let $Y_0(t) = Y_0$ and define the Picard approximation.

$$\begin{aligned} Y_n(t) = Y_0 + &\int_0^t f\Big(Y_{n-1}(u), \int_0^u \sigma_1(u,s) Y_{n-1}(s) dW(s)\Big) du \\ &+ \int_0^t g\Big(Y_{n-1}(u), \int_0^u \sigma_2(u,s) Y_{n-1}(s) ds\Big) dW(u), \end{aligned} \tag{4.7}$$

for $t \in [0,T]$ and $n = 1, 2, \ldots$. It is evident that $Y_0(.) \in \mathcal{M}^2([0,T]\,;\mathbb{R})$, and through induction, we also get $Y_n(.) \in \mathcal{M}^2([0,T]\,,\mathbb{R})$. Using the proof of Lemma 4.1, we have

$$\begin{aligned} \mathbb{E}|Y_n(t)|^2 \leq & 3\mathbb{E}|Y_0|^2 + 3\mathbb{E}\Big|\int_0^t f\Big(Y_{n-1}(u), \int_0^u \sigma_1(u,s) Y_{n-1}(s) dW(s)\Big) du\Big|^2 \\ &+ 3\mathbb{E}\Big|\int_0^t g\Big(Y_{n-1}(u), \int_0^u \sigma_2(u,s) Y_{n-1}(s) ds\Big) dW(u)\Big|^2 \\ \leq & 3\mathbb{E}|Y_0|^2 + 3T\Big(K'T + K'(1 + \|\sigma_1\|_\infty^2 T)\int_0^t \mathbb{E}|Y_{n-1}(s)|^2 ds\Big) \\ &+ 3\Big(K'T + K'(1 + \|\sigma_2\|_\infty^2 T^2)\int_0^t \mathbb{E}|Y_{n-1}(s)|^2 ds\Big) \\ \leq & 3\mathbb{E}|Y_0|^2 + 3K'T(T+1) \\ &\quad + 3K'T(1 + \|\sigma_1\|_\infty^2 T)\int_0^t \mathbb{E}|Y_{n-1}(s)|^2 ds \\ &\quad + 3K'(1 + \|\sigma_2\|_\infty^2 T^2)\int_0^t \mathbb{E}|Y_{n-1}(s)|^2 ds \\ = & 3\mathbb{E}|Y_0|^2 + 3K'T(T+1) \\ &\quad + 3K'(C_1 T + C_2)\int_0^t \mathbb{E}|Y_n(s)|^2 ds \end{aligned}$$

hence

$$\mathbb{E}|Y_n(t)|^2 \leq 3\big(\mathbb{E}|Y_0|^2 + C_3\big) + 3C\int_0^t \mathbb{E}|Y_{n-1}(s)|^2 ds,$$

where C and C_3 are those from Lemma 4.1. Thus, for each $n \geq 1$, we have

$$\begin{aligned}\max_{1\leq k\leq n} \mathbb{E}|Y_k(t)|^2 \leq& 3\big(\mathbb{E}|Y_0|^2 + C_3\big) + 3C\int_0^t \max_{1\leq k\leq n} \mathbb{E}|Y_{k-1}(s)|^2 ds \\ \leq& 3\big(\mathbb{E}|Y_0|^2 + C_3\big) + 3C\int_0^t \left[\mathbb{E}|Y_0|^2 + \max_{1\leq k\leq n} \mathbb{E}|Y_k(s)|^2\right] ds \\ \leq& 3\Big((1+CT)\mathbb{E}|Y_0|^2 + C_3\Big) + 3C\int_0^t \max_{1\leq k\leq n} \mathbb{E}|Y_k(s)|^2 ds.\end{aligned}$$

By the Grönwall's inequality, we obtain

$$\max_{1\leq k\leq n} \mathbb{E}|Y_k(t)|^2 \leq M_1,$$

where $M_1 := 3\Big((1+CT)\mathbb{E}|Y_0|^2 + C_3\Big)\exp(3CT)$. Since k is arbitrary, we conclude

$$\mathbb{E}|Y_n(t)|^2 \leq M_1, \quad \text{for } t \in [0,T], \; n \geq 1. \tag{4.8}$$

Similar to the proof of Lemma 4.1, one has

$$\begin{aligned}\mathbb{E}|Y_1(t) - Y_0(t)|^2 =& \mathbb{E}|Y_1(t) - Y_0|^2 \\ \leq& 2\mathbb{E}\left|\int_0^t f\Big(Y_0(u), \int_0^u \sigma_1(u,s)Y_0(s)dW(s)\Big)du\right|^2 \\ &+ 2\mathbb{E}\left|\int_0^t g\Big(Y_0(u), \int_0^u \sigma_2(u,s)Y_0(s)ds\Big)dW(u)\right|^2 \\ \leq& 2T\int_0^t \mathbb{E}\left|f\Big(Y_0(u), \int_0^u \sigma_1(u,s)Y_0(s)dW(s)\Big)\right|^2 du \\ &+ 2\int_0^t \mathbb{E}\left|g\Big(Y_0(u), \int_0^u \sigma_2(u,s)Y_0(s)ds\Big)\right|^2 du \\ \leq& 2K'T\int_0^t \left(1 + \mathbb{E}|Y_0|^2 + \mathbb{E}\left|\int_0^u \sigma_1(u,s)Y_0(s)dW(s)\right|^2\right) du \\ &+ 2K'\int_0^t \left(1 + \mathbb{E}|Y_0|^2 + \mathbb{E}\left|\int_0^u \sigma_2(u,s)Y_0(s)ds\right|^2\right) du \\ \leq& C_0,\end{aligned}$$

where

$$C_0 := 2K'\Big(T(1+T) + \Big(T\underbrace{\big(1+T\|\sigma_1\|_\infty^2\big)}_{C_1} + \underbrace{\big(1+T^2\|\sigma_2\|_\infty^2\big)}_{C_2}\Big)\mathbb{E}|Y_0|^2\Big).$$

and

$$
\begin{aligned}
&\mathbb{E}|Y_2(t)-Y_1(t)|^2\\
\le& 2\mathbb{E}\bigg|\int_0^t\bigg(f\Big(Y_1(u),\int_0^u\sigma_1(u,s)Y_1(s)dW(s)\Big)-f\Big(Y_0(u),\int_0^u\sigma_1(u,s)Y_0(s)dW(s)\Big)\bigg)du\bigg|^2\\
&+2\mathbb{E}\bigg|\int_0^t\bigg(g\Big(Y_1(u),\int_0^u\sigma_2(u,s)Y_1(s)ds\Big)-g\Big(Y_0(u),\int_0^u\sigma_2(u,s)Y_0(s)ds\Big)\bigg)dW(u)\bigg|^2\\
&\le 2T\int_0^t K\Big(\mathbb{E}\big|Y_1(u)-Y_0(u)\big|^2+\|\sigma_1\|_\infty^2\int_0^u\mathbb{E}\big|Y_1(s)-Y_0(s)\big|^2ds\Big)\\
&\quad+2K\Big(\mathbb{E}\big|Y_1(u)-Y_0(u)\big|^2+T\|\sigma_2\|_\infty^2\int_0^u\mathbb{E}\big|Y_1(s)-Y_0(s)\big|^2ds\Big)du\\
&\le K\big(TC_1+C_2\big)\int_0^t\mathbb{E}|Y_1(s)-Y_0(s)|^2ds\\
&\le 2C\int_0^t C_0ds=2CC_0T.
\end{aligned}
$$

We claim that for $n\ge 0$,

$$
\mathbb{E}|Y_n(t)-Y_{n-1}(t)|^2\le \tfrac{C_0(2Ct)^{n-1}}{(n-1)!}. \tag{4.9}
$$

By induction, we need to show that (4.9) still holds for $n+1$. Note that

$$
\begin{aligned}
|Y_{n+1}(t)-Y_n(t)|^2\le 2\mathbb{E}\bigg|\int_0^t\bigg[&f\Big(Y_n(u),\int_0^u\sigma_1(u,s)Y_n(s)dW(s)\Big)\\
&-f\Big(Y_{n-1}(u),\int_0^u\sigma_1(u,s)Y_{n-1}(s)dW(s)\Big)\bigg]du\bigg|^2\\
&+2\mathbb{E}\bigg|\int_0^t\bigg[g\Big(Y_n(u),\int_0^u\sigma_2(u,s)Y_n(s)ds\Big)\\
&-g\Big(Y_{n-1}(u),\int_0^u\sigma_2(u,s)Y_{n-1}(s)ds\Big)\bigg]dW(u)\bigg|^2.
\end{aligned} \tag{4.10}
$$

Using (4.9), and similar to the proof of Lemma 4.1, we show that

$$
\begin{aligned}
\mathbb{E}|Y_{n+1}(t)-Y_n(t)|^2\le& 2C\int_0^t\mathbb{E}|Y_n(s)-Y_{n-1}(s)|^2ds\\
\le& 2C\int_0^t\tfrac{C_0(2Cs)^{n-1}}{(n-1)!}ds\\
=& 2C\tfrac{C_0(2C)^{n-1}s^n}{n!}\bigg|_0^t\\
=& \tfrac{C_0(2Ct)^n}{n!}.
\end{aligned}
$$

By Chebyshev inequality, we get

$$\mathbb{P}\left\{|Y_n(t) - Y_{n-1}(t)|^2 > \tfrac{1}{2^n}\right\} \leq \tfrac{C_0(2CT)^{n-1}}{(n-1)!},$$

Since $\sum_0^{+\infty} \frac{C_0(2CT)^{n-1}}{(n-1)!} < \infty$. Thus, applying the Borel-Cantelli lemma, we can show that

$$\forall \omega \in \Omega, \quad \exists n_0 = n_0(\omega), \qquad |Y_n(t) - Y_{n-1}(t)|^2 \leq \tfrac{1}{2^n}, \quad \text{for} \quad n \geq n_0.$$

With probability 1, it follows that,

$$Y_0(t) + \sum_{k=1}^{n-1} |Y_k(t) - Y_{k-1}(t)| = Y_n(t)$$

are convergent uniformly for $t \in [0, T]$. $Y(t)$ denote the limit. $Y(t)$ is obviously continuous and $\mathcal{F}_t$-adapted. On the other hand, it can be seen from (4.9) that , $(Y_n(t))_{n\geq 1}$ is a Cauchy sequence in $\mathcal{L}^2([0,T], \mathbb{R})$ for any t.

$$\begin{aligned}\left(\mathbb{E}|Y_n(t) - Y_m(t)|^2\right)^{\frac{1}{2}} &= \left\|Y_n(t) - Y_m(t)\right\|_{L^2} \\ &\leq \sum_{k=m}^{n-1} \left\|Y_n(t) - Y_{n-1}(t)\right\|_{L^2} \\ &\leq \sum_{k=m}^{n-1} \left(\tfrac{C_0(2CT)^{n-1}}{(n-1)!}\right)^{\frac{1}{2}}.\end{aligned}$$

Letting $n, m \to \infty$, therefore

$$\left(\mathbb{E}|Y_n(t) - Y_m(t)|^2\right)^{\frac{1}{2}} \longrightarrow 0,$$

$(Y_n(t))_{n\geq 1}$ is a Cauchy sequence in $\mathcal{L}^2([0,T], \mathbb{R})$. Therefore, we also have $Y_n(t) \longrightarrow Y(t)$ in $\mathcal{L}^2([0,T], \mathbb{R})$.

$$\mathbb{E}|Y(t)|^2 \leq M_1, \quad \text{for } t \in [0,T],$$

where M_1 depends on $\sigma_1, \sigma_2, T, K'$ and Y_0, resulting in $Y(\cdot) \in \mathcal{M}^2([0,T]; \mathbb{R})$. It has to be demonstrated that $Y(t)$ satisfies (4.2). Note that

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^t f\Big(Y_n(u),\int_0^u \sigma_1(u,s)Y_n(s)dW(s)\Big)du \\
&\qquad\qquad - \int_0^t f\Big(Y(u),\int_0^u \sigma_1(u,s)Y_n(s)dW(s)\Big)du\bigg|^2 \\
&+\mathbb{E}\bigg|\int_0^t g\Big(Y_n(u),\int_0^u \sigma_2(u,s)Y_n(s)ds\Big)dW(u) \\
&\qquad\qquad - \int_0^t g\Big(Y(u),\int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u)\bigg|^2 \\
&\leq C\int_0^t \mathbb{E}|Y_n(s)-Y(s)|^2 ds.
\end{aligned}$$

Letting $n\to\infty$ in (4.7), we get

$$\begin{aligned}
Y(t) = Y_0 &+ \int_0^t f\Big(Y(u),\int_0^u \sigma_1(u,s)Y(s)dW(s)\Big)du \\
&+ \int_0^t g\Big(Y(u),\int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u).
\end{aligned}$$

□

4.3.2 Hölder continuity of the analytic solutions

We now show Hölder Continuity property by the analytical solution of SVIDEs (4.2).

Theorem 4.2. *Assume that (A_2) holds. Then, the solution $Y(t)$ is Hölder continuous with exponent $\delta = \frac{1}{2}$.*

Proof. For $0\leq r<t\leq T$,

$$\begin{aligned}
Y(t)-Y(r) = &\int_r^t f\Big(Y(u),\int_0^u \sigma_1(u,s)Y(s)dW(s)\Big)du \\
&+ \int_r^t g\Big(Y(u),\int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u).
\end{aligned}$$

Using expectation, we obtain

$$\begin{aligned}
\mathbb{E}\big|Y(t)-Y(r)\big|^2 \leq 2\mathbb{E}&\bigg|\int_r^t f\Big(Y(u),\int_0^u \sigma_1(u,s)Y(s)dW(s)\Big)du\bigg|^2 \\
&+2\mathbb{E}\bigg|\int_r^t \Big(Y(u),\int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u)\bigg|^2.
\end{aligned} \tag{4.11}$$

Using Cauchy's inequality, Itô isometry and (A_2), we have

$$\begin{aligned}
&\mathbb{E}\left|\int_r^t f\Big(Y(u),\int_0^u \sigma_1(u,s)Y(s)dW(s)\Big)du\right|^2\\
&\qquad \le (t-r)\int_r^t K'\Big[1+\mathbb{E}|Y(u)|^2+\mathbb{E}\Big|\int_0^u \sigma_1(u,s)Y(s)dW(s)\Big|^2\Big]\\
&\qquad \le (t-r)K'\Big[(t-r)+\int_r^t \mathbb{E}|Y(s)|^2ds+T\|\sigma_1\|_\infty^2\int_r^t \mathbb{E}|Y(s)|^2ds\Big] \qquad (4.12)\\
&\qquad \le (t-r)K'\Big[(t-r)+\underbrace{\big(1+T\|\sigma_1\|_\infty^2\big)}_{C_1}\int_0^t \mathbb{E}|Y(s)|^2ds\Big]\\
&\qquad \le K'T\Big(1+C_1M_1\Big)\big(t-r\big).
\end{aligned}$$

Also, we get

$$\begin{aligned}
&\mathbb{E}\left|\int_r^t g\Big(Y(u),\int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u)\right|^2\le \qquad (4.13)\\
&\quad \le \int_r^t K'\Big[1+\mathbb{E}|Y(u)|^2+\mathbb{E}\Big|\int_0^u \sigma_2(u,s)Y(s)ds\Big|^2\Big]du\\
&\quad \le K'\int_r^t\Big[1+\mathbb{E}|Y(u)|^2+T\|\sigma_2\|_\infty^2\int_0^t \mathbb{E}|Y(s)|^2ds\Big]du\\
&\quad \le K'\int_r^t\Big[1+\underbrace{\big(1+T^2\|\sigma_2\|_\infty^2\big)}_{C_2}M_1\Big]du\\
&\quad \le K'\Big(1+C_2M_1\Big)\big(t-r\big).
\end{aligned}$$

By substituting (4.12) and (4.13) into (4.11), we have

$$\mathbb{E}|Y(t)-Y(r)|^2\le \widetilde{M}|t-r|,$$

where $\qquad \widetilde{M}:=2K'\Big[T(1+C_1M_1)+(1+C_2M_1)\Big].$

Consequently, $Y(t)$, $t\in[0,T]$ is Hölder continuous with exponent $1/2$.

□

4.4 Numerical analysis of the class of SVIDE

Let $I_h:=\big\{t_n=nh, n=0,1,...,N\big\}$, $I=[0,T]$. For $n=0,1,...,N-1$, we have defined the numerical results of SVIDE

4.4.1 θ-Euler Maruyama method

We apply the $\theta-$ EM method to SVIDEs (4.1) (see [38]-[77] and references therein),

$$
\begin{aligned}
X_{n+1} = X_n + h\Big[\Theta f\Big(X_{n+1}, \sum_{i=0}^{n-1} \sigma_1(t_{n+1}, t_{i+1})X_{i+1}\Delta W_{i+1}\Big) \\
+ (1-\Theta) f\Big(X_n, \sum_{i=0}^{n-1} \sigma_1(t_n, t_i)X_i\Delta W_i\Big)\Big] \\
+ g\Big(X_n, \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_n, s)X_i ds\Big)\Delta W_n,
\end{aligned} \tag{4.14}
$$

where $\Theta \in [0, 1]$ with initial data $X_0 = Y_0$, where $n \in \mathbb{N}$, $t_n = nh$, $\Delta W_n = W(t_{n+1}) - W(t_n)$. By induction, we rewrite (4.15) in the following form:

$$
\begin{aligned}
X_{n+1} = X_0 + \sum_{j=0}^{n} h\Big(\Theta f\Big(X_{j+1}, \sum_{i=0}^{j-1} \sigma_1(t_{j+1}, t_{i+1})X_{i+1}\Delta W_{i+1}\Big) \\
+ (1-\Theta) f\Big(X_j, \sum_{i=0}^{j-1} \sigma_1(t_j, t_i)X_i\Delta W_i\Big)\Big) \\
+ \sum_{j=0}^{n} g\Big(X_j, \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j, s)X_i ds\Big)\Delta W_j.
\end{aligned} \tag{4.15}
$$

Remark 4.1. The scheme (4.14) is called the $\Theta-$ EM and the choice $\Theta = 0$ gives Euler Maruyama method in [183]

$$
X_{n+1} = X_n + hf\Big(X_n, \sum_{i=0}^{n-1} \sigma_1(t_n, t_i)X_i\Delta W_i\Big) + g\Big(X_n, \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_n, s)X_i ds\Big)\Delta W_n. \tag{4.16}
$$

$\theta = \frac{1}{2}$ gives the trapezoidal solver, and $\theta = 1$ gives the implicit, or backward Euler method in [44].

Now, we examine the stability properties with respect to SVIDE (4.14) in the following result.

Theorem 4.3. *Assume that (A_2) holds. Let $\{X_n\}$ be the numerical solution of the $\theta-$EM method (4.15). Then there exists a positive constant K_0, which depends on $\sigma_1, \sigma_2, T, Y_0$ and K' , but not on h, such that*

$$
\mathbb{E}|X_n|^2 \leq \widetilde{K}_0.
$$

Proof. For all $0 \leq t_{n+1} \leq T$, we have

$$
\begin{aligned}
\mathbb{E}|X_{n+1}|^2 \leq 3\mathbb{E}|X_0|^2 + 3\mathbb{E}\bigg| \sum_{j=0}^{n} h \Big[\theta f \Big(X_{j+1}, \sum_{i=0}^{j-1} \sigma_1(t_{j+1}, t_{i+1}) X_{i+1} \Delta W_{i+1} \Big) \\
+ (1-\theta) f \Big(X_j, \sum_{i=0}^{j-1} \sigma_1(t_j, t_i) X_i \Delta W_i \Big) \Big] \bigg|^2 \\
+ 3\mathbb{E}\bigg| \sum_{j=0}^{n} g \Big(X_j, \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j, s) X_i ds \Big) \Delta W_j \bigg|^2 \\
\leq 3\mathbb{E}|X_0|^2 + 3I_1 + 3I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 :=& \mathbb{E}\bigg| \sum_{j=0}^{n} h \Big[\theta f \Big(X_{j+1}, \sum_{i=1}^{j-1} \sigma_1(t_{j+1}, t_{i+1}) X_{i+1} \Delta W_{i+1} \Big) \\
&+ (1-\theta) f \Big(X_j, \sum_{i=0}^{j-1} \sigma_1(t_j, t_i) X_i \Delta W_i \Big) \Big] \bigg|^2,
\end{aligned}
$$

and

$$
I_2 := \mathbb{E}\bigg| \sum_{j=1}^{n} g \Big(X_j, \sum_{i=1}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j, s) X_i ds \Big) \Delta W_j \bigg|^2.
$$

By Cauchy's inequality, Itô isometry and (A_2), we get

$$
\begin{aligned}
I_1 =& \mathbb{E}\Bigg|\sum_{j=0}^{n} h\Bigg[\theta f\Bigg(X_{j+1}, \sum_{i=0}^{j-1}\sigma_1(t_{j+1}, t_{i+1})X_{i+1}\Delta W_{i+1}\Bigg) \\
&+ (1-\theta) f\Bigg(X_j, \sum_{i=0}^{j-1}\sigma_1(t_j, t_i)X_i\Delta W_i\Bigg)\Bigg]\Bigg|^2 \\
&\leq 2(n+1)\sum_{j=0}^{n} h^2\Bigg[\theta^2\mathbb{E}\Bigg|f\Bigg(X_{j+1}, \sum_{i=0}^{j-1}\sigma_1(t_{j+1}, t_{i+1})X_{i+1}\Delta W_{i+1}\Bigg)\Bigg|^2 \\
&+ (1-\theta)^2\mathbb{E}\Bigg|f\Bigg(X_j, \sum_{i=0}^{j-1}\sigma_1(t_j, t_i)X_i\Delta W_i\Bigg)\Bigg|^2\Bigg] \\
&\leq \underbrace{2(n+1)h^2}_{2t_{n+1}h\leq 2Th} K'\sum_{j=0}^{n}\Bigg[2 + \mathbb{E}|X_{j+1}|^2 + \mathbb{E}\Bigg|\sum_{i=0}^{j-1}\sigma_1(t_{j+1}, t_{i+1})X_{i+1}\Delta W_{i+1}\Bigg|^2 \\
&+ \mathbb{E}|X_j|^2 + \mathbb{E}\Bigg|\sum_{i=0}^{j-1}\sigma_1(t_j, t_i)X_i\Delta W_i\Bigg|^2\Bigg]
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&\leq 2ThK'\Bigg[2(n+1) + \sum_{i=1}^{n}\mathbb{E}|X_i|^2 + T\|\sigma_1\|_\infty^2\sum_{i=1}^{n}\mathbb{E}|X_i|^2 \\
&+ \sum_{i=1}^{n}\mathbb{E}|X_i|^2 + T\|\sigma_1\|_\infty^2\sum_{i=1}^{n}\mathbb{E}|X_i|^2 + \mathbb{E}|X_0|^2 \\
&+ T\|\sigma_1\|_\infty^2\mathbb{E}|X_0|^2 + \mathbb{E}|X_{n+1}|^2 + T\|\sigma_1\|_\infty^2\mathbb{E}|X_{n+1}|^2\Bigg] \\
&\leq 2ThK'\Bigg[2(n+1) + 2C_1\sum_{i=1}^{n}\mathbb{E}|X_i|^2 + C_1\big(\mathbb{E}|X_0|^2 + \mathbb{E}|X_{n+1}|^2\big)\Bigg] \\
&\leq 2T^2K'(2 + C_1\mathbb{E}|X_0|^2) + 4ThK'C_1\sum_{i=1}^{n}\mathbb{E}|X_i|^2 + 2T^2K'C_1\mathbb{E}|X_{n+1}|^2.
\end{aligned}
\tag{4.18}
$$

Similar to [183].

Using Cauchy's inequality, Minkowski inequality and (A_2), we show that

$$I_2 = \sum_{j=0}^{n} \mathbb{E}\Big| g\Big(X_j, \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j,s) X_i ds\Big) \Delta W_j \Big|^2$$

$$\begin{aligned}
&+ 2\mathbb{E}\Big[\sum_{0\le i_1<i_2\le n} g\Big(X_{i_1}, \sum_{i=0}^{i_1-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_{i_1},s) X_i ds\Big)\Delta W_{i_1} \\
&\times g\Big(X_{i_2}, \sum_{i=0}^{i_2-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_{i_2},s) X_i ds\Big)\Delta W_{i_2}\Big] \\
&\le hK' \sum_{j=0}^{n} \Big[1 + \mathbb{E}|X_j|^2 + \mathbb{E}\Big| \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j,s) X_i ds \Big|^2\Big] \\
&+ 2\mathbb{E}\Big[\sum_{0\le i_1<i_2\le n} g_{X_{i_1}}(t_{i_1})\Delta W_{i_1} . g_{X_{i_2}}(t_{i_2})\Delta W_{i_2}\Big].
\end{aligned}$$

Since X_{i_1} is $\mathcal{F}_{t_{i_1}}$-adapted, ΔW_{i_1} is $\mathcal{F}_{t_{i_1+1}}$-adapted, X_{i_2} is $\mathcal{F}_{t_{i_2}}$-adapted, and $i < j$, so $X_{i_1}\Delta W_{i_1} X_{i_2}$ is $\mathcal{F}_{t_{i_2}}$-adapted. Namely, $g_{X_{i_2}}(t_{i_2})\Delta W_{i_1} . g_{X_{i_2}}(t_{i_2})$ is $\mathcal{F}_{t_{i_2}}$-adapted, therefore, $g_{X_{i_2}}(t_{i_2})\Delta W_{i_1} . g_{X_{i_2}}(t_{i_2})$ is independent of ΔW_{i_2}, and by $\Delta W_{i_2} \rightsquigarrow N(0, t_{i_2+1} - t_{i_2})$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{0\le i_1<i_2\le n} g_{X_{i_2}}(t_{i_2})\Delta W_{i_1} . g_{X_{i_2}}(t_{i_2})\Delta W_{i_2}\Big] = \\
&= \sum_{0\le i_1<i_2\le n} \mathbb{E}\big[g_{X_{i_2}}(t_{i_2})\Delta W_{i_1} . g_{X_{i_2}}(t_{i_2})\Delta W_{i_2}\big] \\
&= \sum_{0\le i_1<i_2\le n} \mathbb{E}\big[g_{X_{i_2}}(t_{i_2})\Delta W_{i_1} . g_{X_{i_2}}(t_{i_2})\big]\mathbb{E}\big[\Delta W_{i_2}\big] = 0,
\end{aligned}$$

we have

$$\begin{aligned}
I_2 \le& h \sum_{j=0}^{n} \mathbb{E}\Big| g\Big(X_j, \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j,s) X_i ds\Big)\Big|^2 \\
\le& hK' \sum_{j=0}^{n} \Big[1 + \mathbb{E}|X_j|^2 + \mathbb{E}\Big| \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \sigma_2(t_j,s) X_i ds \Big|^2\Big] \\
\le& hK'\Big[n + 1 + \sum_{i=0}^{n} \mathbb{E}|X_i|^2 + T^2 \|\sigma_2\|_\infty^2 \sum_{i=0}^{n} \mathbb{E}|X_i|^2\Big] \\
\le& TK' + C_2 TK' \mathbb{E}|X_0|^2 + hC_2K' \sum_{i=1}^{n} \mathbb{E}|X_i|^2.
\end{aligned} \tag{4.19}$$

Substituting (4.17) and (4.19) into (25),

$$\begin{aligned}
&\mathbb{E}|X_{n+1}|^2 \leq \\
&\leq 3\mathbb{E}|X_0|^2 + 3\Big(2T^2K'(2 + C_1\mathbb{E}|X_0|^2) \\
&\quad + 4ThK'C_1\sum_{i=1}^{n}\mathbb{E}|X_i|^2 + 2T^2K'C_1\mathbb{E}|X_{n+1}|^2\Big) \\
&\quad + 3\Big(TK' + C_2TK'\mathbb{E}|X_0|^2 + hC_2K'\sum_{i=1}^{n}\mathbb{E}|X_i|^2\Big) \\
&\leq 3TK'\Big(\big(1 + 4T\big) + (1 + 2TC_1 + C_2)\mathbb{E}|X_0|^2 \\
&\quad + 3hK'\big(2TC_1 + C_2\big)\sum_{i=1}^{n}\mathbb{E}|X_i|^2 + 2TC_1\mathbb{E}|X_{n+1}|^2\Big).
\end{aligned}$$

So only if it's if $C' := 1 - 6T^2K'C_1 \neq 0$

$$\begin{aligned}
\mathbb{E}|X_{n+1}|^2 \leq &\tfrac{3TK'}{C'}\Big(\big(1 + 4T\big) + (1 + 2TC_1 + C_2)\mathbb{E}|X_0|^2 \\
&\quad + 3hK'\big(2TC_1 + C_2\big)\sum_{i=1}^{n}\mathbb{E}|X_i|^2\Big).
\end{aligned}$$

Through The discrete Grönwall's inequality, we get

$$\mathbb{E}|X_{n+1}|^2 \leq \widetilde{K}_0,$$

where

$$\begin{aligned}
\widetilde{K}_0 :=&\frac{3TK'}{C'}\Big(\big(1 + 4T\big) + (1 + 2TC_1 + C_2)\mathbb{E}|X_0|^2\Big) \\
&.\exp\Big(\frac{3TK'}{C'}(2TC_1 + C_2)\Big).
\end{aligned}$$

□

Similar to [91, 94, 183, 190], the convergence order of the $\theta-$EM method can be enhanced by including more terms in the numerical approximation.

4.4.2 Strong convergence of the $\theta-$ Euler Maruyama method

In order to obtain the convergence result for the θ-Euler Maruyama method (4.14), we now introduce time continuous interpolations of the discrete numerical approximations.

Define $u_n := t_n$ and $X_h(s) := X_n$, for $s \in [t_n, t_{n+1})$ with $0 \le n \le N-1$.

Let $t \in [t_n, t_{n+1})$ with $0 \le n \le N-1$ and X(t) be the continuous form of X_n with $X(t_n) = X_n$, we obtain

$$
\begin{aligned}
X(t) = X(t_n) &+ \int_{t_n}^{t} \Bigg(\theta f\Big(X_h(u_{n+1}), \int_0^{u_h} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \Big) \\
&+ (1-\theta) f\Big(X_h(u_n), \int_0^{u_h} \sigma_1(u_n, s) X_h(s) dW(s) \Big) \Bigg) du \\
&+ \int_{t_n}^{t} g\Big(X_h(u_n), \int_0^{u_h} \sigma_2(u_n, s) X_h(s) ds \Big) dW(u) \\
= X(t_0) &+ \int_0^{t} \Bigg(\theta f\Big(X_h(u_{n+1}), \int_0^{u_h} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \Big) \\
&+ (1-\theta) f\Big(X_h(u_n), \int_0^{u_h} \sigma_1(u_n, s) X_h(s) dW(s) \Big) \Bigg) du \\
&+ \int_0^{t} g\Big(X_h(u_n), \int_0^{u_h} \sigma_2(u_n, s) X_h(s) ds \Big) dW(u).
\end{aligned} \tag{4.20}
$$

The following theorem illustrates the convergence order of (4.20), and its proof proceeds similarly to [183] for the situation when $\theta = 0$.

Lemma 4.2. *Assume that (A_2) holds. Let $\{X_n\}$ be the numerical solution of the $\theta-$Euler Maruyama method 4.14. Then there exists a positive constant $\widetilde{K}_1$, which depends on σ_1, σ_2, K' and T, but not on h, such that*

$$
\mathbb{E}\big| X(t) - X(t_n) \big|^2 \le \widetilde{K}_1 h.
$$

Proof. It is easy to see that

$$
\begin{aligned}
\mathbb{E}\big| X(t) - X(t_n) \big|^2 \le 2\mathbb{E}\Bigg| &\int_{t_n}^{t} \Bigg(\theta f\Big(X_h(u_{n+1}), \int_0^{u_n} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \Big) \\
&+ (1-\theta) f\Big(X_h(u_n), \int_0^{u_n} \sigma_1(u_n, s) X_h(s) dW(s) \Big) \Bigg) du \Bigg|^2 \\
&+ 2\mathbb{E}\Bigg| \int_{t_n}^{t} g\Big(X_h(u_n), \int_0^{u_n} \sigma_2(u_n, s) X_h(s) ds \Big) dW(u) \Bigg|^2 \\
\le 2J_1 + 2J_2,&
\end{aligned}
$$

$$
\begin{aligned}
J_1 :=& \mathbb{E}\bigg| \int_{t_n}^{t} \Bigg(\theta f\Big(X_h(u_{n+1}), \int_0^{u_n} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \Big) \\
&+ (1-\theta) f\Big(X_h(u_n), \int_0^{u_n} \sigma_1(u_n, s) X_h(s) dW(s) \Big) \Bigg) du \bigg|^2,
\end{aligned}
$$

and

$$
J_2 := \mathbb{E}\bigg| \int_{t_n}^{t} g\Big(X_h(u_n), \int_0^{u_n} \sigma_2(u_n, s) X_h(s) ds \Big) dW(u) \bigg|^2.
$$

By $(x+y)^2 \leq 2x^2 + 2y^2$, Cauchy's inequality, Itô isometry and (A_2), we get

$$
\begin{aligned}
J_1 \leq& 2h \int_{t_n}^{t} \Bigg(\theta \mathbb{E}\bigg| f\Big(X_h(u_{n+1}), \int_0^{u_n} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \Big) \bigg|^2 && (4.21) \\
&+ (1-\theta) \mathbb{E}\bigg| f\Big(X_h(u_n), \int_0^{u_n} \sigma_1(u_n, s) X_h(s) dW(s) \Big) \bigg|^2 \Bigg) du \\
\leq& 2h \Bigg[\theta^2 \int_{t_n}^{t} \Bigg(\mathbb{E}\bigg| f\Big(X_h(u_{n+1}), \int_0^{u_n} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \Big) \bigg|^2 \Bigg) du \\
&+ (1-\theta)^2 \int_{t_n}^{t} \Bigg(\mathbb{E}\bigg| f\Big(X_h(u_n), \int_0^{u_n} \sigma_1(u_n, s) X_h(s) dW(s) \Big) \bigg|^2 \Bigg) du \Bigg] \\
\leq& 2h \int_{t_n}^{t} K' \Bigg[2 + \mathbb{E}|X_{n+1}|^2 + \mathbb{E}\bigg| \int_0^{t_n} \sigma_1(u_{n+1}, s) X_h(s) dW(s) \bigg|^2 \\
&+ \mathbb{E}|X_n|^2 + \mathbb{E}\bigg| \int_0^{t_n} \sigma_1(u_n, s) X_h(s) dW(s) \bigg|^2 \Bigg] du \\
\leq& 2h \int_{t_n}^{t} K' \Bigg[2 + \mathbb{E}|X_{n+1}|^2 + \|\sigma_1\|_\infty^2 \int_0^{t_n} \mathbb{E}\big| X_h(s) \big|^2 ds && (4.22) \\
&+ \mathbb{E}|X_n|^2 + \|\sigma_1\|_\infty^2 \int_0^{t_n} \mathbb{E}\big| X_h(s) \big|^2 ds \Bigg] du \\
\leq& 2h^2 K' \Bigg[2 + \mathbb{E}|X_{n+1}|^2 + \|\sigma_1\|_\infty^2 \int_0^{t_n} \mathbb{E}\big| X_h(s) \big|^2 ds && (4.23) \\
&+ \mathbb{E}|X_n|^2 + \|\sigma_1\|_\infty^2 \int_0^{t_n} \mathbb{E}\big| X_h(s) \big|^2 ds \Bigg] \\
\leq& 2h^2 K' \Big(2 + 2\big(1 + \|\sigma_1\|_\infty^2 T\big) K_0 \Big) := 4h^2 K' \big(1 + C_1 K_0\big).
\end{aligned}
$$

By (A_2), Cauchy's inequality and Itô isometry, we obtain

$$
\begin{aligned}
J_2 \leq & \int_{t_n}^{t} \mathbb{E}\bigg|g\Big(X_n, \int_0^{t_n} \sigma_2(u_n, s)X_h(s)ds, \Big)\bigg|^2 dW(u) \\
\leq & \int_{t_n}^{t} K'\left[1 + \mathbb{E}|X_n|^2 + \mathbb{E}\left|\int_0^{t_n} \sigma_2(u_n, s)X_h(s)ds\right|^2\right]du \\
\leq & \int_{t_n}^{t} K'\left[1 + \mathbb{E}|X_n|^2 + \int_0^{t_n} \|\sigma_2\|_\infty^2 ds \int_0^{t_n} \mathbb{E}\big|X_h(s)\big|^2 ds\right]du \\
\leq & hK'\left[1 + \mathbb{E}|X_n|^2 + T\|\sigma_2\|_\infty^2 \int_0^{t_n} \mathbb{E}\big|X_h(s)\big|^2 ds\right] \\
\leq & hK'\left[1 + \big(1 + \|\sigma_2\|_\infty^2 T^2\big)\mathbb{E}|X_n|^2\right] \\
\leq & hK'(1 + C_2K_0).
\end{aligned}
\tag{4.24}
$$

From 4.21 and 4.24 we get

$$
\mathbb{E}\big|X(t) - X(t_n)\big|^2 \leq \widetilde{K}_1 h,
$$

where

$$
\widetilde{K}_1 := 2K'\Big((2T + 1) + (2C_1T + C_2)M_0\Big).
$$

□

Theorem 4.4. *Suppose(A_1) and $\sigma_i \in C^1(D)$, for $i = 1, 2$ satisfy (A_3). Let $X\ (t)$ and $Y\ (t)$ are The numerical solution of the θ-Euler-Maruyama method and the analytical solution (4.2), respectively. Then there exists a positive constant $\widetilde{K}_2$, which depends on $\sigma_1, \sigma_2, K, K'$, and T, but not on h, such as*

$$
\mathbb{E}|X(t) - Y(t)|^2 \leq \widetilde{K}_2 h.
$$

Proof. By (A_1), Cauchy's inequality and the Itô isometry, we have

$$
\mathbb{E}|X(t) - Y(t)|^2 \leq 2L_1 + 2L_2,
\tag{4.25}
$$

where

$$
\begin{aligned}
L_1 :=& \mathbb{E}\bigg| \int_0^t \theta\Big(f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big) \\
&- f\Big(X_h(u_{n+1}), \int_0^{u_h} \sigma_1(u_h,s)X_h(s)dW(s)\Big)\Big) \\
&+ (1-\theta)\Big(f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big) \\
&- f\Big(X_h(u_n), \int_0^{u_n} \sigma_1(u_h,s)X_h(s)dW(s)\Big)\Big) du\bigg|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 := \mathbb{E}\bigg| &\int_0^t g\Big(Y(u), \int_0^u \sigma_2(u,s)Y(s)ds\Big)dW(u) \\
&- \int_0^t g\Big(X_h(u_h), \int_0^{u_h} \sigma_2(u_h,s)X_h(s)ds\Big)dW(u)\bigg|^2.
\end{aligned}
$$

By Cauchy inequality and Itô isometry, we obtain

$$
\begin{aligned}
L_1 \leq& T\int_0^t \mathbb{E}\bigg|\theta\Big(f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big) \\
&- f\Big(X_h(u_{n+1}), \int_0^{u_h} \sigma_1(u_{n+1},s)X_h(s)dW(s)\Big)\Big) \\
&+ (1-\theta)\Big(f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big) \\
&- f\Big(X_h(u_n), \int_0^{u_h} \sigma_1(u_n,s)X_h(s)dW(s)\Big)\Big)\bigg|^2 du \\
\leq& 2T\bigg(\theta^2 \int_0^t \mathbb{E}\bigg| f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big) \\
&- f\Big(X_h(u_{n+1}), \int_0^{u_h} \sigma_1(u_{n+1},s)X_h(s)dW(s)\Big)\bigg|^2 du \\
&+ (1-\theta)^2 \int_0^t \mathbb{E}\bigg| f\Big(Y(u), \int_0^u \sigma_1(u,s)Y(s)dW(s)\Big) \\
&- f\Big(X_h(u_n), \int_0^{u_h} \sigma_1(u_n,s)X_h(s)dW(s)\Big)\bigg|^2 du\bigg).
\end{aligned}
$$

Next, using (A_1) and $(x+y)^2 \leq x^2+y^2$, one has

$$
\begin{aligned}
L_1 \leq\ & 2KT\int_0^t \bigg[\mathbb{E}\big|Y(u)-X_h(u_{n+1})\big|^2+\mathbb{E}\bigg|\int_0^u \sigma_1(u,s)Y(s)dW(s) \\
& -\int_0^{u_h}\sigma_1(u_{n+1},s)X_h(s)dW(s)\bigg|^2\bigg]du \\
& +2KT\int_0^t \bigg[\mathbb{E}\big|Y(u)-X_h(u_n)\big|^2+\mathbb{E}\bigg|\int_0^u \sigma_1(u,s)Y(s)dW(s) \\
& -\int_0^{u_n}\sigma_1(u_n,s)X_h(s)dW(s)\bigg|^2\bigg]du \\
\leq\ & 2KT\int_0^t \bigg[\mathbb{E}\big|Y(u)-X(u)+X(u)-X_h(u_{n+1})\big|^2 \\
\leq\ & \mathbb{E}\bigg|\int_{u_h}^u \sigma_1(u_{n+1},s)X_h(s)dW(s) \\
& +\int_0^u\bigg(\sigma_1(u,s)Y(s)dW(s)-\sigma_1(u_{n+1},s)X_h(s)dW(s)\bigg|^2\bigg) \\
& +\mathbb{E}\big|Y(u)-X(u)+X(u)-X_h(u_n)\big|^2 \\
& +\mathbb{E}\bigg|\int_{u_h}^u \sigma_1(u_n,s)X_h(s)dW(s) \\
& +\int_0^u\bigg(\sigma_1(u,s)Y(s)dW(s)-\sigma_1(u_n,s)X_h(s)dW(s)\bigg|^2\bigg)\bigg]du \\
\leq\ & 4KT\int_0^t \bigg[\mathbb{E}\big|Y(u)-X(u)\big|^2+\mathbb{E}\big|X(u)-X_h(u_{n+1})\big|^2 \\
& +\mathbb{E}\bigg|\int_{u_h}^u \sigma_1(u_{n+1},s)X_h(s)dW(s)\bigg|^2 \\
& +\mathbb{E}\bigg|\int_0^u\Big(\sigma_1(u,s)Y(s)-\sigma_1(u_{n+1},s)X_h(s)\Big)dW(s)\bigg|^2 \\
& +\mathbb{E}\big|Y(u)-X(u)\big|^2+\mathbb{E}\big|X(u)-X_h(u_n)\big|^2 \\
& +\mathbb{E}\bigg|\int_{u_h}^u \sigma_1(u_n,s)X_h(s)dW(s)\bigg|^2 \\
& +\mathbb{E}\bigg|\int_0^u\Big(\sigma_1(u,s)Y(s)-\sigma_1(u_n,s)X_h(s)\Big)dW(s)\bigg|^2\bigg]du.
\end{aligned}
$$

By Hölder's inequality, Itô isometry, Theorem 4.3 and Lemma 4.2, we have

$$\int_0^t \mathbb{E}\big|X(u)-X_h(u_n)\big|^2 du \leq TK_1h,$$

$$\int_0^t \mathbb{E}\big|X(u)-X_h(u_{n+1})\big|^2 du \leq TK_1h,$$

and

$$\begin{aligned}\int_0^t \mathbb{E}\bigg|\int_{u_h}^u \sigma_1(u_{n+1},s)X_h(s)dW(s)\bigg|^2 du &\leq \int_0^t\int_{u_h}^u \sigma_1(u_{n+1},s)\mathbb{E}|X_h(s)|^2 ds\,du\\ &\leq \int_0^t \|\sigma_1\|_\infty^2 hK_0 du\\ &\leq \|\sigma_1\|_\infty^2 hK_0T,\\ \int_0^t \mathbb{E}\bigg|\int_{u_h}^u \sigma_1(u_{n+1},s)X_h(s)dW(s)\bigg|^2 du &\leq \|\sigma_1\|_\infty^2 hK_0T.\end{aligned}$$

By (A_3), we show that

$$\begin{aligned}&\int_0^t \mathbb{E}\bigg|\int_0^u \Big(\sigma_1(u,s)Y(s)-\sigma_1(u_n,s)X_h(s)\Big)dW(s)\bigg|^2 du\\ &\leq \int_0^t\int_0^u \mathbb{E}\Big|\sigma_1(u,s)Y(s)-\sigma_1(u_n,s)X_h(s)\Big|^2 ds\,du\\ &\leq \int_0^t\int_0^u \mathbb{E}\Big|\sigma_1(u,s)Y(s)-\sigma_1(u_n,s)X_h(s)+\sigma_1(u,s)X_h(s)-\sigma_1(u,s)X_h(s)\Big|^2 ds\,du\\ &\leq 2\int_0^t\int_0^u \mathbb{E}\big|\sigma_1(u,s)\big|^2\mathbb{E}\big|Y(s)-X_h(s)\big|^2 ds\,du\\ &\quad +2\int_0^t\int_0^u \mathbb{E}\big|\sigma_1(u,s)-\sigma_1(u_n,s)\big|^2\mathbb{E}\big|X_h(s)\big|^2 ds\,du\\ &\leq 4\int_0^t\int_0^u \mathbb{E}\big|\sigma_1(u,s)\big|^2\mathbb{E}\big|Y(s)-X(s)\big|^2 ds\,du\\ &\leq +4\int_0^t\int_0^u \mathbb{E}\big|\sigma_1(u,s)\big|^2\mathbb{E}\big|X(s)-X_h(s)\big|^2 ds\,du\\ &\leq +2\int_0^t\int_0^u \mathbb{E}\big|\sigma_1(u,s)-\sigma_1(u_n,s)\big|^2\mathbb{E}\big|X_h(s)\big|^2 ds\,du\\ &\leq 2K''hK_0'T^2+4\|\sigma_1\|_\infty^2 hK_1'T^2+4T\|\sigma_1\|_\infty^2\int_0^t \mathbb{E}\big|Y(s)-X(s)\big|^2 ds.\end{aligned}$$

Thus

$$L_1 \leq L_{11}hT + L_{12}\int_0^u \mathbb{E}\big|Y(s)-X(s)\big|^2 ds, \tag{4.26}$$

where

$$L_{11} := 8KT\Big[\big(K_0' + 4TK_1'\big)\|\sigma_1\|_\infty^2 + K_1 + 2TK''K_0\Big],$$

and

$$L_{12} := 8KT\big[1 + 4T\|\sigma_1\|_\infty^2\big].$$

Using 4.3, we get

$$\begin{aligned}L_2 := \mathbb{E}\Big| \int_0^t g\big(Y(u), \int_0^u \sigma_2(u,s)Y(s)ds\big)dW(u)-\\ - \int_0^t g\big(X_h(u_h), \int_0^{u_h} \sigma_2(u_h,s)X_h(s)ds)\big)dW(u)\Big|^2.\end{aligned}$$

$$\begin{aligned}
L_2 \le& \int_0^t \mathbb{E}\Big|g\Big(Y(u), \int_0^u \sigma_2(u,s)Y(s)ds\Big) - g\Big(X_h(u_h), \int_0^{u_h} \sigma_2(u_h,s)X_h(s)ds\Big)\Big|^2 du\\
&\le K\int_0^t \big[\mathbb{E}\big|Y(u) - X_h(u_h)\big|^2 + \mathbb{E}\Big|\int_0^u \sigma_2(u,s)Y(s)ds - \int_0^{u_h} \sigma_2(u_h,s)X_h(s)ds\Big|^2\big]du\\
&\le K\int_0^t \Big[\mathbb{E}\big|Y(u) - X(u) + X(u) - X_h(u_h)\big|^2 + \mathbb{E}\Big|\int_0^u \sigma_2(u,s)Y(s)ds -\\
&\quad \int_0^{u_h} \sigma_2(u_h,s)X_h(s)ds\Big|^2\Big]du\\
&\le 2KT\int_0^t \Big[\mathbb{E}\big|X(u) - X_h(u_h)\big|^2 + \mathbb{E}\big|Y(u) - X(u)\big|^2\\
&\quad + \mathbb{E}\Big|\int_{u_h}^u \sigma_2(u_h,s)X_h(s)ds\Big|^2 + \mathbb{E}\Big|\int_0^u \sigma_2(u,s)Y(s)ds - \int_0^u \sigma_2(u_h,s)X_h(s)ds\Big|^2\Big]du\\
&\le 2KT\int_0^t \Big[\mathbb{E}\big|X(u) - X_h(u_h)\big|^2 + \mathbb{E}\big|Y(u) - X(u)\big|^2\\
&\quad + \mathbb{E}\Big|\int_{u_h}^u \sigma_2(u_h,s)X_h(s)ds\Big|^2 + \mathbb{E}\Big|\int_0^u \big(\sigma_2(u,s)Y(s) - \sigma_2(u_h,s)X_h(s)\big)ds\Big|^2\Big]du.
\end{aligned} \tag{4.27}$$

By Cauchy's inequality, Itô isometry, Theorem 4.3 and Lamma 4.2, we got

$$\begin{aligned}
\int_0^t \mathbb{E}\Big|\int_{u_h}^u \sigma_2(u_h,s)X_h(s)ds\Big|^2 du \le& \int_0^t \Big[\int_{u_h}^u |\sigma_2(u_h,s)|^2 ds \int_{u_h}^u \mathbb{E}|X_h(s)|^2 ds\Big]du\\
\le& \int_0^t \Big[\int_{u_h}^u \|\sigma_2\|_\infty^2 ds \int_{u_h}^u K_0 ds\Big]du\\
\le& h^2T\|\sigma_2\|_\infty^2 K_0,
\end{aligned}$$

by Cauchy's inequality, Theorem 4.3, Lamma 4.2 and $(x+y+z)^2 \leq 2x^2+2y^2+2z^2$, we can obtain

$$
\begin{aligned}
&\int_0^t \Big[\mathbb{E}\Big|\int_0^u \big(\sigma_2(u_h,s)X_h(s)-\sigma_2(u,s)Y(s)\big)ds\Big|^2\Big]du \\
&\quad= \int_0^t [\mathbb{E}|\int_0^u \big(\sigma_2(u_h,s)X_h(s) - \sigma_2(u,s)[Y(s) - X_h(s) + X_h(s)-X(s) + X(s)]\big)ds|^2]du \\
&\quad\leq 2\int_0^t \int_0^u \big|(\sigma_2(u_h,s) - \sigma_2(u,s)\big|^2\mathbb{E}\big|X_h(s)\big|^2 ds\,du \\
&\qquad+ 2\int_0^t \int_0^u \big|\sigma_2(u,s)\big|^2\mathbb{E}\big|Y(s)-X(s)\big|^2 ds\,du \\
&\qquad+ 2\int_0^t \int_0^u \big|\sigma_2(u,s)\big|^2\mathbb{E}\big|X(s)-X_h(s)\big|^2 ds\,du \\
&\quad\leq 2\int_0^t \int_0^u K''h^2K_0\,ds\,du + 2\int_0^t\int_0^u \|\sigma_2\|_\infty^2\mathbb{E}\big|Y(s) - X(s)\big|^2 ds\,du \\
&\qquad+ 2\int_0^t \int_0^u \|\sigma_2\|_\infty^2 K_1 h\,ds\,du \\
&\quad\leq 2T^2h^2K''K_0 + 2T^2h\|\sigma_2\|_\infty^2K_1 + 2T\|\sigma_2\|_\infty^2\int_0^u \mathbb{E}\big|Y(s)-X(s)\big|^2 ds,
\end{aligned}
$$

Thus

$$
L_2 \leq L_{21}hT + L_{22}\int_0^u \mathbb{E}\big|Y(s)-X(s)\big|^2 ds, \tag{4.28}
$$

where

$$
L_{21} := 2KT\Big[\big(hK_0+2TK_1\big)\|\sigma_2\|_\infty^2 T + 2ThK_0K''\Big],
$$

and

$$
L_{22} := 2KT\big[1+2T\|\sigma_2\|_\infty^2\big].
$$

By compensating (4.26) and (4.28) in (4.25), we have

$$
\mathbb{E}|X(t)-Y(t)|^2 \leq 2\big(L_{11}+L_{21}\big)hT + 2(L_{12}+L_{22})\int_0^u \mathbb{E}\big|Y(s)-X(s)\big|^2 ds,
$$

L_{11}, L_{12}, L_{21} and L_{22} knew it before. By Grönwall's inequality, we have

$$
\mathbb{E}|X(t)-Y(t)|^2 \leq \widetilde{K}_2 h,
$$

where

$$
\widetilde{K}_2 := 2T\big(L_{11}+L_{21}\big)\exp\big(2(L_{12}+L_{22})T\big).
$$

$\square$

4.5 Numerical experiments

In this section, we present two numerical examples to verify the theoretical results, we use discrete Brownian paths over $[0, 1]$ with $\Delta t = 2^{-10}$. Let $Y_h^i(T)$ represent the numerical solution of the θ-Euler-Maruyama method along the i sample path at $t = T$ with step size $h \in [2^3, 2^4, 2^5, 2^6]$. We take the numerical solution $Y_\Delta^i(T)$ to be an approximation of the analytic solution and compare this with the numerical approximation over $M_{mean} = 1000$ sample paths. The mean-square error is

$$\text{Error}_h := \left(\frac{1}{M_{mean}} \sum_{i=1}^{M_{mean}} \left|Y_h^i(T) - Y_{\Delta t}^i(T)\right|^2\right)^{1/2}.$$

while the strong convergence order is defined numerically by

$$\text{Order} = \log_2 \frac{\text{Error}_h}{\text{Error}_{h/2}}.$$

Consider the following stochastic Volterra integro-diferential equation:

$$\begin{aligned} dY(t) = &\left(Y(t) + a\cos\left(\int_0^t \sigma_1(t,s)Y(s)\,dW(s)\right)\right)dt \\ &+ \left(Y(t) + b\sin\left(g\left(Y(t), \int_0^t \sigma_2(t,s)Y(s)\,ds\right)\right)\right)dW(t), \end{aligned} \tag{4.29}$$

with initial data $Y(0) = 1$ and functions $f(x, y) = x + a\cos(y)$, $g(x, y) = x + b\sin(y)$.

Now, we present the following examples:

Example 4.1. In equation (4.29), we take $a = 1$, $b = 1$, $\sigma_1(t, s) = \sin(2t - s)$, $\sigma_2 = (t, s) = t - s + 1$.

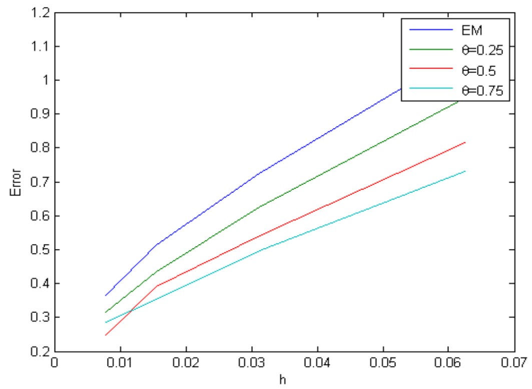
4.1 presents a comparison between the θ- Euler-Maruyama technique and the Euler-Maruyama technique for the average values of the mean square error and the values of the strong convergence order. Additionally, curves 4.1a and 4.1b are displayed.

The curves 4.1a and 4.1b showing the mean square error and strong convergence order curves of the Euler-Maruyama and θ-EM methods with $\theta = 0.25,\ 0.5,\ 0.75$, respectively, and based on the results presented in table 4.1.

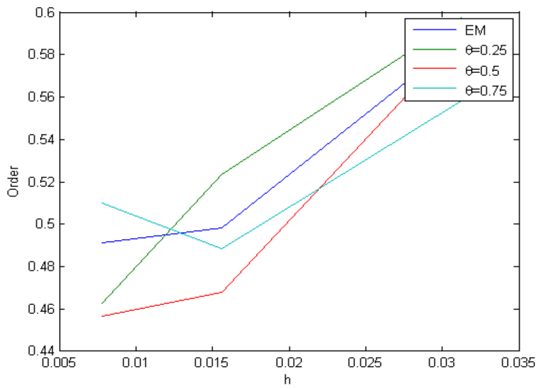
In 4.2, we have presented the solution curves. The blue curve shows the approximation of the analytic solution using the Euler-Maruyama method, the red curve shows the numerical solution of the Euler-Maruyama method, and the green curve shows the numerical solution of the θ-Euler-Maruyama method by changing the value of θ ($\theta = 0,\ 0.25,\ 0.5$ and 0.75), in proportion to example 4.1.

Stepsize	Euler-Maruyama method		θ-Euler-Maruyama method					
			$\theta = 0.25$		$\theta = 0.5$		$\theta = 0.75$	
	Error	Order	Error	Order	Error	Order	Error	Order
$2^3\Delta t$	0.36491		0.31615		0.24880		0.28555	
$2^4\Delta t$	0.51287	0.49104	0.43569	0.46267	0.39184	0.45652	0.35430	0.50998
$2^5\Delta t$	0.72441	0.49822	0.62635	0.52368	0.54196	0.46794	0.49710	0.48856
$2^6\Delta t$	1.08861	0.58761	0.94770	0.59744	0.81481	0.58826	0.73190	0.55811

Table 4.1: The Means square errors and Strong convergence order of the EM and $\theta-$EM methods with $\theta \in [0.25, 0.5, 0.75]$ for example 4.1.



(a) Mean square error of the EM and $\theta-$EM method with $\theta \in [0.25, 0.5, 0.75]$ for the example 4.1.

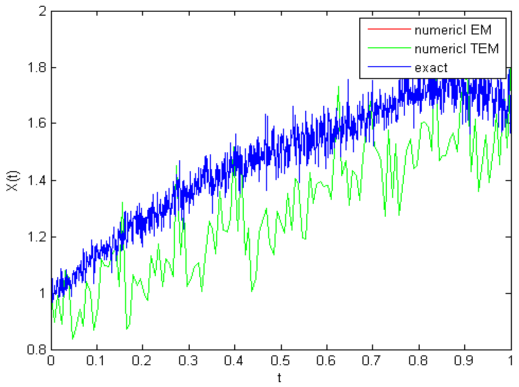


(b) Strong convergence order of the EM and $\theta-$EM method with $\theta \in [0.25, 0.5, 0.75]$ for the example 4.1.

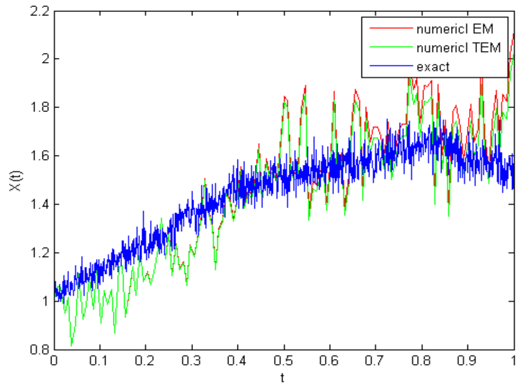
Example 4.2. In equation (4.29), we take $a = 0.5$, $b = 0.2$, $\sigma_1(t,s) = \sin(2st - s)$, $\sigma_2 = (t,s) = \cos(t^2 - s + 1)$.

As with the previous example, we provided the main result of the second example by using the same approach. table 4.2, corresponding to example 4.2, presents the mean square error and the strong convergence results of the Euler-Maruyama and θ-Euler-Maruyama methods, and also reinforces the results obtained in the example 4.1.

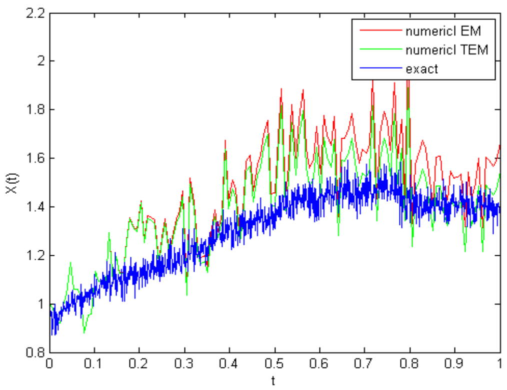
In example 4.4, the solution curves are depicted. The blue curve represents the approximation of the analytical solution using the Euler-Maruyama method, the red curve represents the numerical solution of the Euler-Maruyama method, and the green curve



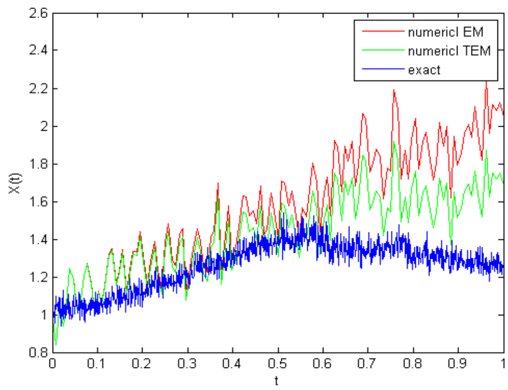
(a) $\theta = 0$



(b) $\theta = 0.25$



(c) $\theta = 0.5$



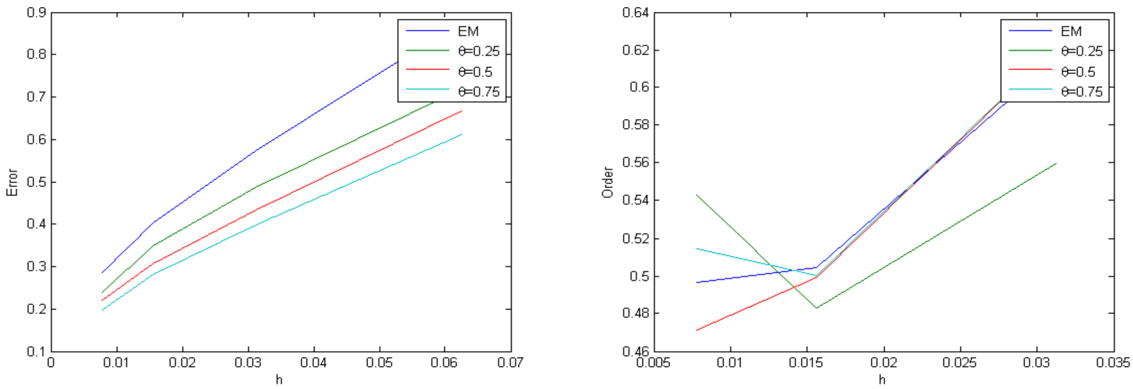
(d) $\theta = 0.75$

Figure 4.2: Approximate solution and numerical solutions by the EM and θ-EM method with $h = 2^3.\Delta t$ for the example 4.1.

represents the numerical solution of the θ-Euler-Maruyama method.

Stepsize	Euler-Maruyama method		θ-Euler-Maruyama method					
			$\theta = 0.25$		$\theta = 0.5$		$\theta = 0.75$	
	Error	Order	Error	Order	Error	Order	Error	Order
$2^3\Delta t$	0,28632		0.23911		0,22111		0,19686	
$2^4\Delta t$	0,40390	0,49637	0.34838	0.54296	0,30660	0,47154	0,28122	0,51449
$2^5\Delta t$	0,57300	0,50453	0.48685	0.48279	0,43328	0,49895	0,39785	0,50051
$2^6\Delta t$	0,87765	0,61508	0.71764	0.55977	0,66639	0,62105	0,61200	0,62130

Table 4.2: The Means square errors and Strong convergence order of the EM and $\theta-$EM methods with $\theta \in \{0.25, 0.5, 0.75\}$ for the example 4.2.

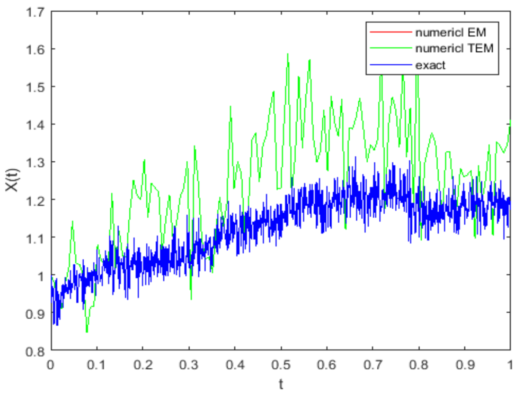


(a) Mean square error of the EM and $\theta-$EM methods with $\theta \in \{0, 0.25, 0.5, 0.75\}$ for the example 4.2.

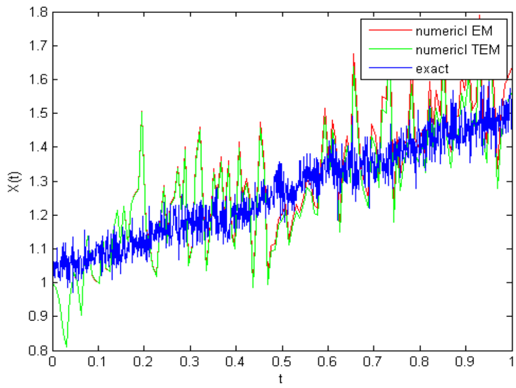
(b) Strong convergence order of the EM and $\theta-$EM methods with $\theta \in [0, 0.25, 0.5, 0.75]$ for the example 4.2.

The strong convergence results of the θ-Euler-Maruyama method of stochastic Volterra integro-differential equations in the examples 4.1 and 4.2 are shown in tables 4.1 and 4.2. From these tables, we can see that the θ-Euler-Maruyama method of the stochastic Volterra integro-differential equations (SVIDEs) is convergent of order $1/2$.

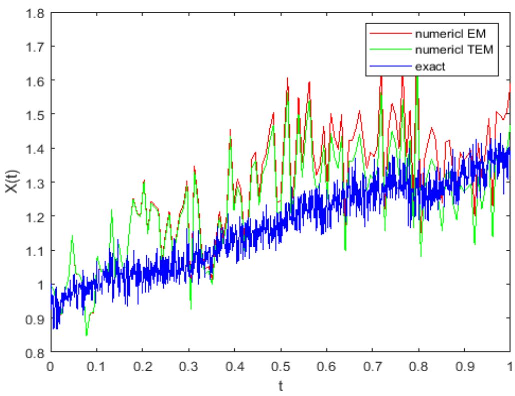
As a result, we have shown that the θ-EM method is more efficient than the EM method for the numerical approximation of the solution of a stochastic Volterra integro-differential equations for different values of θ ($\theta = 0.25, 0.5, 0.75$).



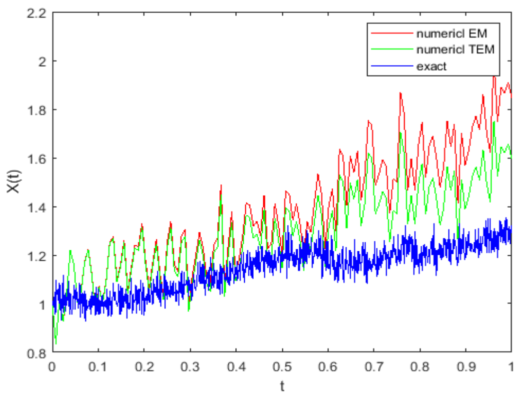
(a) $\theta = 0$



(b) $\theta = 0.25$



(c) $\theta = 0.5$



(d) $\theta = 0.75$

Figure 4.4: Approximate solution and numerical solutions by the EM and θ-EM method with $h = 2^3\Delta t$ for the example 4.2.

Appendix A

Some Fundamental Spaces

Let $J := [0, T]$ be an interval of $\mathbb{R}$. Let $(\mathbb{X}, |.|)$ be a real Banach space.

Throughout this section (Ω, F, μ) is a measure space where μ is σ-finite, and $\mathbb{X}$ is a separable Banach space with norm $|.|_{\mathbb{X}}$. We endow E with the Borel σ-field $\mathfrak{B}(X)$.

Definition A.1. [59]Let $X : \Omega \to \mathbb{X}$ be a simple random variable $X = \sum\limits_{i=1}^{n} x_i \mathbf{1}_{A_i}$, where $x_i \in \mathbb{X}$, $A_i \in F$, $\mu(A_i) < \infty$. The Bochner integral of X is defined as

$$\int_\Omega X(\omega) d\mu(\omega) := \sum_{i=1}^{n} x_i \mu(A_i).$$

Let X be in $L^1(\Omega, \mathbb{X})$. The Bochner integral of X is defined as

$$\int_\Omega X(\omega) d\mu(\omega) := \lim_{k\to+\infty} \int_\Omega X_k(\omega) d\mu(\omega),$$

where $X_k : \Omega \to \mathbb{X}$ are simple random variables such that

$$\lim_{k\to+\infty} \int_\Omega |X(\omega) - X_k(\omega)|_E \, d\mu(\omega) = 0.$$

Proposition A.1. *(i) Let $X \in L^1(\Omega, \mathbb{X})$. Then the Bochner integral of X is well defined and does not depend on the choice of the sequence. Moreover*

$$\left| \int_\Omega X(\omega) d\mu(\omega) \right| \le \int_\Omega |X(\omega)| \, d\mu(\omega).$$

(ii) Let (Ω, F, μ) be a complete measure space, $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and $A : D(A) \subset \mathbb{X} \to \mathbb{Y}$ be a closed operator. If $X \in L^1(\Omega, \mathbb{X})$ and $AX \in L^1(\Omega, \mathbb{Y})$, then

$$\int_\Omega AX(\omega) d\mu(\omega) = A \int_\Omega X(\omega) d\mu(\omega).$$

Proof. See [48].

The Banach space $L^p([0,T],\mathbb{X})$ is the space of Borel measurable functions on $[0,T]$ that are Bochner integrable to the p-th power. The norm on this space is given by:

$$\|f\|_{L^p([0,T],\mathbb{X})} = \left(\int_0^T \|f(t)\|_{\mathbb{X}}^p \, dt\right)^{1/p}.$$

The space of square-integrable functions, $L^2([0,T],\mathbb{X})$, equipped with the scalar product

$$(f,g)_{L^2(\mathbb{X})} = \int_0^T (f(t),g(t))_{\mathbb{X}} \, dt,$$

is a Hilbert space.

The Banach space of essentially bounded functions on $[0,T]$ is denoted by $L^\infty([0,T],\mathbb{X})$ and consists of functions f for which the norm

$$\|f\|_{L^\infty([0,T],\mathbb{X})} = \text{ess sup}\,\|f(t)\|_{\mathbb{X}},\ t \in [0,T],$$

is finite.

A.1 Banach Spaces

Definition A.2. [58, 59] Given a linear space X over $\mathbb{R}$ (or $\mathbb{C}$), a norm $\|\cdot\| : X \to \mathbb{R}$ assigns a real number to each element of X, and satisfies certain properties:

(i) $\|x\| \geq 0$,

(ii) $\|x\| = 0$ if and only if $x = 0$,

(iii) $\|\lambda x\| = |\lambda|\,\|x\|$, for any scalar λ,

(iv) $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

A linear space X with a norm $\|\cdot\|$ is denoted by $(X, \|\cdot\|)$ and is called a normed linear space.

Example A.1. (Examples of normed spaces)

(i) The Euclidean space $\mathbb{R}^n$ with the norm

$$\|\zeta\|_2 = \left(\sum_{i=1}^{n} (\zeta_i)^2\right)^{\frac{1}{2}}, \quad \zeta = (\zeta_1, \cdots, \zeta_n),$$

$(\mathbb{R}^n, \|\|_2)$ is a normed space.

Given any $p \geq 1$, we can define a norm on $\mathbb{R}^n$ by

$$\|\zeta\|_p = \left(\sum_{i=1}^{n} |\zeta_i|^p\right)^{\frac{1}{p}}.$$

Also a norm on $\mathbb{R}^n$ for the case $p = \infty$. It is defined by

$$\|\zeta\|_\infty = \max_{1 \leq i \leq n} |\zeta_i|.$$

(ii) The space $\mathcal{C}[a, b]$ has a similar norm for any $p \leq 1$, with the norm

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Definition A.3. [26, 59] Let $\mathbb{X}$ be a normed space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. If there exists an element $x \in \mathbb{X}$ $\lim_{n\to\infty}\|x_n - x\| = 0$ for some $x \in \mathbb{X}$, the sequence $\{x_n\}$ is said to converge to some element $x \in \mathbb{X}$, then $x_n \to \mathbb{X}$.

Definition A.4. [12, 59] Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed space X. $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence for all $\varepsilon \geq 0$ there exists $\eta \in \mathbb{K}$ such that

$$\|x_n - x_m\| < \varepsilon, \qquad \text{for every } n, m \geq \eta.$$

A space $\mathbb{X}$ is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$ has a limit $x \in \mathbb{X}$.

Definition A.5. [26] A Banach space is a normed space $\mathbb{X}$ that is complete with respect to the norm.

A.2 Hilbert Spaces

Definition A.6. [179] An Inner product (or scalar product) on a complex vector space K, is a mapping $(.,.) : K \times K \to \mathbb{R}$ such that, for all $x, y, z \in K$ and all $\lambda \in \mathbb{C}$

(i) $(x, y) = \overline{(y, x)}$,

(ii) $(\lambda x, y) = \lambda(x, y)$,

(iii) $(x + z, y) = (x, y) + (z, y)$,

(iv) $(x, x) > 0$, when $x \neq 0$.

An inner product space (or pre-Hilbert Spaces) is a pair $(K, (\cdot, \cdot))$, where K is complex vector space and $(\cdot, \cdot)$ is an inner product on K.

Remark A.1. inner product space is a normed vector space,
with norm given by $\|x\| = (x, y)^{\frac{1}{2}}$.

Definition A.7. A Hilbert space $\mathbb{H}$ is a pre-Hilbert space which is complete with respect to the norm induced by the inner product $(\cdot, \cdot)$.

Appendix B

Some inequalities

Let X and Y be random variables defined on L^p. For $p \geq 1$, let $L^p = L^p(\Omega, \mathcal{F}, P)$ denote the family of $\mathbb{R}^n$-valued random variables X such that $\mathbb{E}\left[|X|^p\right] < \infty$. In L^1, we have the inequality $|\mathbb{E}\left[X\right]| \leq \mathbb{E}\left[|X|\right]$. Furthermore, the following four inequalities are very useful:

(i) Hölder's inequality

$$\big|\mathbb{E}\big(XY\big)\big| \leq \big(\mathbb{E}\left|X\right|^p\big)^{\frac{1}{p}}\big(\mathbb{E}\left|Y\right|^q\big)^{\frac{1}{q}},$$

with $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p, Y \in L^q$.

If $p = q = 2$, then Cauchy's inequality, defined by

$$\big|\mathbb{E}\big(XY\big)\big| \leq \big(\mathbb{E}\left|X\right|^2\big)^{\frac{1}{2}}\big(\mathbb{E}\left|Y\right|^2\big)^{\frac{1}{2}}.$$

A simple application of Hölder's inequality implies

$$\big(\mathbb{E}\left|X\right|^r\big)^{\frac{1}{r}} \leq \big(\mathbb{E}\left|X\right|^p\big)^{\frac{1}{p}},$$

if $0 < r < p < \infty, X \in L^p$.

(ii) Chebyshev's inequality

$$P\Big\{\omega : |X(\omega) \geq c|\Big\} \leq c^{-p}\mathbb{E}\left|X\right|^p,$$

if $c > 0$, $p > 0$, $X \in L^p$.

Theorem B.1. *[107] (Grönwall's inequality) Let $T > 0$ and $K \leq 0$. Let $f(.)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let $g(.)$ be a nonnegative integrable function on $[0, T]$. If*

$$f(t) \leq K + \int_0^t g(s) f(s) ds, \quad \text{for all } 0 \leq t \leq T,$$

then

$$f(t) \leq K \exp\left(\int_0^t g(s) ds\right), \quad \text{for all } 0 \leq t \leq T.$$

Conclusion and perspective

This thesis has made contributions to the study of specific classes of stochastic integro-differential equations (SIDEs) with non-instantaneous impulsive (NII) and nonlocal conditions. Various qualitative and quantitative properties of the solution were examined.The existence, uniqueness, controllability, and stability of the solution are verified. These results relied on a range of advanced mathematical tools, including fixed point theories, measures of non-compactness, resolvent operator, fractional calculus, and stochastic calculus.

We investigated the existence of a mild solution and the exact controllability for a class of SIDEs with non-instantaneous impulsive (NII) and non-local conditions, driven by fractional Brownian motion (FBM) (2.1). These were achieved by applying a generalized Darbo's Fixed Point Theorem. Additionally, non-compactness measures, semigroup theories, and resolvent operators were utilized. The results were illustrated and supported by two examples 2.1 and 2.2.

In addition, as a second result of this work, we studied a new class of Caputo FSIDEs with NII and non-local conditions (3.1). The existence and uniqueness of a mild solution were proven using Krasnoselski's Fixed Point Theorem, the Banach contraction principle, semigroup theories, and fractional and stochastic calculus. The UHR stability of the solutions was also verified. Example 3.4 was used to illustrate and support these results.

We have also explored the numerical solutions of VIDEs (4.1), where the strong convergence of the θ-EM method was studied. Meanwhile, the existence, uniqueness, boundedness, and Hölder continuity of the theoretical solution were established using Lipschitz theory, linear growth conditions, and the mean value theorem. The strong convergence order δ and mean square error between the numerical and theoretical solutions were explored. The results were supported by illustrative examples (4.1, 4.2), where the values of the strong convergence order and mean square error for different values of θ were provided in tables (table 4.1,table 4.2) for examples 4.1 and 4.2, respectively. Furthermore, the results were presented graphically for the strong convergence order and mean square error for different values of θ (respectively for examples 4.1 and 4.2 in figures 4.1a, 4.3a, 4.1b , 4.3b). A convergence rate of $\delta = \frac{1}{2}$ was achieved, and the approximate and numerical solutions displayed in curves (figure 4.2, figure 4.4) for both examples.

Finally, we are in the process of studying the optimal control of a class of SIDEs with NII and non-local conditions. Future research may expand on these results by exploring stochastic differential inclusions with delays, fractional stochastic differential equations with variable order, and infinite delays. Additionally, further investigation into advanced numerical techniques may be conducted to validate the results and broaden their potential applications.

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